O’ZBEKISTON RESPUBLIKASI XALQ TA’LIMI VAZIRLIGI

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TENGSIZLIKLAR-I.

ISBOTLASHNING KLASSIK USULLARI

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Fizika –matematika fanlari doktori, professor A. A’zamov umumiy tahriri ostida.

Qo’llanmada asosiy klassik sonli tengsizliklar hamda ularning qo’llanishiga doir turli matematik olimpiadalardagi masalalar keltirilgan.

Qo’llanma umumiy o’rta ta’lim maktablari, akademik litseylar va kasb–hunar kollejlarining iqtidorli o’quvchilari, matematika fani o’qituvchilari hamda pedagogika oliy o’quv yurtlari talabalari uchun mo’ljallangan.

Qo’llanmadan sinfdan tashqari mashg’ulotlarda, o’quvchilarni turli matematik musobaqalarga tayyorlash jarayonida foydalanish mumkin.

Taqrizchilar: TVDPI matematika kafedrasi mudiri, f.–m.f.n., dotsent Sh.B. Bekmatov

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Ushbu qo’llanma Respublika ta’lim markazi qoshidagi matematika fanidan ilmiy-metodik kengash tomonidan nashrga tavsiya etilgan. (15 iyun 2008 y., 8 -sonli bayyonnoma)

Qo’llanmaning yaratilishi Vazirlar Mahkamasi huzuridagi Fan va texnologiyalarni rivojlantirishni muvofiqlashtirish Q’omitasi tomonidan moliyalashtirilgan (ХИД 1-16 – sonli innovatsiya loyihasi**)**

§1. Sonli tengsizliklar haqida.

1. Sonli tengsizliklar va ularning xossalari.

Ta'rif: Agar $a-b$ ayirma musbat son bo'lsa, a soni b sonidan katta deyiladi va bu munosabat $a>b$ shaklida yoziladi. Agar $a-b$ ayirma manfiy bo'lsa, a soni b sonidan kichik deyiladi va $a<b$ shaklida yoziladi.

Istalgan a va b sonlar uchun quyidagi uchta munosabatdan faqat bittasi o'rinli:

1. $a-b>0 \Leftrightarrow a>b$;
2. $a-b<0 \Leftrightarrow a<b$;
3. $a-b=0 \Leftrightarrow a=b$.

Sonli tengsizliklar quyidagi xossalarga ega:

1^0. Agar $a>b$ va $b>c$ bo'lsa, $a>c$ bo'ladi (tengsizlik munosabatini tranzitivlik xossasi).

2^0. Agar $a>b$ va $c\in R$ bo'lsa, $a+c>b+c$ bo'ladi.

3^0. Agar $a>b$ va $c>0$ bo'lsa, $a\cdot c>b\cdot c$ bo'ladi.

4^0. Agar $a>b$ va $c<0$ bo'lsa, $a\cdot c<b\cdot c$ bo'ladi.

5^0. Agar $a>b$ va $c>d$ bo'lsa, $a+c>b+d$ bo'ladi.

6^0. Agar $a>b>0$ va $c>d>0$ bo'lsa, $a\cdot c>b\cdot d$ bo'ladi.

7^0. Agar $a>b>0$ va $n\in N$ bo'lsa, $a^n>b^n$ bo'ladi ($n-$toq son bo'lganda $b>0$ shart ortiqcha).

2.Tengsizliklarni isbotlashning usullari haqida.

1–misol. Istalgan a, b va c sonlari uchun $2a^2+b^2+c^2\geq 2a(b+c)$ ekanligini isbotlang.

Yechilishi. Istalgan a, b va c sonlari uchun $\left(2a^2+b^2+c^2\right)-2a(b+c)$ ayirmani manfiy emasligini ko'rsatamiz:

$$\left(2a^2+b^2+c^2\right)-2a(b+c)=\left(a^2-2ab+b^2\right)+\left(a^2-2ac+c^2\right)=$$
$$=\left(a-b\right)^2+\left(a-c\right)^2.$$

Istalgan sonning kvadrati nomanfiy son bo'lgani uchun $\left(a-b\right)^2\geq 0$ va $\left(a-c\right)^2\geq 0$. Demak, $\left(2a^2+b^2+c^2\right)-2a(b+c)$ istalgan a, b va c sonlari uchun manfiy emas. Shuning uchun berilgan tengsizlik istalgan a, b va c sonlari uchun o'rinli. Jumladan, tenglik belgisi $a=b=c$ bo'lgandagina bajariladi. Δ

Tengsizlikning to'g'riligini ko'rsatish uchun uning har ikkala qismining ayirmasini musbat yoki manfiyligini aniqlash, ya'ni 1-misoldagidek ta'rifdan foydalanib isbotlashga harakat qilish ayrim hollarda juda qiyin bo'ladi. Shuning uchun tengsizliklarni isbotlashda tengsizliklarning xossalaridan foydalaniladi.

2-misol. Musbat a, b va c sonlari uchun $\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\geq 6$ tengsizlikni isbotlang.

Yechilishi: Tengsizlikning chap qismida shakl almashtirish bajarib, uni quyidagi ko'rinishda yozamiz:

$$\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)\geq 6. \qquad (1)$$

Ikkita musbat son uchun o'rta arifmetik va o'rta geometrik qiymatlar orasidagi Koshi tengsizligidan foydalanamiz:

$$\frac{a}{b}+\frac{b}{a}\geq 2\sqrt{\frac{a}{b}\cdot\frac{b}{a}}=2, \quad \frac{a}{c}+\frac{c}{a}\geq 2, \quad \frac{b}{c}+\frac{c}{b}\geq 2 .$$

Bu tengsizliklarni hadma-had qo'shib, (1) tengsizlikni hosil qilamiz.

§2. O'rtacha qiymatlar va ular orasidagi munosabatlar.

1. O'rtacha qiymatlar.

$a=\{a_1, a_2, \dots, a_n\}$ musbat sonlar ketma-ketligi uchun

o'rta arifmetik qiymat $A(a)=A_n=\frac{a_1+a_2+\dots+a_n}{n}$,

o'rta geometrik qiymat $G(a)=G_n=\sqrt[n]{a_1a_2\dots a_n}$,

o'rta kvadratik qiymat $K(a)=K_n=\sqrt{\frac{a_1^2+a_2^2+\dots+a_n^2}{n}}$ va

o'rta garmonik qiymat $N(a)=N_n=\frac{n}{a_1^{-1}+a_2^{-1}+\dots+a_n^{-1}}$ larni aniqlaymiz.

Xususan x, y musbat sonlar uchun bu o'rta qiymatlar quyidagicha aniqlanadi:

$$A_2=\frac{x+y}{2};\quad G_2=\sqrt{xy};\quad K_2=\sqrt{\frac{x^2+y^2}{2}};\quad N_2=\frac{2xy}{x+y}.$$

2. O'rta arifmetik va o'rta geometrik qiymatlar haqida Koshi tengsizligi va uning turli isbotlari.

Teorema. $A_n \geq G_n$ va $A_n = G_n$ tenglik faqat va faqat $a_1=a_2=\dots=a_n$ tenglik bo'lganda o'rinli.

*1-Isboti***.** $x\geq 1$ da $e^{x-1}\geq x$ ekanligi ma'lum, $e^{x-1}=x$ tenglik esa faqat $x=1$ da bajariladi. Bundan:

$$1=e^0=exp\left(\sum_{i=1}^{n}\frac{a_i}{A(a)}-1\right)=\prod_{i=1}^{n}\exp(\frac{a_i}{A(a)}-1)\geq\prod_{i=1}^{n}\frac{a_i}{A(a)}=\left(\frac{G(a)}{A(a)}\right)^n.$$

Demak, $A_n \geq G_n$ va tenglik esa faqat $\frac{a_i}{A_n(a)}=1$, $i=1, 2, \dots, n$ bulganda bajariladi.

Bundan esa $a_1=a_2=\dots=a_n=A_n$ ekanligi kelib chiqadi.

$A_n \geq G_n$ ekanligini isbotlaymiz: $n=2$ da $\sqrt{a_1 \cdot a_2} \leq \frac{a_1+a_2}{2}$. Bu tengsizlik ixtiyoriy musbat a_1 va a_2 sonlar uchun o'rinli bo'lgan $\left(\sqrt{a_1}-\sqrt{a_2}\right)^2 \geq 0$ tengsizlikdan oson hosil qilinadi. Berilgan tengsizlikni ixtiyoriy *p* ta natural sonlar uchun to'g'ri deb, *p*+1 ta natural sonlar uchun to'g'riligini isbotlaymiz. Bu sonlar $a_1, a_2, ..., a_n, a_{n+1}$ bo'lsin. a_{n+1} ularning orasida eng kattasi bo'lsin. Ya'ni, $a_{n+1} \geq a_1, ... a_{n+1} \geq a_n$. Shuning uchun $a_{n+1} \geq \frac{a_1+a_2+...+a_n}{n}$. Quyidagicha belgilash kiritamiz:

$$A_n = \frac{a_1+a_2+...+a_n}{n}, \quad A_{n+1} = \frac{a_1+a_2+...+a_n+a_{n+1}}{n+1} = \frac{n \cdot A_n + a_{n+1}}{n+1}.$$

$a_{n+1} \geq A_n$ bo'lgani uchun $a_{n+1} = A_n + \alpha$ deb yozish mumkin, bu yyerda $\alpha \geq 0$. U holda $A_{n+1} = \frac{n \cdot A_n + A_n + \alpha}{n+1} = A_n + \frac{\alpha}{n+1}$. Bu tenglikni ikkala qismini (*p*+1) – darajaga ko'tarib, quyidagini topamiz:

$$\left(A_{n+1}\right)^{n+1} = \left(A_n + \frac{\alpha}{n+1}\right)^{n+1} = \left(A_n\right)^{n+1} + C_{n+1}^1 \left(A_n\right)^n \frac{\alpha}{n+1} + ... \geq$$
$$\geq \left(A_n\right)^{n+1} + \left(A_n\right)^n \cdot \alpha = \left(A_n\right)^n \cdot \left(A_n + \alpha\right) = \left(A_n\right)^n \cdot A_{n+1}.$$

Farazga ko'ra, $\left(A_n\right)^n \geq a_1 \cdot a_2 \cdot ... \cdot a_n$. Buni e'tiborga olib, $\left(A_{n+1}\right)^{n+1} \geq \left(A_n\right)^n \cdot a_{n+1} \geq a_1 \cdot a_2 \cdot ... \cdot a_n \cdot a_{n+1}$. Bundan $A_{n+1} \geq \sqrt[n+1]{a_1 \cdot a_2 \cdot ... \cdot a_n \cdot a_{n+1}}$. Tenglik $a_1 = a_2 = ... = a_n$ bo'lganda o'rinli bo'ladi.

2-isbot. Teoremaning isboti quyidagi tasdiqqa asoslangan:

Agar nomanfiy $b_1, b_2, ..., b_n$ sonlar $b_1 \cdot b_2 \cdot ... \cdot b_n = 1$ tenglikni qanoatlantirsa, u holda $b_1 + b_2 + ... + b_n \geq n$.

Bu tasdiqni masalani matematik induktsiya usulida isbotlaymiz.

$n = 1$ da masala ravshan. $n = k$ da $b_1 \cdot b_2 \cdot ... \cdot b_k = 1$ tenglikni qanoatlantiruvchi ixtiyoriy $b_1, b_2, ..., b_k$ – nomanfiy sonlar uchun $b_1 + b_2 + ... + b_k \geq k$ tengsizlik o'rinli bo'lsin. $n = k+1$ da $b_1 \cdot b_2 \cdot ... \cdot b_{k+1} = 1$ tenglikni qanoatlantiruvchi ixtiyoriy $b_1, b_2, ..., b_{k+1}$ – nomanfiy sonlar uchun $b_1 \cdot b_2 \cdot ... \cdot b_k \cdot b_{k+1} \geq 1$ tengsizlikni qanoatlantirishini ko'rsatamiz.

Umumiylikka zarar etkazmasdan $b_k \leq 1 \leq b_{k+1}$ deb hisoblaymiz. Unda $b_1 \cdot b_2 \cdot ... \cdot b_{k-1} \cdot \left(b_k \cdot b_{k+1}\right) = 1$ bo'lgani uchun induktsiya faraziga ko'ra $b_1 + b_2 + ... + b_{k-1} + b_k \cdot b_{k+1} \geq k$ bo'ladi. Endi $b_k + b_{k+1} \geq b_k \cdot b_{k+1} - 1$ ekanligini isbotlash etarli. Bu $\left(1 + b_k\right) \cdot \left(b_{k+1} - 1\right) \geq 0$ tengsizlikka teng kuchli $b_k \leq 1 \leq b_{k+1}$ bo'lgani uchun ohirgi tengsizlik o'rinli ekanligi ravshan.

3-isbot. Teoremaning isboti quyidagi ma'lum tasdiqqa asoslangan:

$x \geq 1$ da $e^{x-1} \geq x$, shu bilan birga $e^{x-1} = x$ tenglik esa faqat $x=1$ da bajariladi. Bundan:

$$1 = e^0 = exp\left(\sum_{i=1}^{n} \frac{a_i}{A(a)} - 1\right) = \prod_{i=1}^{n} \exp\left(\frac{a_i}{A(a)} - 1\right) \geq \prod_{i=1}^{n} \frac{a_i}{A(a)} = \left(\frac{G(a)}{A(a)}\right)^n.$$

Demak, $A(a) \geq G(a)$ va tenglik esa faqat $\frac{a_i}{A(a)} = 1$, $i=1, 2, ..., n$ bo'lganda bajariladi. Bundan esa $a_1 = a_2 = ... = a_n = A(a)$ ekanligi kelib chiqadi.

1-misol. $x, y > 0$ bo'lsa, $x^2 + y^2 + 1 \geq xy + x + y$ tengsizlikni isbotlang.

Yechilishi:

$$x^2+y^2+1\geq xy+x+y \Rightarrow \frac{x^2}{2}+\frac{x^2}{2}+\frac{y^2}{2}+\frac{y^2}{2}+$$

$$+\frac{1}{2}+\frac{1}{2}=x^2+y^2+1$$

$$+\begin{cases}\frac{x^2}{2}+\frac{y^2}{2}\geq xy,\\ \frac{y^2}{2}+\frac{1}{2}\geq y, \\ \frac{x^2}{2}+\frac{1}{2}\geq x.\end{cases} \Rightarrow x^2+y^2+1\geq xy+x+y.$$

2-misol. $x>0$ bo'lsa, $2^{\sqrt[12]{x}}+2^{\sqrt[4]{x}}\geq 2\cdot 2^{\sqrt[6]{x}}$ tengsizlikni isbotlang.

Yechilishi. $2^{\sqrt[12]{x}}+2^{\sqrt[4]{x}}\geq 2\cdot 2^{\sqrt[12]{x}}\cdot 2^{\sqrt[4]{x}}=2\cdot 2^{x^{\frac{1}{12}+\frac{1}{4}}}=2\cdot 2^{x^{\frac{1}{6}}}=2\cdot 2^{\sqrt[6]{x}}$.

Misollar:

1. Agar $x,\ y>0$ bo'lsa, $x^4+y^4+8\geq 8xy$ ni isbotlang.
2. $x_1,\ x_2,\ x_3, x_4, x_5>0$ bo'lsa quyidagini isbotlang:

$$x_1^2+x_2^2+x_3^2+x_4^2+x_5^2\geq x_1\left(x_2+x_3+x_4+x_5\right).$$

3. $x,y,z>0$ bo'lsa, $x^2+y^2+z^2\geq xy+yz+xz$ ni isbotlang.
4. $a,\ b,\ c>0$ bo'lsa, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$ ni isbotlang.
5. $a,\ b,\ c>0$ bo'lsa, $(a+1)(b+1)(c+a)(b+c)\geq 16abc$ ni isbotlang.

3. O'rta geometrik va o'rta garmonik qiymatlar orasidagi tengsizlik.

Teorema. $G(a)\geq H(a)$ ekanligini, jumladan, $H(a)=G(a)$ tenglik faqat va faqat $a_1=a_2=\ldots=a_n$ shart bajarilsa to'g'riligini isbotlang.

Isboti. Koshi tengsizligidan foydalanib (1-masalaga qarang) foydalanib

$(H(a))^{-1} = \frac{a_1^{-1} + a_2^{-1} + ... + a_n^{-1}}{n} \geq \sqrt[n]{a_1^{-1} a_2^{-1} ... a_n^{-1}} = (G(a))^{-1}$ tenglikga ega bo'lamiz. Jumladan, $H(a) = G(a)$ tenglik faqat $a_1 = a_2 = ... = a_n$ da bajariladi.

1-misol. Agar $a, b, c > 0$ bo'lsa, $\frac{3}{1/a + 1/b + 1/c} \leq \frac{a+b+c}{3}$ tengsizlikni isbotlang.

Yechilishi: $9 \leq (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$,

$$\begin{cases} a+b+c \geq 3\sqrt[3]{abc}, \\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3 \cdot \frac{1}{\sqrt[3]{abc}}. \end{cases} \Rightarrow (a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq \frac{9\sqrt[3]{abc}}{\sqrt[3]{abc}} = 9.$$

2-misol. Agar $a, b, c > 0$, $ab^2c^3 = 1$ bo'lsa, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 6$ ni isbotlang.

Yechilishi: $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{b} + \frac{1}{c} + \frac{1}{c} + \frac{1}{c} \geq 6 \frac{1}{\sqrt[6]{ab^2c^3}} = 6$.

Misollar

1. $x, y, z > 0$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\frac{1}{(x+y+z)^2} + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \geq \frac{28\sqrt{3}}{9\sqrt{xyz(x+y+z)}}.$$

2. Agar $x_1, x_2, ..., x_n > 0$ va $x_1 + x_2 + ... + x_n = 1$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\frac{x_1}{\sqrt{x_1 + x_2 + ... + x_n}} + \frac{x_2}{\sqrt{(1+x_1)(x_2 + ... + x_n)}} + ... + \frac{x_n}{\sqrt{(1 + x_1 + ... + x_{n-1})x_n}} \geq 1.$$

3. $x, y, z > 0$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\frac{x^2-xy}{x+y}+\frac{y^2-yz}{y+z}+\frac{z^2-xz}{x+z}\geq 0.$$

4. Agar $a, b, c > 0$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{a+c}}+\sqrt{\frac{c}{a+b}}\geq 2.$$

5. Agar $a, b, c, d > 0$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$a^{p+2}+b^{p+2}+c^{p+2}\geq a^{p+2}bc+b^{p+2}ac+c^{p+2}ab.$$

4. O'rta arifmetik va o'rta kvadratik qiymatlar orasidagi tengsizlik.

Teorema. K (a) ≥ A(a) tengsizlik o'rinli ekanligini, jumladan,
K(a) = A(a) tenglik faqat $a_1=a_2=...=a_n$ holdagina o'rinli bo'lishini isbotlang.

Isboti: Koshi tengsizligidan foydalanib (1-masalaga qarang) foydalanib $2a_ia_j\leq a_i^2+a_j^2,\ 1\leq i<j\leq n$ tengsizlikni hosil qilamiz.

Demak,

$$(a_1+a_2+...+a_n)^2=a_1^2+a_2^2+...+a_n^2+2\sum_{1\leq i<j\leq n}a_ia_j\leq$$

$$\leq a_1^2+a_2^2+...+a_n^2+\sum_{1\leq i<j\leq n}(a_i^2+a_j^2)=n(a_1^2+a_2^2+...+a_n^2).$$

Eslatib o'tamiz, *K (a) = A(a)* tenglik faqat $a_1=a_2=...=a_n$ o'rinli bo'ladi.

*1-misol***.**

$H(a)\geq min\{a_1, a_2, ..., a_n\}$ va $max\{a_1, a_2, ..., a_n\}\geq K(a)$ tengsizliklarni isbotlang.

Yechimi: Umumiylikni chegaralamagan holda

$$min\{a_1, a_2, ..., a_n\}=a_1,\ max\{a_1, a_2, ..., a_n\}=a_n$$

deb hisoblash mumkin. U holda

$$H(a)=\frac{n}{a_1^{-1}+a_2^{-1}+\ldots+a_n^{-1}}\geq\frac{n}{a_1^{-1}+a_1^{-1}+\ldots+a_1^{-1}}=a_1,$$

$$K(a)=\sqrt{\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}}\leq\sqrt{\frac{a_n^2+a_n^2+\ldots+a_n^2}{n}}=a_n \quad \text{bo'ladi.}$$

Izoh 1. yuqoridagi misollardan

$$max\{a_1, a_2, \ldots, a_n\}\geq K(a)\geq A(a)\geq G(a)\geq H(a)\geq min\{a_1, a_2, \ldots, a_n\}$$

ekanligi kelib chiqadi.

2-misol. $3\left(a^2+b^2+c^2\right)\geq\left(a+b+c\right)^2$ tengsizlikni isbotlang.

Yechilishi:

$$3a^2+3b^2+3c^2\geq a^2+b^2+c^2 2ab+2bc+2ac\Rightarrow$$

$$\Rightarrow a^2+b^2+c^2\geq ab+bc+ac\Rightarrow\left(a-b\right)^2+\left(b-c\right)^2+\left(c-a\right)^2\geq 0.$$

3-misol. $6\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)\geq\left(a+b\right)^2\left(a+b+c\right)^2$ tengsizlikni isbotlang.

Yechilishi:

$$\times\begin{cases}2\left(a^2+b^2\right)\geq\left(a+b\right)^2\\ 3\left(a^2+b^2+c^2\right)\geq\left(a+b+c\right)^2\end{cases}\Rightarrow 2a^2+2b^2\geq a^2+b^2+2ab\Rightarrow\left(a-b\right)^2\geq 0.$$

$$6\left(a^2+b^2\right)\left(a^2+b^2+c^2\right)\geq\left(a+b\right)^2\left(a+b+c\right)^2.$$

Misollar

1. Agar $a, b, c>0$ va $a+b+c=1$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\sqrt{a+\left(b-c\right)^2}+\sqrt{b+\left(a-c\right)^2}+\sqrt{c+\left(a-b\right)^2}\geq\sqrt{3}.$$

2. Agar $a, b, c>0$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\frac{a^3+b^3+c^3}{a^2b+b^2c+c^2a}+\frac{8abc}{\left(a+b\right)\left(b+c\right)\left(a+c\right)}\geq 4.$$

3. Agar $a, b, c \in R$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$(a+b)^4+(b+c)^4+(a+c)^4 \geq \frac{4}{7}\left(a^4+b^4+c^4\right).$$

4. Agar $x, y, z > 0$ va $x+y+z=3$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\sqrt{x}+\sqrt{y}+\sqrt{z} \geq xy+xz+yz.$$

5. Agar $a, b, c > 0$ va $a^2+b^2+c^2=1$ bo'lsa, u holda quyidagi tengsizlikni isbotlang:

$$\frac{bc}{a-a^3}+\frac{ac}{b-b^3}+\frac{ab}{c-c^3} \geq \frac{5}{2}.$$

§4. Umumlashgan Koshi tengsizligi.

Teorema. $a_1, a_2, ..., a_n, p_1, p_2, ..., p_n$ – musbat sonlar bo'lsin.

$$a_1^{p_1} a_2^{p_2} ... a_n^{p_n} \leq \left(\frac{a_1 p_1 + a_2 p_2 + ... + a_n p_n}{p_1 + p_2 + ... + p_n}\right)^{p_1+p_2+...+p_n} \qquad (1)$$

ekanligini isbotlang, tenglik esa faqat $a_1 = a_2 = ... = a_n$ da bajariladi.

Isboti: $s = \dfrac{a_1 p_1 + a_2 p_2 + ... + a_n p_n}{p_1 + p_2 + ... + p_n}$ belgilash kiritamiz.

$e^{x-1} \geq x$ $(x \geq 1)$ tengsizlikka ko'ra $s\, e^{(a_i-1)/s} \geq a_i$, $i=1, 2, ..., n.$

Bu tengsizliklarni barchasini ko'paytirib chiqamiz:

$$a_1^{p_1} a_2^{p_2} ... a_n^{p_n} \leq s^{p_1+p_2+...+p_n} \exp\left(\frac{a_1 p_1 + a_2 p_2 + ... + a_n p_n}{s} - p_1 + p_2 + ... + p_n\right) =$$

$$= s^{p_1+p_2+...+p_n}.$$

Tenglik faqat $s = a_1 = a_2 = ... = a_n$ da bajarilishi esa 1-masaladagidek isbotlanadi.

$$a_1^{p_1} a_2^{p_2} \dots a_n^{p_n} \le \left(\frac{a_1 p_1 + a_2 p_2 + \dots + a_n p_n}{p_1 + p_2 + \dots + p_n} \right)^{p_1 + p_2 + \dots + p_n}$$

Misol. Quyidagi tengsizlikni isbotlang:

$$\left(\frac{a^3 + 4b^4 + 3c^6}{8} \right)^8 \ge a^3 b^{16} c^{18}.$$

Yechilishi: Koshi tengsizligining umumiy holiga ko'ra p ning o'rnida 3 kelyapti.

1) $x \in R;\ p, q \in Q$ bo'lsa, $\sin^p x \cdot \cos^q x \le \dfrac{p^{\frac{p}{2}} q^{\frac{q}{2}}}{(p+q)^{\frac{p+q}{2}}}$ ni isbotlang.

2) $\left(\dfrac{3a^2 + 4b^3 + 5c^4}{12} \right)^{12} \ge a^6 b^{12} c^{20}$

3) $a, b, c > 0$ bo'lsa, $\dfrac{a^2}{b^2} + \dfrac{b^2}{c^2} + \dfrac{c^2}{a^2} + 4\left(\dfrac{ab + bc + ac}{a^2 + b^2 + c^2} \right)^2 \ge 7$ isbotlang.

4) $a, b, c > 0$ bo'lsa, $\sqrt{\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a}} + \sqrt{\dfrac{ab + bc + ac}{a^2 + b^2 + c^2}} \ge \sqrt{3} + 1$ isbotlang.

5) $a, b, c > 0$ bo'lsa, $\sqrt{\dfrac{a+b}{c} + \dfrac{b+c}{a} + \dfrac{a+c}{b}} + 2\sqrt{\dfrac{ab + bc + ac}{a^2 + b^2 + c^2}} \ge \sqrt{6} + 2$

isbotlang.

§5. Umumlashgan Yung tengsizligi.

Teorema.

$$a_1 a_2 \dots a_n \le \frac{a_1^{r_1}}{r_1} + \frac{a_2^{r_2}}{r_2} + \dots + \frac{a_n^{r_n}}{r_n} \qquad (2)$$

tengsizlik urinli, bu yyerda $a_1, a_2, \ldots, a_n, r_1, r_2, \ldots, r_n$ lar musbat sonlar, jumladan, $\frac{1}{r_1}+\frac{1}{r_2}+\ldots+\frac{1}{r_n}=1$.

Isboti: 5-masaladagi (1) tenglikda a_i ni $a_i^{r_i}$ ga , r_i ni esa $1/r_i$ $(i=1, 2, \ldots, n)$ ga almashtirib

$$a_1 a_2 \ldots a_n \le \frac{a_1^{r_1}}{r_1}+\frac{a_2^{r_2}}{r_2}+\ldots+\frac{a_n^{r_n}}{r_n}$$ ni olamiz.

Izoh. *n=2* holida esa Yung klassik tengsizligiga ega bo'lamiz:

$$\frac{1}{p}a^p+\frac{1}{q}b^q \ge ab \quad (a\ge 0,\ b\ge 0), \tag{3}$$

bu yyerda p, q sonlar $\frac{1}{p}+\frac{1}{q}=1$ tenglikni qanoatlantiruvchi musbat sonlar.

1-misol. Agar $a, b, c>0$ va $ab+bc+ac=abc$ bo'lsa, $abc \le \frac{a^b}{b}+\frac{b^c}{c}+\frac{c^a}{a}$

Yechilishi: Shartga ko'ra $ab+bc+ac=abc \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$

$abc \le \frac{a^b}{b}+\frac{b^c}{c}+\frac{c^a}{a}$ tengsizlik Yung tengsizligining xususiy holidan kelib chiqadi.

2-misol. Agar $a, b, c>0$ bo'lsa, $18a^2+12b^3+6c^6 \ge 36abc$ ni isbotlang.

Yechilishi: $18a^2+12b^3+6c^6 \ge 36abc$ tengizlikni ikkala tomonini 36 ga bo'lamiz $\frac{a^2}{2}+\frac{b^3}{3}+\frac{c^6}{6} \ge abc$; $\frac{1}{r_1}+\frac{1}{r_2}+\ldots+\frac{1}{r_n}=1$ bo'lsa, Yung tengsizligi o'rinli.

$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}=1 \Rightarrow \frac{a^2}{2}+\frac{b^3}{3}+\frac{c^6}{6} \ge abc$ tengsizlik o'rinli.

Misollar

1) Agar $x,\ y,\ z \geq 0$ va $x+y+z=2$ bo'lsa, $\sqrt{x^3y+y^3z+z^3x}+\sqrt{xy^3+yz^3+zx^3} \leq 2$ ni isbotlang.

2) Agar $a,\ b,\ c \geq 0$ va $a^2+b^2+c^2=3$ bo'lsa, $(2-ab)(2-bc)(2-ac) \geq 1$ ni isbotlang.

3) Agar $a,\ b,\ c \geq 0$ va $a+b+c=2$ bo'lsa, $\sqrt{a^2b+b^2c+c^2a}+\sqrt{ab^2+bc^2+ca^2} \leq 3$ ni isbotlang.

4) Agar $a,\ b,\ c \geq 0$ va $a+b+c=1$ bo'lsa, $\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2} \geq \sqrt{3}$ ni isbotlang.

5) Agar $a,\ b,\ c \geq 0$ bo'lsa, $\frac{a+b}{c}+\frac{b+c}{a}+\frac{a+c}{b} \geq 4\left(\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\right)$ ni isbotlang.

§6. Gel'der tengsizligi.

Teorema. $\frac{1}{p}+\frac{1}{q}=1$ shartni qanoatlantiruvchi barcha musbat p, q sonlar va a_j, b_j, $j = 1, ..., n$ sonlar uchun

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}} \qquad (4)$$

tengsizlik har doim to'g'ri.

Isboti. $\sum_{i=1}^{n} |a_i|^p \neq 0, \sum_{i=1}^{n} |b_i|^q \neq 0$ deb faraz qilamiz (aks holda (4) tengsizlik bajarilishi ravshan). Yung tengsizligini qo'llab

$$\left|\sum_{i=1}^{n}\frac{a_i}{\left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}}}\frac{b_i}{\left(\sum_{k=1}^{n}|b_k|^q\right)^{\frac{1}{q}}}\right| \le \left|\sum_{i=1}^{n}\frac{|a_i|}{\left(\sum_{k=1}^{n}|a_k|^p\right)^{\frac{1}{p}}}\frac{|b_i|}{\left(\sum_{k=1}^{n}|b_k|^q\right)^{\frac{1}{q}}}\right| \le$$

$$\le \left|\sum_{i=1}^{n}\frac{|a_i|^p}{p\sum_{k=1}^{n}|a_k|^p}\frac{|b_i|^q}{q\sum_{k=1}^{n}|b_k|^q}\right| = \frac{1}{p}+\frac{1}{q}=1$$

ga ega bo'lamiz. Bu yyerdan (4) tengsizlik kelib chiqadi.

*Izoh***.** Gyol'der tengsizligining $p = q= 2$ dagi $\left|\sum_{i=1}^{n}a_ib_i\right| \le \sqrt{\sum_{i=1}^{n}a_i^2}\sqrt{\sum_{i=1}^{n}b_i^2}$

Koshi-Bunyakovskiy-Shvarts tengsizligi deb ataluvchi bir muhim hususiy holini aytib o'tamiz.

1-misol (Minkovskiy tengsizligi). Ixtiyoriy musbat a_j, b_j $(j = 1,...,n)$ sonlar va natural p son uchun

$$\left(\sum(a_k+b_k)^p\right)^{1/p} \le \left(\sum a_k^p\right)^{1/p}\left(\sum b_k^p\right)^{1/p} \qquad (5)$$

tengsizlikni isbotlang

Yechilishi. $(a_k+b_k)^p = a_k(a_k+b_k)^{p-1} + b_k(a_k+b_k)^{p-1}$ $(k=1, 2, ..., n)$ tengsizlikni qo'shib,

$$\sum(a_k+b_k)^p = \sum a_k(a_k+b_k)^{p-1} + \sum b_k(a_k+b_k)^{p-1} \text{ ni olamiz.}$$

(4) tengsizlikka ko'ra

$$\sum a_k(a_k+b_k)^{p-1} \le \left(\sum a_k^p\right)^{1/p}\left(\sum(a_k+b_k)^{q(p-1)}\right)^{1/q},$$

$$\sum b_k(a_k+b_k)^{p-1} \le \left(\sum b_k^p\right)^{1/p}\left(\sum(a_k+b_k)^{q(p-1)}\right)^{1/q}$$

larga ega bo'lamiz, bu yyerdan $q(p-1)= p$ tenglik yordamida (6) tengsizlik kelib chiqadi.

Misollar

1) $\left(a_1b_1+a_2b_2\right)\le\left(a_1^2+a_2^2\right)^{\frac{1}{2}}\cdot\left(b_1^2+b_2^2\right)^{\frac{1}{2}}$ Gyol'der tengsizligining $p=q=2$ holiga ko'ra o'rinli.

2) $\left(a_1b_1+a_2b_2+a_3b_3\right)\le\left(a_1^{\frac{5}{3}}+a_2^{\frac{5}{3}}+a_3^{\frac{5}{3}}\right)^{\frac{5}{2}}\cdot\left(b_1^{\frac{5}{3}}+b_2^{\frac{5}{3}}+b_3^{\frac{5}{3}}\right)^{\frac{5}{2}}$ Gyol'der tengsizligining $n=3,\ p=\frac{5}{3},\ q=\frac{5}{2}$ holiga ko'ra o'rinli.

3) Agar $a,\ b,\ c\ge 0$ va $a^3+b^3+c^3=3$ bo'lsa, $a^4b^4+b^4c^4+c^4a^4\le 3$ ni isbotlang.

4) Agar $a,\ b,\ c\ge 0$ bo'lsa, $a^2+b^2+c^2+2abc+1\ge 2\left(ab+bc+ac\right)$ isbotlang.

5) Agar $a,\ b,\ c\ge 0$ va $abc=1$ bo'lsa, $\frac{a+b+c}{3}\ge\sqrt[5]{\frac{a^2+b^2+c^2}{3}}$ ni isbotlang.

Amaliyot uchun masalalar.

1-masala. Tengsizliklarni isbotlang:

$$n!>\left(\frac{n}{3}\right)^n,\quad n\in N; \tag{1}$$

$$n!<\left(\frac{n+1}{2}\right)^n,\quad \forall n\in N:n\ge 2; \tag{2}$$

$$n!>n^{\frac{n}{2}},\quad \forall n\in N:n\ge 3; \tag{3}$$

$$n!>2^{n-1},\quad \forall n\in N:n\ge 3; \tag{4}$$

$$\left(\frac{n}{e}\right)^n < n < e\left(\frac{n}{2}\right)^n, \quad \forall n \in N. \qquad (5)$$

(1) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n=1$ da: $1! > \left(\frac{1}{3}\right)^1$ ga egamiz. Induktsiya bazasi isbotlandi.

Induktiv o'tish. $n=k$ da $k! > \left(\frac{k}{3}\right)^k$ tengsizlik to'g'ri deb faraz qilamiz. $n=k+1$ da tengsizlik bajarilishini isbotlaymiz:

$$(k+1)! > \left(\frac{(k+1)}{3}\right)^{(k+1)}. (k+1)! = k!(k+1) > \left(\frac{k}{3}\right)^k (k+1)$$

ga egamiz.

$\left(\frac{k+1}{3}\right)^{(k+1)}$ songa ko'paytiramiz va bo'lamiz:

$$\left(\frac{k+1}{3}\right)^{k+1} \frac{3^{(k+1)} \cdot k^k \cdot (k+1)}{(k+1)^{(k+1)} \cdot 3^k} = \; = \left(\frac{k+1}{3}\right)^{k+1} \frac{3}{(1+\frac{1}{k})^k} > \left(\frac{k+1}{3}\right)^{k+1}.$$

Bu yyerda quyidagi joriy hisoblashlarni bajaramiz:

$$(1+\frac{1}{k})^k = 1 + \frac{1}{k} + \frac{k(k-1)}{2!} \cdot \frac{1}{k^2} + \ldots + \frac{k(k-1) \cdot \ldots \cdot (k-k+1)}{k!} \cdot \frac{1}{k^k} =$$

$$= 1 + 1 + \frac{1}{2!}\underbrace{\left(1-\frac{1}{k}\right)}_{<1} + \ldots + \frac{1}{k!}\underbrace{\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right) \cdot \ldots \cdot \left(1-\frac{k-1}{k}\right)}_{<1} <$$

$$=1+1+\frac{1}{2!}+\underbrace{\frac{1}{3!}}_{=\frac{1}{2\cdot 3}<\frac{1}{2^2}}+\ldots+\underbrace{\frac{1}{k!}}_{=\frac{1}{1\cdot 2\cdot 3\cdot\ldots\cdot k}<\frac{1}{2^{k-1}}}<1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{k-1}}<$$

$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{k-1}}+\frac{1}{2^k}+\ldots<1+\frac{1}{1-\frac{1}{2}}=3 \Rightarrow (1+\frac{1}{k})^k<3 \Rightarrow \frac{3}{(1+\frac{1}{k})^k}>1.$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy n natural son uchun (1) tengsizlik bajariladi deb xulosa qilamiz.

(2) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n=2$ da:

(2) tengsizlikning chap tomoni: $2!=2$;

(2) tengsizlikning o'ng tomoni: $\left(\frac{2+1}{2}\right)^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}=2,25$. $2<2,25$ demak, induktsiya bazasi isbot bo'ldi.

Induktiv o'tish. $n=k$ da $k!<\left(\frac{k+1}{2}\right)^k$, $k\geq 2$ tengsizlik to'g'ri deb faraz qilamiz. $n=k+1$ da $(k+1)!<\left(\frac{k+2)}{3}\right)^{k+1}$, $k\geq 2$ tengsizlik bajarilishini isbotlash kerak. $(k+1)!=k!\cdot(k+1)<\left(\frac{k+1}{2}\right)^k\cdot(k+1)=$ ga egamiz. $\left(\frac{k+2}{2}\right)^{k+1}$ songa ko'paytiramiz va bo'lamiz:

$$=\left(\frac{k+2}{2}\right)^{k+1}\frac{(k+1)^k\cdot(k+1)\cdot 2^{k+1}}{2^k\cdot(k+2)^{k+1}}=\left(\frac{k+2}{2}\right)^{k+1}\frac{2\cdot(k+1)^{k+1}}{(k+2)^{k+1}}<\frac{2\cdot(k+1)^{k+1}}{(k+2)^{k+1}}<1$$

tengsizlik bajarilishini isbotlaymiz.

$$\frac{2\cdot(k+1)^{k+1}}{(k+2)^{k+1}}=\frac{2}{\left(\dfrac{k+2}{k+1}\right)^{k+1}}=2\cdot\frac{1}{\left(1+\dfrac{1}{k+1}\right)^{k+1}}$$

$$(1+\frac{1}{k+1})^{k+1}=1+\frac{k+1}{k+1}+\underbrace{\frac{(k+1)\cdot k}{2!}\cdot\frac{1}{(k+1)^2}+...+\left(\frac{1}{k+1}\right)^k}_{>0}>2.$$

$$\Rightarrow(1+\frac{1}{k+1})^{k+1}<\frac{1}{2}\ \Rightarrow 2\cdot\left(\frac{k+2}{k+1}\right)^{k+1}<2\cdot\frac{1}{2}=1\ .$$

$$<\left(\frac{k+2}{2}\right)^{k+1}\cdot 1=\left(\frac{k+2}{2}\right)^{k+1}.$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy $n\geq 2$ natural son uchun (2) tengsizlik bajariladi deb xulosa qilamiz.

(3) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n=3$ da:

(3) tengsizlikning chap tomoni: $3!=6=\sqrt{36}$;

(3) tengsizlikning o'ng tomoni: $3^{\frac{3}{2}}=\sqrt{27}$. $\sqrt{36}<\sqrt{27}$ demak, induktsiya bazasi isbot bo'ldi.

Induktiv o'tish. $n=k$ da $k!>k^{\frac{k}{2}}$, $k\geq 3$ tengsizlik to'g'ri deb faraz qilamiz.

$n=k+1$ da $(k+1)!>k^{\frac{k+1}{2}}$, $k\geq 3$ tengsizlik bajarilishini isbotlash kerak.

$$(k+1)!=k!\cdot(k+1)>k^{\frac{k}{2}}\cdot(k+1)\cdot\frac{(k+1)^{\frac{k+1}{2}}}{(k+1)^{\frac{k+1}{2}}}=(k+1)^{\frac{k+1}{2}}\cdot\frac{k^{\frac{k}{2}}\cdot(k+1)^{\frac{1}{2}}\cdot k^{\frac{1}{2}}}{(k+1)^{\frac{k}{2}}\cdot k^{\frac{1}{2}}}>$$

(6) masalada $n^{n+1} > (n+1)^n, \ \forall n \geq 3$ tengsizlik isbotlangan edi. Bu tengsizlikdan kelib chiqadi: $n^{\frac{n+1}{2}} > (n+1)^{\frac{n}{2}}, \ \forall n \geq 3$. U holda $n=k$ da

$$> (k+1)^{\frac{k+1}{2}} \cdot \frac{(k+1)^{\frac{k}{2}} \cdot (k+1)^{\frac{1}{2}}}{(k+1)^{\frac{k}{2}} \cdot k^{\frac{1}{2}}} = (k+1)^{\frac{k+1}{2}} \cdot \underbrace{(1+\frac{1}{k})^{\frac{1}{2}}}_{>1} > (k+1)^{\frac{k+1}{2}}.$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy $n \geq 3$ natural son uchun (3) tengsizlik bajariladi deb xulosa qilamiz.

(4) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n = 3$ da:

(4) tengsizlikning chap tomoni: $3! = 6 = \sqrt{36}$;

(4) tengsizlikning o'ng tomoni: $2^{3-1} = 4$. $6 > 4$ demak, induktsiya bazasi isbot bo'ldi.

Induktiv o'tish. (4) tengsizlik $n = k$ da bajariladi deb faraz qilamiz: $k! > 2^{k-1}, \ \forall k \geq 3$. $n = k+1$ da $(k+1)! > 2^{k+1-1}, \ k \geq 3$ tengsizlik bajarilishini isbotlash kerak.

$$(k+1)! = k! \cdot (k+1) > 2^{k-1} \cdot (k+1) = 2^k \cdot \underbrace{\frac{k+1}{2}}_{>1} > 2^k, \quad k \geq 3 .$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy $n \geq 3$ natural son uchun (4) tengsizlik bajariladi deb xulosa qilamiz.

(5) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n = 1$ da: $\left(\frac{1}{e}\right)^1 < 1! < e\left(\frac{1}{2}\right)^1$. Induktsiya bazasi isbot bo'ldi.

Induktiv o'tish. $n=k$ da (5) tengsizlik to'g'ri deb faraz qilamiz: $\left(\frac{k}{e}\right)^k < k! < e\left(\frac{k}{2}\right)^k$. $n=k+1$ da $\left(\frac{k+1}{e}\right)^{k+1} < (k+1)! < e\left(\frac{k+1}{2}\right)^{k+1}$ tengsizlik bajarilishini isbotlash kerak. Bu tengsizlikning chap tomonini isbotlaymiz.

$$(k+1)! = k!\cdot(k+1) > (k+1)\cdot\left(\frac{k}{e}\right)^k = \left(\frac{k+1}{e}\right)^{k+1}\cdot\frac{(k+1)\cdot\left(\frac{k}{e}\right)^k}{\left(\frac{k+1}{e}\right)^{k+1}} =$$

$$= \left(\frac{k+1}{e}\right)^{k+1}\frac{(k+1)\cdot k^k\cdot e^{k+1}}{(k+1)^{k+1}} = \left(\frac{k+1}{e}\right)^{k+1}\cdot\frac{e}{\underbrace{\left(1+\frac{1}{k}\right)^k}_{e}} > \left(\frac{k+1}{e}\right)^{k+1}.$$

(5) tengsizlikning chap tomonini isbotlaymiz.

$$(k+1)! = k!\cdot(k+1) < e\left(\frac{k}{2}\right)^k = e\left(\frac{k+1}{2}\right)^{k+1}\cdot\frac{\left(\frac{k}{2}\right)^k}{\left(\frac{k+1}{2}\right)^{k+1}} = e\left(\frac{k+1}{2}\right)^{k+1}\times$$

$$\times\underbrace{\frac{2}{(k+1)}}_{\le 1}\cdot\underbrace{\left(\frac{k}{k+1}\right)^{k+1}}_{<1} < e\left(\frac{k+1}{2}\right)^{k+1}.$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy n natural son uchun (5) tengsizlik bajariladi deb hulosa qilamiz.

Eslatib o'tamiz, (2) tengnsizlik va $\left(1+\frac{1}{n}\right)^n < e$ tengsizlikdan foydalanib, $n>1$ da

$$k! < \left(\frac{n+1}{2}\right)^n = e\cdot\left(\frac{n}{2}\right)^n \cdot \frac{\left(\frac{n+1}{2}\right)^n}{e\cdot\left(\frac{n}{2}\right)^n} = e\cdot\left(\frac{n}{2}\right)^n \cdot \frac{\left(1+\frac{1}{n}\right)^n}{e} < e\cdot\left(\frac{n}{2}\right)^n.$$

2-masala. Tengsizliklarni isbotlang:

$$x_n = \underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{n+1 \text{ та илдиз}} < \sqrt{2}+1, \quad n\in N; \tag{6}$$

$$x_n = \underbrace{\sqrt{4+\sqrt{4+...+\sqrt{4}}}}_{n \text{ та илдиз}} < 3, \quad n\in N. \tag{7}$$

(6) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n=1$ da: $x_n = \sqrt{2+\sqrt{2}} < \sqrt{2+2\sqrt{2}+1} =$ $= \sqrt{(\sqrt{2}+1)^2} = \sqrt{2}+1$. Induktsiya bazasi isbotlandi.

Induktiv o'tish. $n=k$ da $x_k = \underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{k+1 \text{ та илдиз}} < \sqrt{2}+1$ tengsizlik to'g'ri deb faraz qilamiz. $n=k+1$ da tengsizlik bajarilishini isbotlash kerak: $x_{k+1} = \underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{k+2 \text{ та илдиз}} < \sqrt{2}+1.$

$$x_{k+1} = \underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{k+1 \text{ та илдиз} < \sqrt{2}+1} < \sqrt{2+\sqrt{2}+1} < \sqrt{2+2\sqrt{2}+1} = \sqrt{2}+1$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy n natural son uchun (6) tengsizlik bajariladi deb xulosa qilamiz.

(7) tengsizlikni isbotlaymiz.

Induktsiya bazasi. $n=1$ da: $x_1 = \sqrt{4} < \sqrt{9}$. Induktsiya bazasi isbotlandi.

Induktiv o'tish. $n=k$ da $x_k=\underbrace{\sqrt{4+\sqrt{4+...+\sqrt{4}}}}_{k\ \text{та илдиз}}<3$ tengsizlik to'g'ri deb faraz qilamiz. $n=k+1$ da: $x_{k+1}=\sqrt{4+\underbrace{\sqrt{4+\sqrt{4+...+\sqrt{4}}}}_{k\ \text{та илдиз}(<3)}}<\sqrt{4+3}<3$.

Matematik induktsiya printsipiga asoslanib, ixtiyoriy n natural son uchun (7) tengsizlik bajariladi deb xulosa qilamiz.

3-masala.

$$\frac{5^n}{n!}\le\frac{5^5}{5!}\left(\frac{5}{6}\right)^{n-5},\quad \forall n\in N: n\ge 6, \tag{8}$$

tengsizlikni isbotlang.

Induktsiya bazasi. $n=6$ da: $\frac{5^6}{6!}\le\frac{5^5}{5!}\left(\frac{5}{6}\right)^{6-5}$.Induktsiya bazasi isbotlandi.

Induktiv o'tish. $\frac{5^k}{k!}\le\frac{5^5}{5!}\left(\frac{5}{6}\right)^{k-5}$, $k\ge 6$ tengsizlik bajariladi deb faraz qilamiz. $\frac{5^{k+1}}{(k+1)!}\le\frac{5^5}{5!}\left(\frac{5}{6}\right)^{k+1-5}$ tengsizlik bajarilishini isbotlash kerak.

$$\frac{5^{k+1}}{(k+1)!}\le\underbrace{\frac{5^k}{k!}}_{\le\frac{5^5}{5!}\left(\frac{5}{6}\right)^{k-5}}\cdot\underbrace{\frac{5}{k+1}}_{<\frac{5}{6}}\le\frac{5^5}{5!}\left(\frac{5}{6}\right)^{k-5}\frac{5}{6}=\frac{5^5}{5!}\left(\frac{5}{6}\right)^{k+1-5}.$$

Matematik induktsiya printsipiga asoslanib ixtiyoriy $n\ge 6$ natural son uchun (8) tengsizlik bajariladi deb xulosa qilamiz.

4-masala. Ixtiyoriy n natural son uchun

$$|\sin kx|\le k|\sin x|, \tag{9}$$

tengsizlik o'rinli bo'lishini isbotlang.

Induktsiya bazasi. $n=1$ da: $\left|\sin 1x\right| \le 1\cdot\left|\sin x\right|$. Induktsiya bazasi isbotlandi.

Induktiv o'tish. $n=k$ da $\left|\sin kx\right| \le k\left|\sin x\right|$ tengsizlik bajariladi deb faraz qilamiz. $\left|\sin(k+1)x\right| \le (k+1)\left|\sin x\right|$ tengsizlik bajarilishini isbotlash kerak.

$$\left|\sin(k+1)x\right| = \left|\sin kx\cos x + \sin x\cos kx\right| \le \underbrace{\left|\sin kx\right|}_{\le k\left|\sin x\right|}\cdot\underbrace{\left|\cos x\right|}_{\le 1} + \sin x\underbrace{\left|\cos kx\right|}_{\le 1} \le$$

$$\le (k+1)\left|\sin x\right|.$$

Matematik induktsiya printsipiga asoslanib, ixtiyoriy n natural son uchun (9) tengsizlik bajariladi deb xulosa qilamiz.

5-masala. Ixtiyoriy n natural son uchun

$$\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{3n+1}>1, \qquad (10)$$

tengsizlik o'rinli bo'lishini isbotlang.

$S_k=\dfrac{1}{k+1}+\dfrac{1}{k+2}+...+\dfrac{1}{3k+1}$ deb belgilab olamiz.

Induktsiya bazasi. $n=1$ da: $S_1=\dfrac{1}{1+1}+\dfrac{1}{1+2}+...+\dfrac{1}{3\cdot 1+1}=\dfrac{13}{12}>1$.

Induktsiya bazasi isbotlandi.

Induktiv o'tish. $n=k$ da $S_k=\dfrac{1}{k+1}+\dfrac{1}{k+2}+...+\dfrac{1}{3k+1}>1$ tengsizlik bajariladi deb faraz qilamiz.

$$S_{k+1}=\frac{1}{k+2}+\frac{1}{k+3}+...+\frac{1}{3k+1}+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}>1$$

tengsizlik bajarilishini isbotlash kerak.

$$S_{k+1}=\frac{1}{k+2}+\frac{1}{k+3}+...+\frac{1}{3k+1}+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}+\left(\frac{1}{k+1}-\frac{1}{k+1}\right)=$$

$$=\underbrace{\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+...+\frac{1}{3k+1}}_{=S_1>1}+\underbrace{\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}-\frac{1}{k+1}}_{>0}>1.$$

$$\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}-\frac{1}{k+1}=\frac{1}{3k+2}+\frac{1}{3k+4}-\frac{2}{3k+3}=$$

$$=\frac{(3k+4)(3k+3)+(3k+2)(3k+3)-(6k+4)(3k+4)}{(3k+2)(3k+3)(3k+4)}=\frac{2}{(3k+2)(3k+3)(3k+4)}>0$$

ekanligidan ">0" tengsizlik kelib chiqadi. Matematik induktsiya printsipiga asoslanib, ixtiyoriy n natural son uchun (10) tengsizlik bajariladi deb xulosa qilamiz.

6-masala. $x^2+y^2+z^2\ge xy+xz+yz$ tengsizlikni isbotlang, bu yyerda x, y, z - musbat sonlar.

Yechilishi. Ma'lum $x^2+y^2\ge 2xy,\ x^2+z^2\ge 2xz,\ y^2+z^2\ge 2yz$ tengsizliklarni qo'shib, ushbu $(x^2+y^2)+(x^2+z^2)+(y^2+z^2)\ge 2(x^2+y^2+z^2)\ge 2(xy+xz+yz)$ tengsizlikni olamiz.

7-masala. $x^4+y^4+z^4\ge xyz(x+y+z)$ tengsizlikni isbotlang, bu yerda x, y, z - musbat sonlar.

Yechilishi. 1-masalaga ko'ra: $x^4+y^4+z^4=(x^2)^2+(y^2)^2+(z^2)^2\ge x^2y^2+y^2z^2+x^2z^2$ ga egamiz. Bu yerdan esa $x^2y^2+y^2z^2+x^2z^2\ge xyyz+yzzx+zxxy=xyz(x+y+z)$ ni olamiz.

8-masala. $x^4+y^4+z^4+u^4\ge 4xyzu$ tengsizlikni isbotlang, bu yerda x, y, z, u - musbat sonlar.

Yechilishi. $x^4+y^4\ge 2x^2y^2,\ z^4+u^4\ge 2z^2u^2$ ga egamiz. Demak, $x^4+y^4+z^4+u^4\ge 2x^2y^2+2z^2u^2$. Bundan tashqari $x^2y^2+z^2u^2\ge 2xyzu$. Demak, $x^4+y^4+z^4+u^4\ge 4xyzu$.

9-masala. $\frac{1}{2}(x+y)^2+\frac{1}{4}(x+y)\geq x\sqrt{y}+y\sqrt{x}$ tengsizlikni isbotlang, bu yerda x, y - musbat sonlar.

Yechilishi. Birinchidan, $\frac{1}{2}(x+y)^2+\frac{1}{4}(x+y)=\frac{1}{2}(x+y)(x+y+\frac{1}{2})$.

Ikkinchidan, $\frac{x+y}{2}\geq\sqrt{xy}$, $x+y+\frac{1}{2}=x+\frac{1}{4}+y+\frac{1}{4}\geq\sqrt{x}+\sqrt{y}$.

Demak, $\frac{1}{2}(x+y)^2+\frac{1}{4}(x+y)=\sqrt{xy}(\sqrt{y}+\sqrt{x})\geq x\sqrt{y}+y\sqrt{x}$.

10-masala. $x\geq 0$, $y\geq 0$ va $x+y=2$ bo'lsin.

$x^2y^2(x^2+y^2)\leq 2$ tengsizlikni isbotlang.

Aniqlik uchun $x=1+\varepsilon,\ y=1-\varepsilon,\ 0\leq\varepsilon\leq 1$ deb olamiz. U holda $x^2y^2(x^2+y^2)=(1-\varepsilon)^2(1+\varepsilon)^2((1-\varepsilon)^2+(1+\varepsilon)^2)=(1-\varepsilon^2)^2(2+2\varepsilon^2)=$

$=2(1-\varepsilon^2)(1-\varepsilon^2)(1+\varepsilon^2)=2(1-\varepsilon^2)(1-\varepsilon^4)\leq 2$

11-masala. a va b bir xil ishorali sonlar bo'lsin.

$\sqrt[3]{\frac{a^2b^2(a+b)^2}{4}}\leq\frac{a^2+10ab+b^2}{12}$ ekanligini isbotlang.

Qachon tenglik bajariladi? $ab>0$ ekanligini hisobga olib va Koshi tengsizligidan foydalanib quyidagiga ega bo'lamiz:

$$\sqrt[3]{ab\cdot ab\cdot\frac{(a+b)}{2}}\leq\frac{ab+ab+\frac{a^2+2ab+b^2}{4}}{3}=\frac{a^2+10ab+b^2}{12}.$$

Tenglik esa $a=b$ bo'lganda bajarilishini eslatib o'tamiz.

12-masala. a,b va c birdan katta sonlar bo'lsin.

$\log_a(\frac{b^2}{ac}-b+ac)\log_b(\frac{c^2}{ab}-c+ab)\log_b(\frac{a^2}{bc}-a+bc)\geq 1$ tengsizlikni isbotlang.

$a>1,\ b>1,\ c>1$ va $\frac{b^2}{ac}+ac\geq 2b,\ \frac{c^2}{ab}+ab\geq 2c,\ \frac{a^2}{bc}+bc\geq 2a$ bo'lgani uchun

$$\log_a(\frac{b^2}{ac}-b+ac)\log_b(\frac{c^2}{ab}-c+ab)\log_b(\frac{a^2}{bc}-a+bc)\geq \log_a(2b-b)\times$$

$$\times\log_b(2c-c)\log_c(2a-a)=\log_a b\log_b c\log_c a=1.$$

13-masala. a va b musbat sonlar bo'lsin.

$$\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{a}}\leq\sqrt[3]{2(a+b)(\frac{1}{a}+\frac{1}{b})}$$

tengsizlikni isbotlang.

Yechilishi. Berilgan tengsizlikni kubga ko'tarish va soddalashtirishlardan so'ng quyidagiga ega bo'lamiz:

$3(\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{a}})\leq 4+\frac{a}{b}+\frac{b}{a}$. Koshi tengsizligiga ko'ra $1+1+\frac{a}{b}\geq 3\sqrt[3]{\frac{a}{b}}$ va $1+1+\frac{b}{a}\geq 3\sqrt[3]{\frac{b}{a}}$ tengsizliklarni olamiz va ularni qo'shib, qidirilayotgan tengsizlikni olamiz.

14-masala. $\frac{2n}{3n+1}\leq\sum_{k=n+1}^{2n}\frac{1}{k}\leq\frac{3n+1}{4(n+1)}$ $(n\in N)$ ekanligini isbotlang.

Yechilishi.

$$\sum_{k=n+1}^{2n}\frac{1}{k}=\sum_{k=n+1}^{2n}\frac{1}{3n+1-k}=\frac{1}{2}\sum_{k=n+1}^{2n}(\frac{1}{k}+\frac{1}{3n+1-k})=\frac{1}{2}\sum_{k=n+1}^{2n}\frac{1}{k}\leq\frac{3n+1}{k(3n+1-k)}\ ;$$

$1)\,k(3n+1-k)=\frac{(3n+1)^2}{4}-(\frac{3n+1}{4}-k^2)\leq\frac{(3n+1)^2}{4}$ tengsizlikdan

$\frac{1}{2}\sum_{k=n+1}^{2n}\frac{3n+1}{k(3n+1-k)}\geq\frac{4n(3n+1)}{2(3n+1)^2}=\frac{2n}{3n+1}$ ekanligi kelib chiqadi;

2) $k(3n+1-k)-2n(n+1)=(2n-k)(k-(n+1))\geq 0 \quad (n+1\leq k\leq 2n)$

tengsizlikdan $\frac{1}{2}\sum_{k=n+1}^{2n}\frac{3n+1}{k(3n+1-k)}\leq\frac{n(3n+1)}{4n(n+1)}=\frac{3n+1}{4(n+1)}$ ekanligi kelib chiqadi.

Demak $\frac{2n}{3n+1}\leq\sum_{k=n+1}^{2n}\frac{1}{k}\leq\frac{3n+1}{4(n+1)} \quad (n\in N)$.

15-masala. a,b va c musbat sonlar bo'lsin. $2(a^4+b^4+c^4)<(a^2+b^2+c^2)^2$ tengsizlik a,b va c faqat biror uchburchak tashkil qilgandagina bajarilishi mumkinligini isbotlang.

Yechilishi. Ravshanki, bizning tengsizligimiz $a^4+b^4+c^4-2a^2b^2+2b^2c^2+2a^2c^2<0$ tengsizlikka teng kuchli. Oxirgi tengsizlikning chap tomonini almashtiramiz:

$$a^4+b^4+c^4-2a^2b^2+2b^2c^2+2a^2c^2=a^2+b^2-c^2-4a^2b^2=$$
$$=(a^2+b^2-c^2-2ab)(a^2+b^2-c^2+2ab)=((a-b)^2-c^2)((a+b)^2-c^2)=$$
$$=(a-b+c)(a-b-c)(a+b+c)(a+b-c)$$

Demak, berilgan tengsizlik a,b va c biror uchburchak tashkil qilganda aniq ravishda bajariladigan ushbu $(a-b+c)(a-b-c)(a+b+c)(a+b-c)>0$ tengsizlikka teng kuchli.

Endi faraz qilamiz, bu tengsizlik bajariladi, biroq a, b va c biror uchburchak tashkil qilmaydi. U holda $a-b+c, a-b-c, a+b+c, a+b-c$ sonlardan kamida ikkitasi manfiy. $a+b-c<0$ va $b+c-a<0$ bo'lsin. Bu yerdan masalaning shartiga zid bo'lgan $2b<0$ tengsizlikni olamiz.

16-masala. $b_1, b_2, \ldots, b_n$ – $a_1, a_2, \ldots, a_n$ ketma-ketlikning biror o'rin almashtirishi bo'lsin.

$$(a_1+\frac{1}{b_1})(a_2+\frac{1}{b_2})\ldots(a_n+\frac{1}{b_n})\geq 2^n$$

tengsizlikni isbotlang.

Tengsizlikning isboti Koshi tengsizligidan darhol kelib chiqadi:

$$(a_1+\frac{1}{b_1})(a_2+\frac{1}{b_2})...(a_n+\frac{1}{b_n})\geq 2\sqrt{\frac{a_1}{b_1}}2\sqrt{\frac{a_2}{b_2}}...2\sqrt{\frac{a_n}{b_n}}=\sqrt{\frac{a_1a_2...a_n}{b_1b_2...b_n}}=2^n.$$

17-masala. $x^{10}+x^6+x^5+x^3+x^2+x+1>0$ tengsizlikni isbotlang.

Yechilishi. Ravshanki, tengsizligimiz $x\geq 0$ da bajariladi, shuning uchun x manfiy bo'lgan qiymatlarni qarash etarli. $x\leq -1$ bo'lsin, holda $x^{10}+x^5\geq 0$, $x^6+x^3\geq 0$, $x^2+x\geq 0$ va $1>0$ tengsizliklarni qo'shib, izlanayotgan tengsizlikni olamiz.

Endi $-1<x<0$ bo'lsin. Qism hollarni qaraymiz:

a) $x^5+x+1>0$. U holda

$$x^{10}+x^6+x^5+x^3+x^2+x+1=x^{10}+x^6+x^2+1+x^3+(x^5+x+1)>0.$$

b) $x^5+x+1\leq 0$. U holda

$$x^{10}+x^6+x^5+x^3+x^2+x+1=x^5(x^5+x+1)+(x^2+x^3)+(1+x)>x^5(x^5+x+1)\geq 0.$$

18-masala. $x>-1,\ y>-1$ va $z>-1$ bo'lsin.

$S=\frac{1+x^2}{1+y+z^2}+\frac{1+y^2}{1+z+x^2}+\frac{1+z^2}{1+x+y^2}\geq 2$ tengsizlikni isbotlang.

$\frac{1+x^2}{1+y+z^2}\geq\frac{1+x^2}{1+|y|+z^2}$ tengsizlik yordamida $x,\ y$ va z ning manfiy bo'lmagan qiymatlarini qarash etarli.

$$S=\frac{1+z+x^2}{1+y+z^2}+\frac{1+x+y^2}{1+z+x^2}+\frac{1+y+z^2}{1+x+y^2}-(\frac{z}{1+y+z^2}+\frac{x}{1+z+x^2}+\frac{y}{1+x+y^2})\geq$$

$$\geq 3\sqrt[3]{\frac{1+z+x^2}{1+y+z^2}\cdot\frac{1+x+y^2}{1+z+x^2}\cdot\frac{1+y+z^2}{1+x+y^2}}-(\frac{z}{1+y+z^2}+\frac{x}{1+z+x^2}+\frac{y}{1+x+y^2})\geq$$

$$\geq 3-(\frac{z}{1+y+z^2}+\frac{x}{1+z+x^2}+\frac{y}{1+x+y^2}).$$

Endi $S_1=\frac{z}{1+y+z^2}+\frac{x}{1+z+x^2}+\frac{y}{1+x+y^2}\le 1$ ekanligini isbotlaymiz. $x=0$ holni qaraymiz. U holda. Demak, $xyz=0$ da $S_1\le 1$.

$xyz\ne 0$ holda

$$S_1=\frac{1}{(x+\frac{1}{x})+\frac{z}{x}}+\frac{1}{(z+\frac{1}{z})+\frac{y}{z}}+\frac{1}{(y+\frac{1}{y})+\frac{x}{y}}\le\frac{1}{2+\frac{z}{x}}+\frac{1}{2+\frac{y}{z}}+\frac{1}{2+\frac{x}{y}}.$$

$\frac{x}{y}=a,\ \frac{y}{z}=b$ va $\frac{z}{x}=c$ deb belgilab olamiz. $abc=1$ va $S_1=\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}$ ekanligi ravshan. U holda

$$\frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}=\frac{(2+b)(2+c)+(2+a)(2+c)+(2+a)(2+b)}{(2+a)(2+b)(2+c)}=$$

$$=\frac{12+4(a+b+c)+(ab+bc+ac)}{8+4(a+b+c)+2(ab+bc+ac)+abc}\le$$

$$\le\frac{12+4(a+b+c)+(ab+bc+ac)}{8+4(a+b+c)+(ab+bc+ac)+3\sqrt[3]{a^2b^2c^2}+abc}=\frac{12+4(a+b+c)+(ab+bc+ac)}{12+4(a+b+c)+(ab+bc+ac)}=1.$$

Demak, $S_1\le 1$.

19-masala. Ixtiyoriy musbat $a_j,\ b_j\ (j=1,2,...,n)$ sonlar uchun

$$\sqrt[n]{a_1...a_n}+\sqrt[n]{b_1...b_n}\le\sqrt[n]{(a_1+b_1)...(a_n+b_n)}$$

tengsizlik o'rinli ekanligini isbotlang.

Yechilishi. Gyuygens tengsizligiga asoslanib $\left(1+\frac{a_1}{b_1}\right)...\left(1+\frac{a_n}{b_n}\right)\ge\left(1+\sqrt[n]{\frac{a_1}{b_1}...\frac{a_n}{b_n}}\right)^n$

yoki $(a_1+b_1)...(a_n+b_n)\ge\left(\sqrt[n]{a_1...a_n}+\sqrt[n]{b_1...b_n}\right)^n$ ni olamiz. Bu yerdan esa $\sqrt[n]{a_1...a_n}+\sqrt[n]{b_1...b_n}\le\sqrt[n]{(a_1+b_1)...(a_n+b_n)}$ kelib chiqadi.

20-masala. Ixtiyoriy $a_1, a_2, \ldots, a_n$ musbat sonlar uchun $(a_1+a_2+\ldots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+..+\frac{1}{a_n}\right)\geq n^2$ tengsizlik o'rinli bo'lishini isbotlang.

Yechilishi. Koshi–Bunyakovskiy–Shvarts tengsizligiga ko'ra

$$n^2=\left(\sqrt{a_1}\cdot\frac{1}{\sqrt{a_1}}+\sqrt{a_2}\cdot\frac{1}{\sqrt{a_2}}+\ldots+\sqrt{a_n}\cdot\frac{1}{\sqrt{a_n}}\right)^2\leq$$

$$\leq\left((\sqrt{a_1})^2+(\sqrt{a_2})^2+\ldots+(\sqrt{a_n})^2\right)\sqrt{a_2}\cdot\left(\left(\frac{1}{\sqrt{a_1}}\right)^2+\left(\frac{1}{\sqrt{a_2}}\right)^2+..+\left(\frac{1}{\sqrt{a_n}}\right)^2\right)$$

ni olamiz.

21-masala. $\frac{\left(a_1+a_2+\ldots+a_n\right)^2}{a_1^2+a_2^2+\ldots+a_n^2}\leq\frac{a_1}{a_1+a_3}+\frac{a_2}{a_3+a_4}+\ldots+\frac{a_n}{a_1+a_2}$

tengsizlikni isbotlang, bu yerda $a_k\geq 0\ (k=1,2,\ldots,n)$.

Yechilishi. Koshi–Bunyakovskiy–Shvarts tengsizligiga ko'ra

$$(a_1+a_2+\ldots+a_n)^2=\left(\sqrt{\frac{a_1}{a_1+a_3}}\cdot\sqrt{a_1(a_1+a_3)}+\ldots+\sqrt{\frac{a_n}{a_1+a_2}}\cdot\sqrt{a_n(a_1+a_2)}\right)^2\leq$$

$$\leq\left(\frac{a_1}{a_1+a_3}+\frac{a_2}{a_3+a_4}+\ldots+\frac{a_n}{a_1+a_2}\right)\left(a_1(a_1+a_3)+\ldots a_n(a_1+a_2)\right)\leq$$

$$\geq\left(\frac{a_1}{a_1+a_3}+\frac{a_2}{a_3+a_4}+\ldots+\frac{a_n}{a_1+a_2}\right)\left(\frac{1}{2}(a_1^2+a_2^2)+\frac{1}{2}(a_1^2+a_3^2)\right)+\ldots+$$

$$+\left(\frac{1}{2}(a_{n-1}^2+a_n^2)+\frac{1}{2}(a_n^2+a_1^2)\right)+\left(\frac{1}{2}(a_n^2+a_1^2)+\frac{1}{2}(a_n^2+a_2^2)\right)=$$

$$=\left(\frac{a_1}{a_1+a_3}+\frac{a_2}{a_3+a_4}+\ldots+\frac{a_n}{a_1+a_2}\right)(2a_1^2++\ldots+2a_n^2).$$

Mashqlar

1. Agar $a, b, c, d > 0,\ c+d \le a,\ c+d \le b$, bo'lsa, u holda $ab+bc \le ab$ tengsizlik o'rinli bo'lishini isbotlang.
2. Agar x, y, z lar xaqiqiy sonlar to'plamiga tegishli bo'lsa, $x^2+y^2+z^2 \ge xy+yz+xz$ tengsizlikni isbotlang.
3. Agar $x+y+z=1$ bo'lsa, $x^2+y^2+z^2 \ge \frac{1}{3}$ ni isbotlang.
4. Agar $ab>0$ bo'lsa, $\frac{a}{b}+\frac{b}{a} \ge 2$ tengsizlikni isbotlang.
5. Xar qanday $n \ge 2$ ($n \in N$) larda $\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}<1.$ tengsizlik o'rinli bo'lishini isbotlang.
6. Xar qanday $n \ge 2$ ($n \in N$) larda $\frac{n}{2}<1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^n-1}$ tengsizlik o'rinli bo'lishini isbotlang
7. $n \in N$ bo'lsa, $\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2n+1)^2}$ tengsizlikni isbotlang.
8. $n \in N$ bo'lsa, $\frac{1}{2}<\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}<\frac{3}{4}$ tengsizlikni isbotlang.
9. Agar $a_1^2+a_2^2+\ldots+a_n^2=b_1^2+b_2^2+\ldots+b_n^2=1$ bo'lsa, $-1 \le a_1b_1+a_2b_2+\ldots+a_nb_n \le 1$ tengsizlikni isbotlang.
10. Agar $a_1a_2\cdot\ldots\cdot a_n=1,\ a_1,a_2,\ldots a_n>0$ bo'lsa, $(1+a_1)(1+a_2)\ldots(1+a_n)\ge 2^n$ tengsizlikni isbotlang.
11. Agar $a+b\ge 1$ bo'lsa, $a^4+b^4 \ge \frac{1}{8}$ tengsizlikni isbotlang.

12. a, b musbat sonlar va birdan farqli bo'lsa, $\left|\log_a b\right| + \left|\log_b a\right| \geq 2$ tengsizlikni isbotlang.

13. $\frac{1}{\log_2 \pi} + \frac{1}{\log_\pi 2} > 2$ tengsizlikni isbotlang.

14. Agar $n \in N$ bo'lsa, $1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} < 3$ tengsizlikni isbotlang.

15. Agar $n \in N$ bo'lsa, $2\left(\sqrt{n+1} - 1\right) < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$ tengsizlikni isbotlang.

16. Agar $a_k = a_{k-1} + a_{k-2}$ $(k = 3, 4, \ldots)$ bo'lsa,
$\frac{1}{2} + \frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \frac{5}{2^5} + \ldots + \frac{a_n}{2^n} < 2$ tengsizlikni isbotlang.

17. Agar $n \in N$ bo'lsa, $1 < \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n+1} < 2$ tengsizlikni isbotlang.

18. Agar $a_i > 0,\ i = 1, 2, \ldots, n$ bo'lsa, $\frac{a_1 + a_2 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdot \ldots \cdot a_n}$ tengsizlikni isbotlang.

19. Agar $a_i > 0,\ i = 1, 2, \ldots, n$ bo'lsa,
$\left(a_1 + a_2 + \ldots + a_n\right)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \geq n^2$ tengsizlikni isbotlang.

20. Agar $a > 0$ bo'lsa, $\frac{1 + a + a^2 + \ldots + a^{2n}}{a + a^2 + \ldots + a^{2n-1}} \geq \frac{n+1}{n}$ tengsizlikni isbotlang.

21. Agar $0<\alpha_1<\alpha_2<\ldots<\alpha_n<\frac{\pi}{2}$ bo'lsa,

$tg\alpha_1<\frac{\sin\alpha_1+\sin\alpha_2+\ldots+\sin\alpha_n}{\cos\alpha_1+\cos\alpha_2+\ldots+\cos\alpha_n}<tg\alpha_n$ tengsizlikni isbotlang.

22. Agar $n\in N$ bo'lsa, quyidagi tengsizlikni isbotlang.

$2!\cdot 4!\cdot 6!\cdot\ldots\cdot(2n)!\geq((n+1)!)^n$

23. Agar $0<\varphi<\frac{\pi}{2}$ bo'lsa, $ctg\frac{\varphi}{2}\geq 1+ctg\varphi$ tengsizlikni isbotlang.

24. $k,l-$butun sonlar va $\alpha\neq\pm\beta+2\pi n$ bo'lsa,

$\left|\frac{\cos k\alpha\cos l\beta-\cos l\alpha\cos k\beta}{\cos\alpha-\cos\beta}\right|\leq\left|k^2-l^2\right|$ tengsizlikni isbotlang.

25. Agar $n>2$ bo'lsa, $(n!)^2>n^n$ tengsizlikni isbotlang.

26. Agar $a,b,p,q>0$ va p,q ratsional sonlar $\frac{1}{p}+\frac{1}{q}=1$ shartni qanoatlantirsa $ab\leq\frac{a^p}{p}+\frac{b^q}{q}$ tengsizlikni isbotlang.

27. Agar $n\in N$ bo'lsa, $2<\left(1+\frac{1}{n}\right)^n<3$ tengsizlikni isbotlang.

28. Agar $n>0$ bo'lsa, quyidagi tengsizlikni isbotlang. $\left(\frac{n}{3}\right)^n<n!$

29.Agar $n>0$ bo'lsa, quyidagi tengsizlikni isbotlang. $(3n)!>(n)^{3n}$

30. Agar $s=a_1+a_2+\ldots+a_n$; $a_i>0,\ i=1,2,\ldots,n$ bo'lsa,

$(1+a_1)(1+a_2)\cdot\ldots\cdot(1+a_n)\leq 1+\frac{s}{1!}+\frac{s^2}{2!}+\ldots+\frac{s^n}{n!}$ tengsizlikni isbotlang.

31. a) $\frac{a}{a^2+a+1} \le \frac{1}{3}$; b) $\frac{a}{a^2-4a+9} \le \frac{1}{2}$;

s) $a^2 + \frac{1}{a^2} \ge a + \frac{1}{a}$; d) $\frac{a^4+16}{a^2+4} \ge 2a$.

32. a) $9a^2 - 30|a| + 25 \ge 0$; b) $b^2 + 25 \ge 10|b|$;

s) $a^2 - 4a + 5 \ge 2|a-2|$; d) $b^2 - 2b + 10 \ge 6|b-1|$.

33. a) $a^4 + b^4 \ge a^3b + ab^3$; b) $a^4 + b^4 + ab(a^2+b^2) \ge 0$;

s) $a^6 + b^6 \ge a^4b^2 + a^2b^4$; d) $a^6 + b^6 \ge a^5b + ab^5$.

34. Agar $a \ge 0$ va $b \ge 0$ bo'lsa, u holda quyidagilarni isbotlang:

a) $a^3 + b^3 \ge a^2b + ab^2$; b) $(a+b)^3 \le 4(a^3+b^3)$;

s) $a^5 + b^5 \ge a^4b + ab^4$; d) $a^5 + b^5 \ge a^3b^2 + a^2b^3$.

35. Agar $a \ge 0$ va $b \ge 0$ bo'lsa, u holda quyidagilarni isbotlang:

a) $a^3 - b^3 \ge a^2b - ab^2$; b) $a^3 - b^3 \ge 3ab(a-b)$;

s) $a^3 - b^3 \ge ab^2 - a^2b$; d) $a^5 - b^5 \ge a^4b - ab^4$.

36. a va b sonlarning ixtiyoriy qiymatlarida tengsizlik o'rinli bo'lishini isbotlang:

a) $a^4 - 2a^3b + 2a^2b^2 - 2ab^3 + b^4 \ge 0$;

b) $a^4 - 4a^3b + 8a^2b^2 - 16ab^3 + 16b^4 \ge 0$.

37. Ixtiyoriy a, b, c, va d sonlar uchun tengsizlik

a) $(a^2-b^2)(c^2-d^2) \le (ac-bd)^2$;

b) $(a^2+b^2)(c^2+d^2) \ge (ac+bd)^2$.

o'rinli bo'lishini isbotlang, jumladan tenglik bajariladi shu holda va faqat shu holdaki, qachonki $ad = bc$.

38. $ab \ge 0$ shartni qanoatlantiruvchi ixtiyoriy a va b sonlar uchun $(a^2 - b^2)^2 \ge (a-b)^4$ tengsizlik o'rinli bo'lishini isbotlang.

39. Agar $a < b$ bo'lsa, $a < \dfrac{a+b}{2} < b$ bo'lishini isbotlang.

40. Agar $a < b < c$ bo'lsa, $a < \dfrac{a+b+c}{3} < b$ bo'lishini isbotlang.

41. Agar $a > 0,\ b > 0,\ c < 0,\ d < 0$ ekanligi ma'lum.

$$abc,\ bcd,\ \frac{ab}{c},\ \frac{ac}{d},\ \frac{c}{ad},\ \frac{b}{cd},\ abcd,\ \frac{ac}{bd},\ \frac{abd}{c}$$

ifodalar qanday ishoralarga ega bo'ladi ?

42. Agar

a) a va b bir xil ishorali sonlar;

b) a va b turli ishorali sonlar ekanligi ma'lum bo'lsa,

ab ko'paytma va $\dfrac{a}{b}$ kasr qanday ishoralarga ega bo'ladi ?

43. Agar

a) $ab > 0$; b) $\dfrac{a}{b} > 0$; s) $ab < 0$; d) $\dfrac{a}{b} < 0$; e) $a^2 b > 0$;

f) $a^2 b < 0$; g) $\dfrac{a}{b^2} < 0$ ekanligi ma'lum bo'lsa, a va b sonlarning ishorasini toping.

44. $a > 2$ ekanligi ma'lum bo'lsa, ifodaning ishorasi qanaqa ?

a) $3a-6$; b) $10-5a$; s) $2a-2$; d) $(a-2)(1-a)$; e) $\dfrac{a-2}{a-1}$;

f) $(a-3)^2(a-1)$; g) $\dfrac{-5}{2-a}$; h) $\dfrac{(a-1)(2-a)}{(5+a)}$.

45. $a < 3$ ekanligi ma'lum bo'lsa, ifodaning ishorasi qanday bo'ladi ?

a) $2a-6$; b) $12-4a$; s) $2a-8$; d) $(a-5)(a-3)$; e) $\frac{a-4}{3-a}$;

f) $(a-1)^2(a-2)$; g) $\frac{2}{3-a}$; h) $\frac{a-1}{(a-2)(3-a)}$.

46. Agar a) $a<1$; b) $a>4$; s) $1<a<4$; d) $a>5$ ekanligi ma'lum bo'lsa $(a-1)(a-4)$ ifoda qaysi ishoraga ega bo'ladi ?

47. Agar $a>1$ va $b>1$ bo'lsa, u holda $ab+1>a+b$ ekanligini isbotlang.

48. Agar $a>b$ va $b<2$ bo'lsa, u holda $b(a+2)>b^2+2a$ ekanligini isbotlang.

49. Agar $a>b>1$ bo'lsa, u holda $a^2b+b^2+a>ab^2+a^2+b$ ekanligini isbotlang.

50. Agar $a<b<2$ bo'lsa, u holda $a^2b+2b^2+4a<ab^2+2a^2+4b$ ekanligini isbotlang.

51. Agar $1<a<b<2$ bo'lsa, u holda $a^2b-ab^2-a^2-ab+2b^2+2a-2b>0$ ekanligini isbotlang.

52. Agar $a\geq b\geq c$ bo'lsa, u holda $a^2(b-c)+b^2(c-a)-c^2(a-b)\geq 0$ ekanligini isbotlang.

53. $\sin x > x-\frac{x^3}{6}$ $(x>0)$ tengsizlikni isbotlang.

54. Sonlarni taqqoslang.

a) $\frac{\ln 2004}{\ln 2005}$ va $\frac{\ln 2005}{\ln 2006}$

b) $\cos(\sin 2006)$ va $\sin(\cos 2006)$

55. $x>0$ uchun $1+2\ln x\leq x^2$ tengsizlik o'rinli bo'lishini isbotlang.

56. $x_1, x_2, \dots, x_n$ musbat sonlar bo'lsin.

$$f(\alpha)=\begin{cases}\left(\dfrac{x_1^{\alpha}+\ldots+x_n^{\alpha}}{n}\right)^{\frac{1}{\alpha}}, & \alpha\neq 0\\ \sqrt[n]{x_1\cdot\ldots\cdot x_n}, & \alpha=0\end{cases}$$

funtsiyaning monoton o'suvchi bo'lishini isbotlang. Bundan tashqari f funktsiya qat'iy o'suvchi bo'ladi, faqat va faqat shu holdaki, qachonki x_j sonlarning hammasi o'zaro teng bo'lmasa.

57. $\sin\alpha\sin\beta\sin\gamma\leq\dfrac{3\sqrt{3}}{8}$ tengsizlikni isbotlang, bu yerda α,β va γ biror uchburchakning ichki burchaklari.

58. $a_1, a_2, \ldots, a_n$ lar $a_k\in(0:\frac{1}{2})$ $(k=1,\ldots,n)$ va $a_1+a_2+\ldots+a_n=1$ xossalarga ega bo'lgan sonlar bo'lsin. $\left(\dfrac{1}{a_1^2}-1\right)\left(\dfrac{1}{a_2^2}-1\right)\ldots\left(\dfrac{1}{a_n^2}-1\right)\geq(n^2-1)^n$ ekanligini isbotlang.

59. Ixtiyoriy a, b, c musbat sonlar uchun

$$a+b+c\leq\frac{a^2+b^2}{2c}+\frac{b^2+c^2}{2a}+\frac{a^2+c^2}{2b}\leq\frac{a^3}{bc}+\frac{b^3}{ac}+\frac{c^3}{ab}$$

tengsizlik bajarilishini isbotlang.

60. Agar $1<a\leq b\leq c$ bo'lsa, u holda

$$\frac{a}{\ln a}+\frac{b}{\ln b}+\frac{c}{\ln c}\leq\frac{1}{3}(a+b+c)\left(\frac{1}{\ln a}+\frac{1}{\ln b}+\frac{1}{\ln c}\right)\leq\frac{a}{\ln c}+\frac{b}{\ln b}+\frac{c}{\ln c}\frac{a^3}{bc}+\frac{b^3}{ac}+\frac{c^3}{ab}\quad ?$$

bo'lishini isbotlang.

61. $\sqrt[n]{2-\sqrt{3}}+\sqrt[n]{2+\sqrt{3}}\geq 2,\ (n=2, 3, 4,\ldots)$ ekanligini isbotlang.

62. Ixtiyoriy a, b, c nomanfiy sonlar uchun

$(a+b)(b+c)(c+a)\geq 8abc$

tengsizlik o'rinli bo'lishini isbotlang.

63. Ixtiyoriy $a_1, a_2, \dots, a_n$ musbat sonlar uchun

$$\sqrt{\frac{a_1+a_2}{a_3}}+\sqrt{\frac{a_2+a_3}{a_4}}+\dots+\sqrt{\frac{a_{n-1}+a_n}{a_1}}+\sqrt{\frac{a_n+a_1}{a_2}}\geq n\sqrt{2}$$

tengsizlik o'rinli bo'lishini isbotlang.

Yensen tengsizligi:

64. $\sqrt{\frac{a_1+a_2}{a_3}}+\sqrt{\left(\sum a_i\right)^2+\left(\sum b_i\right)^2}\leq\sum\sqrt{a_i^2+b_i^2}$ $(a_i, b_i > 0)$ tengsizlikni isbotlang. (Ko'rsatma: $y=\sqrt{1+x^2}$).

65. $\sum_{i=1}^{n}\frac{a_i}{S-a_i}\geq\frac{n}{n-1}$ tengsizlikni isbotlang, bu yerda $S = a_1 + a_2 + \dots a_n$, $a_i > 0$.

66. $(\sum_{i=1}^{n} x_i)^p \geq n^{p-1}\cdot\sum_{i=1}^{n} x_i^p$, $p>1$, $x_i > 0$ tengsizlikni isbotlang.

Koshi-Bunyakovskiy tengsizligi:

67. $(a^2+b^2+c^2)(h_a^2+h_b^2+h_c^2)\geq 36S^2$ tengsizlikni isbotlang, bu yerda a, b, c lar uchburchakning tomonlari; h_a, h_b, h_c lar uchburchakning shu tomonlarga tushirilgan balandliklari; S uchburchakning yuzi.

66. $ab+\sqrt{(1-a^2)(1-b^2)}\leq 1$, $|a|\leq 1$, $|b|\leq 1$ ekanligini isbotlang**.**

68. Agar $a+2b+3c=14$ bo'lsa, $a^2+b^2+c^2\geq 14$ bo'lishini isbotlang.

Koshi tengsizligi:

69. a, b, c nomanfiy sonlar uchun $a^2+b^2+c^2=1$ shart bajariladi. $a+b+c\leq\sqrt{3}$ ekanligini isbotlang.

70. $a,b,c\geq 0$, $a+b+c=1$ berilgan. $(1-a),(1-b)(1-c)\geq 8abc$, $a+b+c=1$ tengsizlikni isbotlang.

71. Isbotlang: $abc\geq(a+b-c)(a+c-b)(b+c-a)$, $(1-b)(1-c)8abc$, $a+b+c=1$

72. $x, y, z \geq 0,\ xyz = 1$ berilgan. $(3x+2y+z)(3y+2z+x)(3z+2x+y) \geq 216$ ni isbotlang.

Bernulli tengsizligi:

75. $\sqrt{1-x}+\sqrt{1+x}+\sqrt[4]{1-x^2}+\sqrt[4]{1+x^2}=4$ tenglamani yeching.

76. $\sqrt[4]{1-x}+\sqrt[4]{1+x}=4$ tenglamani yeching.

77. $\sqrt[4]{1-\frac{x}{3}}+\sqrt[6]{x+1}=\left(1-\frac{x}{24}\right)^4+\left(1+\frac{x}{36}\right)^6$ tenglamani yeching.

78. $(\frac{a}{b})^{10}+(\frac{b}{c})^{10}+(\frac{c}{d})^{10}+(\frac{d}{a})^{10} \geq abcd(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}+\frac{1}{d^4})$ tengsizlikni isbotlang.

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O'ZBEKISTON RESPUBLIKASI XALQ TA'LIMI VAZIRLIGI

Sh. Ismailov, O. Ibrogimov

TENGSIZLIKLAR-II.
ISBOTLASHNING ZAMONAVIY USULLARI

Toshkent- 2008

Sh. Ismailov, O. Ibrogimov. **Tengsizliklar-II. Isbotlashning zamonaviy usullari** / Toshkent, 2008 y.
Fizika –matematika fanlari doktori, professor A. A'zamov umumiy tahriri ostida.

Qo'llanmada tengsizliklarni isbotlashning yangi samarali usullari va ularni qo'llanishiga doir turli matematik olimpiadalardagi masalalar keltirilgan.

Qo'llanma umumiy o'rta ta'lim maktablari, akademik litseylar va kasb–hunar kollejlarining iqtidorli o'quvchilari, matematika fani o'qituvchilari hamda pedagogika oliy o'quv yurtlari talabalari uchun mo'ljallangan.

Qo'llanmadan sinfdan tashqari mashg'ulotlarda, o'quvchilarni turli matematik musobaqalarga tayyorlash jarayonida foydalanish mumkin.

Taqrizchilar: TVDPI matematika kafedrasi mudiri, f.–m.f.n., dotsent Sh.B. Bekmatov
TVDPI boshlang'ich ta'lim metodikasi kafedrasi dotsenti, ped. f.n. Z. S. Dadanov

Ushbu qo'llanma Respublika ta'lim markazi qoshidagi matematika fanidan ilmiy-metodik kengash tomonidan nashrga tavsiya etilgan. (15 iyun 2008 y., 8 -sonli bayyonnoma)

Qo'llanmaning yaratilishi Vazirlar Mahkamasi huzuridagi Fan va texnologiyalarni rivojlantirishni muvofiqlashtirish Q'omitasi tomonidan moliyalashtirilgan (ХИД 1-16 – sonli innovatsiya loyihasi**)**

1-BOB. FUNKSIYANING XOSSALARI YORDAMIDA TENGSIZLIKLARNI ISBOTLASH USULLARI

1-§. Funksiyaning monotonlik xossasi yordamida isbotlanadigan tengsizliklar

Ta'rif. $f(x)$ funksiya $(a;b)$ oraliqda aniqlangan bo'lsin. Agar ixtiyoriy $x_1 \le x_2$ tengsizlikni qanoatlantiradigan $x_1, x_2 \in (a,b)$ nuqtalar uchun $f(x_1) \le f(x_2)$ $(f(x_1) \ge f(x_2))$ tengsizlik bajarilsa, u holda f funksiya (a,b) oraliqda *o'suvchi (kamayuvchi)* funksiya deyiladi, (a,b) oraliq esa *monotonlik oralig'i* deb yuritiladi.

Ta'rif. $f(x)$ funksiya $(a;b)$ intervalda aniqlangan bo'lsin. Agar ixtiyoriy $x_1 < x_2$ tengsizlikni qanoatlantiradigan $x_1, x_2 \in (a,b)$ nuqtalar uchun $f(x_1) < f(x_2)$ $(f(x_1) > f(x_2))$ tengsizlik bajarilsa, u holda f funksiya (a,b) oraliqda *qat'iy o'suvchi (kamayuvchi) funksiya* deyiladi.

Teorema. $f(x)$ funksiya $(a;b)$ oraliqda aniqlangan va differentsiallanuvchi bo'lsin. $f(x)$ funksiya $(a;b)$ intervalda o'suvchi (kamayuvchi) bo'lishi uchun shu intervalda $f'(x) \ge 0$ $(f'(x) \le 0)$ tengsizlik bajarilishi zarur va etarli.

Agar $(a;b)$ intervalda $f'(x) > 0$ $(f'(x) < 0)$ tengsizlik bajarilsa, u holda $f(x)$ funksiya $(a;b)$ intervalda qat'iy o'suvchi (kamayuvchi) bo'ladi.

1-masala. e^{π} va π^{e} sonlarni taqqoslang.

Yechilishi. $f : [e; +\infty) \to \mathbf{R}$, $f(x) = \dfrac{\ln x}{x}$ funksiyani qaraymiz. Uning hosilasi barcha x, $x \in (e;+\infty)$ larda $f'(x) = \dfrac{1-\ln x}{x^2} < 0$ manfiy qiymat qabul qiladi va f funksiya $[e; +\infty)$ da uzluksiz, shunday qilib, f $[e; +\infty)$ da qat'iy kamayadi. Bu yerdan, $e < \pi$ ekanligini hisobga olib

$$f(e) > f(\pi) \Rightarrow \frac{\ln e}{e} > \frac{\ln \pi}{\pi} \Rightarrow \pi \ln e > e \ln \pi$$

ni olamiz. Demak, $e^{\pi} > \pi^{e}$.

2-masala. $x_n = 1 + \frac{1}{2} + \ldots \frac{1}{n}$, $n=1, 2, \ldots$ sonli ketma-ketlikni chegaralanganlikka tekshiring.

Yechilishi. Dastlab

$$\ln(1+x) \le x \quad (x \ge 0) \qquad (1)$$

tengsizlikni isbotlaymiz. Buning uchun $f : [0; +\infty) \to \mathbf{R}$; $f(x) = x - \ln(1+x)$ funksiyani qaraymiz. f funksiya aniqlanish sohasida uzluksiz va barcha x, $x \in (0; +\infty)$ lar uchun $f'(x) = \frac{x}{x+1}$ tenglik o'rinli, bu yerdan

$f'(x) > 0$, $(x \in (0; +\infty))$ ekanligi kelib chiqadi. Shunday qilib f funksiya $D(f)$ aniqlanish sohasida qat'iy o'sadi va demak, $f(x) \ge f(0)$ $(x \ge 0)$ dan (1) tengsizlikning to'g'riligi kelib chiqadi.

(1) tengsizlikdan $x = \frac{1}{n}$ deb olib $(n = 1,2,...)$,

$$\ln\left(1 + \frac{1}{n}\right) \le \frac{1}{n} (n = 1,2,...)$$

yoki

$$\ln(n+1) - \ln n \le \frac{1}{n} \qquad (n = 1,2,...) \qquad (2)$$

ni hosil qilamiz.

(2) tengsizlikdan

$$\ln 2 - \ln 1 \le 1, \ \ln 3 - \ln 2 \le \frac{1}{2}, ..., \ \ln(n+1) - \ln n \le \frac{1}{n} \qquad (3)$$

kelib chiqadi. (3) tengsizliklarni hadma-had qo'shib,

$$\ln(n+1) \le 1 + ... + \frac{1}{n}$$

tengsizlikni olamiz.

Demak, $x_n=1+\frac{1}{2}+\ldots\frac{1}{n}$, n=1, 2, … sonli ketma-ketlik chegaralanmagan.

3-masala (Bernulli tengsizligi). Ixtiyoriy $x > -1$; $\alpha > 1$ uchun

$$(1 + x)^{\alpha} \geq 1 + \alpha x\,, \qquad (4)$$

tengsizlik o'rinli, shu bilan birga tenglik o'rinli faqat $x = 0$ da

Yechilishi. $f(x) = (1 + x)^{\alpha} - 1 - \alpha x$, $(x \in [-1; +\infty))$ funksiyani qaraymiz, bu yerda α – fiksirlangan 1 dan katta son. Bu funksiyaning hosilasini hisoblaymiz:

$$f'(x) = \alpha(1 + x)^{\alpha-1} - \alpha = \alpha((1 + x)^{\alpha-1} - 1) \quad (x > -1).$$

$\alpha > 1$ shartdan, $x \in (-1; 0)$ uchun $f'(x) < 0$ va $x \in (0; +\infty)$ uchun $f'(x) > 0$ ekanligi kelib chiqadi. Demak, f funksiya $[-1; 0]$ da kamayadi va $[0; +\infty)$ da o'sadi. Bundan barcha $x \in [-1;+\infty)\backslash\{0\}$ lar uchun $f(x) > f(0)$ tengsizlik o'rinli, ya'ni,

$$(1 + x)^{\alpha} - 1 - \alpha x > 1 - 1 \text{ va } (1 + x)^{\alpha} > 1 + \alpha x$$

$(x \in [-1; 0) \cup (0; +\infty), \alpha > 1)$ deb hulosa qilamiz.

$(1 + x)^{\alpha} = 1 + \alpha x$ tenglik $x = 0$ da bajarilishini eslatib o'tish qolyapti.

Izoh.

$$(1 + x)^{\alpha} \leq 1 + \alpha x \quad (x \geq -1;\ 0 < \alpha < 1),$$

$$(1 + x)^{\alpha} \geq 1 + \alpha x \quad (x \geq -1;\ \alpha < 0).$$

tengsizliklar shunga o'xshash isbotlanadi.

4-masala. (Yung tengsizligi) Agar $p, q \in \mathbf{R}\backslash\{0, 1\}$ sonlar $\frac{1}{p}+\frac{1}{q}=1$ tenglikni qanoatlantirsa, u holda ixtiyoriy a, b musbat sonlar uchun

$$\frac{1}{p}a^p+\frac{1}{q}b^q \geq ab \quad (p>1), \qquad (5)$$

$$\frac{1}{p}a^p+\frac{1}{q}b^q \leq ab \quad (p<1) \qquad (6)$$

tengsizliklar bajariladi.

Bundan tashqari, tenglik bajariladi faqat va faqat shu holdaki, qachonki $a^p = b^q$ bo'lsa.

Yechilishi. $p > 1$ holni qaraymiz. Ixtiyoriy musbat a sonni fiksirlab, $f:(0,+\infty)\to R;\quad f(b)=\frac{1}{p}a^p+\frac{1}{q}b^q-ab$ funksiyani aniqlaymiz.

Bu funksiyaning hosilasi $f'(b) = b^{q-1} - a$ ga teng. Elementar hisoblashlar yordamida $a^{\frac{1}{q-1}}$ nuqtada f funksiya uzining eng kichik kiymatiga erishishini kurish mumkin, ya'ni

$$f(b) > f(a^{\frac{1}{q-1}}), b > 0. \qquad (7)$$

ko'rsatiladi.

(7) tengsizlikdan $\frac{1}{p}+\frac{1}{q}=1$ ekanligini hisobga olib,

$$\frac{1}{p}a^p+\frac{1}{q}b^q-ab\ge 0 \quad (a>0, b>0; p>1)$$

olinadi. (5) tengsizlik isbotlanadi. (7) dan tenglik $b=a^{\frac{1}{q-1}}$, ya'ni $a^p = b^q$ holda o'rinli ekanligi kelib chiqadi. (6) tengsizlik shunga o'xshash isbotlanadi.

5-masala.

$$|sinx| \le |x| \qquad (x\in\mathbf{R}) \qquad (8)$$

tengsizlikni isbotlang

Yechilishi. Ikkala qismning juftligidan $x \ge 0$ holni qarash etarli. Bundan tashqari, $|sinx| \le 1$ ligidan $0 \le x \le 1$ holni o'rganish etarli. Shu maqsadda f: $[0;1]\to \boldsymbol{R}$, $f(x) = x - \sin x$ funksiyani qaraymiz. f funksiyaning hosilasi

$$f'(x) = 1 - \cos x \quad (x \in [0; 1]).$$

Kosinusning chegaralanganligidan ($|\cos x| \le 1;\ x \in \boldsymbol{R}$) $f'(x) \ge 0$ deb hulosalaymiz. Bu yerdan f funksiya o'zining aniqlanish sohasida monoton o'suvchi bo'lishi kelib chiqadi va shuning uchun

$$f(x) \ge f(0) \quad (x \in [0;1]) \text{ yoki } x - \sin x \ge 0, \quad (x \in [0;1])$$

tengsizlik o'rinli bo'ladi. Bu yerdan esa berilgan tengsizlik kelib chiqadi.

6-masala. Agar $a > b > c$ bo'lsa, u holda

$$a^2(b-c) + b^2(c-a) + c^2(a-b) > 0$$

tengsizlik o'rinli bo'lishini isbotlang.

Yechilishi. $f(t) = (b+t)^2(b-c) + b^2(c-(b+t)) + c^2((b+t)-b)$ ko'rinishdagi $f:[0;+\infty) \to \boldsymbol{R}$ funksiyani qaraymiz, bu yerda a, b, c lar $a > b > c$ tengsizlikni qanoatlantiruvchi haqiqiy parametrlar. f funksiyaning $[0;+\infty)$ da qat'iy o'suvchi bo'lishi avvalgi masalalardagidek isbotlanadi va shunday qilib

$$f(a-b) > f(0)$$

tengsizlik o'rinli. Ohirgi tengsizlik berilgan tengsizlikka tengkuchli.

2-§. Funksiyaning qavariqlik xossasi yordamida isbotlanadigan tengsizliklar

$(a;b)$ – haqiqiy sonlar o'qidagi oraliq berilgan bo'lsin.

Ta'rif: $f:(a;b) \to \boldsymbol{R}$ funksiya $(a;b)$ da quyidan qavariq deyiladi, agar barcha x_1, $x_2 \in (a;b)$ shunday $\lambda_1 \ge 0$, $\lambda_2 \ge 0$ va $\lambda_1 + \lambda_2 = 1$ shartlarni qanoatlantiruvchi λ_1, λ_2 sonlar uchun

$$f(\lambda_1 x_1 + \lambda_2 x_2) \le \lambda_1 f(x_1) + \lambda_2 f(x_2) \qquad (1)$$

tengsizlik o'rinli bo'lsa.

Yuqoridan qavariq funksiyaning ta'rifi esa yuqorida keltirilgan (1) tengsizlik belgisini qarama-qarshisiga almashtirishdan olinadi.

Yensen tengsizligi. $f:(a;b)\to \boldsymbol{R}$ – quyidan (yuqoridan) qavariq funksiya bo'lsin. U holda barcha $x_j \in (a;b)$ $(j = 1,...,n)$ lar va

$$\lambda_1 + ... + \lambda_n = 1$$

tenglikni qanoatlantiruvchi ixtiyoriy $\lambda_j \geq 0$ $(j = 1,...,n)$ sonlar uchun

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \ldots + f(\lambda_n x_n)$$

$$(f(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n) \geq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \ldots + f(\lambda_n x_n)$$

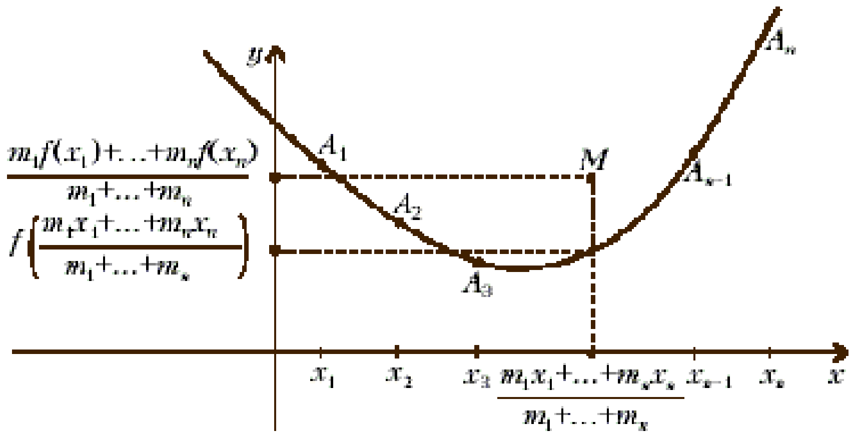
tengsizlik o'rinli.

Isboti: $y = f(x)$ funksiyaning grafigida abtsissalari $x_1, x_2, ..., x_n$ bo'lgan $A_1, A_2, ..., A_n$ nuqtalarni qaraymiz. Bu nuqtalarda $m_1, m_2, ..., m_n$ massali yuklarni joylashtiramiz. Bu nuqtalar massalari markazi

$$\left(\frac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + m_n};\ \frac{m_1 f(x_1) + m_2 f(x_2) + \ldots + m_n f(x_n)}{m_1 + m_2 + \ldots + m_n}\right)$$

$A_1, A_2, ..., A_n$ nuqtalar qavariq funksiyaning grafigi ustida yotganligidan, ularning massalar markazi ham grafik ustida yotadi. Bu esa massalar markazi M ning ordinatasi shu abstsissaga ega bo'lgan nuqtaning ordinatasidan kichik emasligini bildiradi, ya'ni (1-chiz.),

$$f\left(\frac{m_1x_1 + \ldots + m_nx_n}{m_1 + \ldots + m_n}\right) \leq \frac{m_1 f(x_1) + \ldots + m_n f(x_n)}{m_1 + \ldots + m_n}. \qquad (2)$$



1-chiz.

Isbotni tugatish uchun $m_1= a_1, \ldots, m_n= a_n$ olish qolyapti. Biroq, ikkita muhim izoh mavjud. Birinchidan, Yensenning (1) tengsizligini isbotlash jarayonida biz (2) tengsizlikni isbotladik. Aslida bu tengsizliklar teng kuchli.

(1) tengsizlikda

$$a_i = \frac{m_1}{m_1 + \ldots + m_n} \quad (i = 1, 2, \ldots, n)$$

deb olib, biz (2) tengsizlikni olamiz. Shuning uchun tabiiy ravishda bu tengsizliklar Yensen tengsizliklari deb ataladi. (1) tengsizlik ancha ixcham ko'rinadi, biroq tatbiq qilish uchun (2) tengsizlikdan foydalanish qulayroq. Ikkinchidan, agar esli funksiya $f(x)$ funksiya qavariq bo'lsa, u holda uning uchun (1) va (2) Yensen tengsizliklaring ishoralari qarama-qarshisiga o'zgaradi. Buni isbotlash uchun – $f(x)$ qavariq funksiyani qarash etarli.

Teorema. $(a;b)$ oraliqda uzluksiz va ikkinchi tartibli hosilaga ega bo'lgan $f:(a;b) \to \mathbf{R}$ funksiya shu intervalda quyidan (yuqoridan) qavariq bo'lishi uchun $(a;b)$ da $f''(x) \geq 0$ $(f''(x) \leq 0)$ tengsizlikning bajarilishi zarur va etarli.

1-masala (O'rta qiymatlar haqidagi Koshi tengsizligi). Ixtiyoriy nomanfiy $a_1, a_2, ..., a_n$ sonlar uchun

$$\sqrt[n]{a_1 a_2 ... a_n} \le \frac{a_1 + a_2 + ... + a_n}{n} \qquad (2)$$

tengnsizlik o'rinli, ya'ni o'rta geometrik qiymat o'rta arifmetik qiymatdan katta emas.

Yechilishi. Agar a_j sonlardan biri 0 ga teng bo'lsa, u holda (2) tengsizlikning bajarilishi ravshan, shuning uchun barcha a_j sonlar musbat deb hisoblaymiz.

$f(x) = \ln x$ ($x > 0$) funksiyani qaraymiz. f funksiya $(0; +\infty)$ da yuqoridan qavariq ekanligi ravshan. Yensen tengsizligiga asoslanib

$$\ln\left(\sum_{i=1}^{n} \frac{1}{n} a_i\right) \ge \sum_{i=1}^{n} \frac{1}{n} \ln a_i$$

tengsizlikni hosil qilamiz.

2-masala. $x_1, ..., x_n$ – nomanfiy sonlar bo'lsin.

$$f:[0,+\infty) \to R; \quad f(\alpha) = \left(\frac{x_1^\alpha + \ldots + x_n^\alpha}{n}\right)^{\frac{1}{\alpha}}$$

funksiya monoton o'suvchi ekanligini isbotlang.

Yechilishi. $0 < \alpha < \beta$ bo'lsin. $h(x) = x^{\frac{\beta}{\alpha}}, x \ge 0$ ($x \ge 0$) funksiyani qaraymiz. $h''(x) = \frac{\beta}{\alpha}\left(\frac{\beta}{\alpha} - 1\right) x^{\frac{\beta}{\alpha}} > 0 \quad (x > 0)$, shunday qilib h funksiya $[0; +\infty)$ da quyidan qavariq. Yensen tengsizligiga ko'ra

$$h\left(\sum_{i=1}^{n} \frac{1}{n} x_i^\alpha\right) \le \sum_{i=1}^{n} \frac{1}{n} h(x_i^\alpha) \text{ yoki } \left(\sum_{i=1}^{n} \frac{1}{n} x_i^\alpha\right)^{\frac{\beta}{\alpha}} \le \sum_{i=1}^{n} \frac{1}{n} x_i^\beta ,$$

bu yerdan $f(\alpha) \le f(\beta)$ kelib chiqadi.

3-masala. $\sin\alpha + \sin\beta + \sin\gamma \le \frac{3\sqrt{3}}{2}$ tengsizlikni isbotlang, bu yerda

α, β, γ - biror uchburchakning ichki burchaklari.

Yechilishi. $f: [0; \pi] \to \mathbf{R}$; $f(x) = sinx$ funksiyani qaraymiz. $x \in (0; \pi)$ lar uchun $f''(x) = -\sin x$ va $f''(x) < 0$, shuning uchun f funksiya $[0; \pi]$ da yuqoridan qavariq. Yensen tengsizligiga ko'ra

$$f\left(\frac{\alpha}{3}+\frac{\beta}{3}+\frac{\gamma}{3}\right) \geq \frac{f(\alpha)}{3}+\frac{f(\beta)}{3}+\frac{f(\gamma)}{3} \text{ yoki } sin\frac{\pi}{3} \geq \frac{1}{3}(sin\alpha + sin\beta + sin\gamma),$$

bu yerdan $sin\alpha + sin\beta + sin\gamma \leq \dfrac{3\sqrt{3}}{2}$ ni olamiz.

4-masala. Ixtiyoriy musbat a_j, b_j ($j = 1,.., n$) sonlar uchun

$$\left(\frac{a_1+\ldots+a_n}{b_1+\ldots+b_n}\right)^{b_1+\ldots+b_n} \geq \left(\frac{a_1}{b_1}\right)^{b_1} \ldots \left(\frac{a_n}{b_n}\right)^{b_n}$$

tengsizlik o'rinli bo'lishini isbotlang

Yechilishi. $f: [0; +\infty) \to \mathbf{R}$, $f(x) = lnx$ funksiyani qaraymiz. Bu funksiya yuqoridan qavariq. Shunday qilib, Yensen tengsizligiga ko'ra

$$f\left(\sum_{i=1}^{n}\frac{b_i}{b_1+\ldots+b_n}\cdot\frac{a_i}{b_i}\right) \leq \sum_{i=1}^{n}\frac{b_i}{b_1+\ldots+b_n}\cdot f\left(\frac{a_i}{b_i}\right)$$

yoki

$$(b_1+\ldots+b_n)\ln\frac{a_1+\ldots+a_n}{b_1+\ldots+b_n} \geq \sum_{i=1}^{n} b_i \ln\left(\frac{a_i}{b_i}\right)$$

demak,

$$\left(\frac{a_1+\ldots+a_n}{b_1+\ldots+b_n}\right)^{b_1+\ldots+b_n} \geq \left(\frac{a_1}{b_1}\right)^{b_1} \ldots \left(\frac{a_n}{b_n}\right)^{b_n}.$$

5-masala. (Gyuygens tengsizligi). Ixtiyoriy nomanfiy a_j ($j = 1,...,n$) sonlar uchun

$$(1+a_1)\ldots(1+a_n) \geq (1+\sqrt[n]{a_1\ldots a_n})^n$$

tengsizlik o'rinli bo'lishini isbotlang.

Yechilishi. $f:\mathbf{R}\to\mathbf{R}$, $f(x)=ln(1+e^x)$ funksiyani qaraymiz. Barcha $x\in\mathbf{R}$ lar uchun $f''(x)>0$ o'rinli. Shunday qilib, funksiya yuqoridan qavariq. Yensen tengsizligiga ko'ra

$$f\left(\sum_{i=1}^{n}\frac{1}{n}\ln a_i\right)\le\sum_{i=1}^{n}\frac{1}{n}f(\ln a_i)\Leftrightarrow\ln\left(1+\exp\sum_{i=1}^{n}\frac{1}{n}\ln a_i\right)\le\sum_{i=1}^{n}\frac{1}{n}\ln(1+a_i)\Leftrightarrow$$

$$\Leftrightarrow\ln((1+a_1)\dots(1+a_n))\ge n\ln(1+\sqrt[n]{a_1\dots a_n})^n\Leftrightarrow$$

$$\Leftrightarrow(1+a_1)\dots(1+a_n)\ge(1+\sqrt[n]{a_1\dots a_n})^n$$

ni olamiz.

6-masala.

$$\sqrt{(\sum a_i)^2+(\sum b_i)^2}\le\sum\sqrt{a_i^2+b_i^2}\quad(a_i,b_i>0)$$

tengsizlikni isbotlang.

Yechilishi. $f(x)=\sqrt{1+x^2}$ funksiyani qaraymiz. Barcha musbat $x\in\mathbf{R}$ lar uchun

$f''(x)>0$ o'rinli. Shunday qilib, f funksiya o'zining aniqlanish sohasida quyidan qavariq.

$$x_i=\frac{a_i}{b_i},\qquad \alpha_i=\frac{b_i}{\sum b_i}$$

deb olamiz. Bu funksiya uchun Yensen tengsizligini yozamiz.

$$\frac{b_1}{\sum b_i}\sqrt{1+\left(\frac{a_1}{b_1}\right)^2}+\dots+\frac{b_n}{\sum b_i}\sqrt{1+\left(\frac{a_n}{b_n}\right)^2}\ge f\left(\frac{b_1}{\sum b_i}\,\frac{a_1}{b_1}+\dots+\frac{b_n}{\sum b_i}\,\frac{a_n}{b_n}\right)=f\left(\frac{\sum a_i}{\sum b_i}\right),$$

$$\frac{1}{\sum b_i}\left(\sqrt{b_1^2+a_1^2}+\dots+\sqrt{b_n^2+a_n^2}\right)\ge\sqrt{1+\left(\frac{\sum a_i}{\sum b_i}\right)^2}$$

Ohirgi tengsizlikning ikkala qismini $\sum b_i$ ga ko'paytiramiz va kerakli tengsizlikni olamiz.

7-masala. Tengsizlikni isbotlang:

$$\sum_{i=1}^{n}\frac{a_i}{S-a_i}\geq\frac{n}{n-1},$$

bu yerda

$$S=a_1+a_2+..+a_n,\ a_i>0.$$

Yechilishi.

$$f(x)=\frac{x}{S-x}$$

funksiyani qaraymiz. Barcha musbat $x\in\mathbf{R}$ lar uchun $f''(x)>0$ o'rinli. Shunday qilib, f funksiya o'zining aniqlanish sohasida quyidan qavariq.

Bu funksiya uchun Yensen tengsizligini yozamiz.

$$\frac{1}{n}\cdot\frac{a_1}{S-a_1}+...+\frac{1}{n}\cdot\frac{a_n}{S-a_n}\geq f\left(\frac{a_1}{n}+...+\frac{a_n}{n}\right)=f\left(\frac{S}{n}\right)=\frac{S/n}{S-S/n}=\frac{1}{n-1}.$$

Ikkala qismini n ga ko'paytiramiz va kerakli tengsizlikni olamiz.

8-masala. Tengsizlikni isbotlang:

$$\left(\sum_{i=1}^{n}x_i\right)^p<n^{p-1}\cdot\sum_{i=1}^{n}x_i^p,\quad p>1, x_i>0.$$

Isbot.

$$f(x) = x^p$$

funksiya uchun Yensen tengsizligini yozamiz.

$$f\left(\frac{x_1}{n} + \frac{x_2}{n} + \ldots + \frac{x_n}{n}\right) < \frac{1}{n}\left(x_1^p + x_2^p + \ldots + x_n^p\right),$$

$$\frac{(x_1 + x_2 + \ldots + x_n)^p}{n^p} < \frac{1}{n}\left(x_1^p + x_2^p + \ldots + x_n^p\right).$$

Ikkala qismini n ga ko'paytiramiz va kerakli tengsizlikni olamiz.

$$(x_1 + x_2 + \ldots + x_n)^p < n^{p-1}\left(x_1^p + x_2^p + \ldots + x_n^p\right).$$

2-BOB. TRANS-TENGSIZLIK VA UNING TADBIQLARI .

1-§. Trans-tengsizlik haqida .

Teorema. $a_1, ..., a_n$ sonli ketma-ketlik $a_1 \geq a_2 \geq ... \geq a_n$ shartni qanoatlantirsin. $a_1b_1 + a_2b_2 + ... + a_nb_n$ yig'indi eng katta qiymatiga $b_1 \geq b_2 \geq ... \geq b_n$ bo'lganda , eng kichik qiymatiga esa $b_1 \leq b_2 \leq ... \leq b_n$ bo'lganda yerishadi.

Isbot. r, s natural sonlar $r < s \leq n$ shartni qanoatlantirsin .
Isbot qilish uchun

$S = a_1c_1 + a_2c_2 + ... + a_rc_r + ... + a_sc_s + ... + a_nc_n$,

$S' = a_1c_1 + a_2c_2 + ... + a_rc_s + ... + a_sc_r + ... + a_nc_n$ yig'indilarni solishtirish etarli. Ular uchun

$$S' - S = a_rc_s + a_sc_r - a_rc_r - a_sc_s = (a_r - a_s)(c_s - c_r)$$

munosabatga ega bo'lamiz.

Demak, agar $c_s \leq c_r$ bo'lsa $S \leq S'$ va $c_s \geq c_r$ bo'lsa $S \geq S'$. Bu esa talab qilingan tasdiqni isbotlaydi.

Natija. Teoremadan ko'rinib turibdiki, agar

$$a_1 \geq a_2 \geq ... \geq a_n, \quad b_1 \geq b_2 \geq ... \geq b_n$$

bo'lsa, u holda $b_1, ..., b_n$ sonlarning ixtiyoriy $(x_1, ..., x_n)$ o'rin almashtirishi uchun

$$a_1b_1 + a_2b_2 + ... + a_nb_n \geq a_1x_1 + a_2x_2 + ... + a_nx_n \geq a_1b_n + a_2b_{n-1} + ... + a_nb_1 \qquad (1)$$

qo'shtengsizlik o'rinli.

Izoh. Xorijiy adabiyotlarida (1) ko'shtengsizlik "rearrangement inequality" deb yuritiladi. Qizig'i shundaki, hozirgacha (1) ko'shtengsizlik xattoki rus tilida aniq nomga ega emas. Uni nomlash uchun "trans-tengsizlik" terminini qo'llash mualliflardan biriga Xalqaro matematika olimpiadalarida ishtirok etuvchi Rossiya o'quvchilari terma jamoasining ilmiy rahbari dotsent N. Agaxanov taklif qildi.

Ta'rif. (a_1, a_2, a_3) va (b_1, b_2, b_3) uchliklar *bir xil tartiblangan* deyiladi, agar a_1, a_2, a_3 va b_1, b_2, b_3 ketma-ketliklarlardan ikkalasi kamaymaydigan (ya'ni $a_1 \le a_2 \le a_3$ va $b_1 \le b_2 \le b_3$), yoki ikkalasi ortmaydigan (ya'ni $a_1 \ge a_2 \ge a_3$ va $b_1 \ge b_2 \ge b_3$) bo'lsa.

Agar a_1, a_2, a_3 va b_1, b_2, b_3 ketma-ketliklarlardan bittasi kamaymay-digan, boshqasi esa ortmaydigan bo'lsa, u holda (a_1, a_2, a_3) va (b_1, b_2, b_3) uchliklar *turlicha tartiblangan* deyiladi.

Masalan, (-1,1,3) va (2,5,7) uchliklar bir xil tartiblangan, (-1,1,3) va (7,5,2) uchliklar esa turlicha tartiblangan.

a, b, c musbat sonlar uchun $a \ge b \ge c$ yoki $a \le b \le c$ bo'lsa, u holda (a,b,c) va $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ uchliklar turlicha, (a,b,c) va (a^n,b^n,c^n) uchliklar esa bir xil tartiblangan , bu yerda n - ixtiyoriy natural son.

(a_1, a_2, a_3) va (b_1, b_2, b_3) uchliklar berilgan bo'lsin.

(x_1, x_2, x_3) uchlik b_1, b_2, b_3 sonlarning o'rin almashtirishi bo'lsin.

U holda yuqoridagi teoremaning quyidagi muhim bo'lgan natijalarini qayd etamiz.

1) Agar (a_1, a_2, a_3) va (b_1, b_2, b_3) uchliklar bir xil tartiblangan bo'lsa, u holda

$$a_1b_1 + a_2b_2 + a_3b_3 \ge a_1x_1 + a_2x_2 + a_3x_3 \qquad (2)$$

tengsizlik o'rinli.

2) Agar (a_1, a_2, a_3) va (b_1, b_2, b_3) uchliklar turlicha tartiblangan bo'lsa, u holda

$$a_1b_1 + a_2b_2 + a_3b_3 \le a_1x_1 + a_2x_2 + a_3x_3 \qquad (3)$$

tengsizlik o'rinli.

2-§. Trans-tengsizlikni masalalar yechishga tadbiqlari.

1-masala. Ixtiyoriy a,b,c haqiqiy sonlar va n natural son uchun

$$\text{a) } a^2+b^2+c^2 \geq ab+bc+ca$$

$$\text{b) } a^n+b^n+c^n \geq a^{n-1}b+b^{n-1}c+c^{n-1}a$$

tengsizliklarni isbotlang.

Yechilishi. $a^2+b^2+c^2 \geq ab+bc+ca$ tengsizlik $a^n+b^n+c^n \geq a^{n-1}b+b^{n-1}c+c^{n-1}a$ tengsizlikning xususiy holi bo'lgani uchun $a^n+b^n+c^n \geq a^{n-1}b+b^{n-1}c+c^{n-1}a$ tengsizlikni isbotlaymiz.

$a \geq b \geq c$ deb faraz qilamiz. U holda $a^{n-1} \geq b^{n-1} \geq c^{n-1}$, ya'ni (a,b,c) va $(a^{n-1},b^{n-1},c^{n-1})$ uchliklar bir hil tartiblangan bo'ladi.

(2) tengsizlikda

$(a_1,\ a_2,\ a_3)=(a^{n-1},b^{n-1},c^{n-1})$, $(b_1,\ b_2,\ b_3)=(a,b,c)$, $(x_1,\ x_2,\ x_3)=(b,c,a)$ deb olsak,

$a^n+b^n+c^n = a^{n-1}a+b^{n-1}b+c^{n-1}c \geq a^{n-1}b+b^{n-1}c+c^{n-1}a$ tengsizlikka ega bo'lamiz.

2-masala. Ixtiyoriy a,b,c musbat sonlar uchun

$$\text{a) } \frac{a+b+c}{abc} \leq \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

$$\text{b) } \frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geq \frac{b}{a}+\frac{c}{b}+\frac{a}{c}$$

$$\text{c) } \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq a+b+c$$

tengsizliklarni isbotlang.

Yechilishi. $a \geq b \geq c$ deb faraz qilamiz.

a) Ravshanki, $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ va $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ uchliklar bir xil tartiblangan.

(2) tengsizlikda

$(a_1,\ a_2,\ a_3)=(b_1,\ b_2,\ b_3)=\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$, $(x_1,\ x_2,\ x_3)=\left(\frac{1}{b},\frac{1}{c},\frac{1}{a}\right)$ deb olsak,

$$\frac{1}{a}\cdot\frac{1}{a}+\frac{1}{b}\cdot\frac{1}{b}+\frac{1}{c}\cdot\frac{1}{c}\geq\frac{1}{a}\cdot\frac{1}{b}+\frac{1}{b}\cdot\frac{1}{c}+\frac{1}{c}\cdot\frac{1}{a}=\frac{a+b+c}{abc}$$

tengsizlikni hosil qilamiz.

b) Ravshanki, $\left(\frac{a}{b},\frac{b}{c},\frac{c}{a}\right)$ va $\left(\frac{a}{b},\frac{b}{c},\frac{c}{a}\right)$ uchliklar bir xil tartiblangan.

(2) tengsizlikda

$(a_1,\ a_2,\ a_3)=(b_1,\ b_2,\ b_3)=\left(\frac{a}{b},\frac{b}{c},\frac{c}{a}\right)$, $(x_1,\ x_2,\ x_3)=\left(\frac{b}{c},\frac{c}{a},\frac{a}{b}\right)$ deb olsak,

$$\frac{a}{b}\cdot\frac{a}{b}+\frac{b}{c}\cdot\frac{b}{c}+\frac{c}{a}\cdot\frac{c}{a}\geq\frac{a}{b}\cdot\frac{b}{c}+\frac{b}{c}\cdot\frac{c}{a}+\frac{c}{a}\cdot\frac{a}{b}=\frac{a}{c}+\frac{b}{a}+\frac{c}{b}$$

tengsizlikni hosil qilamiz.

c) Ravshanki,

$$a^2\geq b^2\geq c^2 \quad \text{va} \quad \frac{1}{c}\geq\frac{1}{b}\geq\frac{1}{a},$$

ya'ni (a^2,b^2,c^2) va $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ uchliklar turlicha tartiblangan.

(3) tengsizlikda

$(a_1,\ a_2,\ a_3)=(a^2,b^2,c^2)$, $(b_1,\ b_2,\ b_3)=\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$, $(x_1,\ x_2,\ x_3)=\left(\frac{1}{b},\frac{1}{c},\frac{1}{a}\right)$ deb olsak,

berilgan tengsizlikka tengkuchli bo'lgan

$$a^2\frac{1}{a}+b^2\frac{1}{b}+c^2\frac{1}{c}\leq a^2\frac{1}{b}+b^2\frac{1}{c}+c^2\frac{1}{a}$$

tengsizlikni hosil qilamiz.

Izoh. Ko'rinib turibdiki, barcha tengsiziklarda tenglik $a=b=c$ bo'lgandagina o'rinlidir.

3-masala. Ixtiyoriy *a*, *b*, *c* musbat sonlar uchun

$$a^3b+b^3c+c^3a\geq a^2bc+b^2ca+c^2ab$$

tengsizlik o’rinli bo’lishini isbotlang.

Yechilishi. Umumiylikka putur etkazmasdan $a \geq b \geq c$ deb faraz qilamiz. U holda

$$a^2 \geq b^2 \geq c^2 \quad \text{va} \quad \frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a},$$

ya’ni (a^2, b^2, c^2) va $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ uchliklar turlicha tartiblangan.

(3) da

$(a_1,\ a_2,\ a_3)=(a^2, b^2, c^2)$, $(b_1,\ b_2,\ b_3)=\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, $(x_1,\ x_2,\ x_3)=\left(\frac{1}{c}, \frac{1}{a}, \frac{1}{b}\right)$

deb olsak, berilgan tengsizlikka tengkuchli bo’lgan

$$a^2\frac{1}{a} + b^2\frac{1}{b} + c^2\frac{1}{c} \leq a^2\frac{1}{c} + b^2\frac{1}{a} + c^2\frac{1}{b}$$

tengsizlikni hosil qilamiz.

4-masala. a, b, c –musbat sonlar bo’lsin.

$$(a^a b^b c^c)^2 \geq a^{b+c} b^{c+a} c^{a+b}$$

bo’lishini isbotlang

Yechilishi. Umumiylikka putur etkazmasdan $a \geq b \geq c$ deb faraz qilamiz. U holda

$$\ln a \geq \ln b \geq \ln c,$$

ya’ni (a, b, c) va $(\ln a, \ln b, \ln c)$ uchliklar bir xil tartiblangan.

(2) tengsizlikda $(a_1,\ a_2,\ a_3)=(\ln a, \ln b, \ln c)$, $(b_1,\ b_2,\ b_3)=(a, b, c)$, $(x_1,\ x_2,\ x_3)=(b, c, a)$

deb olsak,

$$a\ln a + b\ln b + c\ln c \geq b\ln a + c\ln b + a\ln c$$

tengsizlikka,

$(x_1,\ x_2,\ x_3)=(c, a, b)$ deb olsak,

$$a\ln a + b\ln b + c\ln c \geq c\ln a + a\ln b + b\ln c$$

tengsizlikka ega bo’lamiz.

Ularni hadma-had qo’shib

$$2(a\ln a + b\ln b + c\ln c) \geq (b + c)\ln a + (c + a)\ln a + (c + a)\ln b + (a + b)\ln c$$

yoki

$$\ln(a^a b^b c^c)^2 \geq \ln(a^{b+c} b^{c+a} c^{a+b})$$

ga ega bo'lamiz. Bu yerdan

$$(a^a b^b c^c)^2 \geq a^{b+c} b^{c+a} c^{a+b}$$

tengsizlik kelib chiqadi.

5-masala. (Moskva olimpiadasi –1963**)**. Ixtiyoriy $a,\ b,\ c$ musbat sonlar uchun

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$$

tengsizlik to'g'ri bo'lishini isbotlang.

Yechilishi. Tengsizlik simmetrik bo'lgani uchun umumiylikni chegaralamagan holda $a\geq b\geq c$ deb faraz qilamiz. U holda

$$\frac{1}{b+c}\geq\frac{1}{c+a}\geq\frac{1}{a+b}$$

tengsizlik o'rinli, ya'ni (a,b,c) va $\left(\frac{1}{b+c},\frac{1}{c+a},\frac{1}{a+b}\right)$ uchliklar bir xil tartiblangan bo'ladi.

(2) tengsizlikda $(a_1,\ a_2,\ a_3)=\left(\frac{1}{b+c},\frac{1}{c+a},\frac{1}{a+b}\right)$, $(b_1,\ b_2,\ b_3)=(a,b,c)$, $(x_1,\ x_2,\ x_3)=(b,c,a)$ deb olsak,

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{b}{b+c}+\frac{c}{c+a}+\frac{a}{a+b}$$

tengsizlikka,

$(x_1,\ x_2,\ x_3)=(c,a,b)$ deb olsak,

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{c}{b+c}+\frac{a}{c+a}+\frac{b}{a+b}$$

tengsizlikka ega bo'lamiz.

Oxirgi ikkita tengsizlikni hadma-had qo'shib va 2 ga bo'lib

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$$

ni hosil qilamiz.

6-masala. (XMO–1975**)**. Har biri n ta sondan iborat ikkita $a_1, a_2, \dots, a_n,\quad b_1, b_2, \dots, b_n$ ketma-ketlik berilgan bo'lib, ular $a_1 \leq a_2 \leq \dots \leq a_n,\quad b_1 \leq b_2 \leq \dots \leq b_n$ shartni qanoatlantirsin.

$$(a_1-b_1)^2+(a_2-b_2)^2+\dots+(a_n-b_n)^2 \leq (a_1-c_1)^2+(a_2-c_2)^2+\dots+(a_n-c_n)^2$$

tengsizlikni isbotlang, bu yerda $(c_1, c_2, \dots, c_n)$ – $(b_1, b_2, \dots, b_n)$ ning o'rin almashtirishi.

Yechilishi. Sodda hisob-kitoblardan so'ng berilgan tengsizlikni

$$a_1b_1+a_2b_2+\dots+a_nb_n \geq a_1c_1+a_2c_2+\dots+a_nc_n$$

ko'rinishga keltiramiz. Bu tengsizlik esa (1) qo'sh tengsizlikdagi chap tengsizlikning o'zi.

7-masala. (XMO–1995**)**. $abc=1$ shartni qanoatlantiruvchi ixtiyoriy $a,\ b,\ c$ musbat sonlar uchun

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}\geq\frac{3}{2}$$

tengsizlikni isbotlang.

Yechilishi. $a \geq b \geq c$ deb faraz qilamiz. U holda $\frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a}$ munosabatdan

$$\frac{1}{ac+bc}\geq\frac{1}{ab+bc}\geq\frac{1}{ab+ac}$$

tengsizliklarga ega bo'lamiz.

Shuning uchun

$$\frac{1}{c(ac+bc)} \ge \frac{1}{b(ab+bc)} \ge \frac{1}{a(ab+ac)}.$$

$ab \ge ac \ge bc$ bo'lgani bois (ab, ac, bc) va $\left(\frac{1}{c(ac+bc)}, \frac{1}{b(ab+bc)}, \frac{1}{a(ab+ac)}\right)$

uchliklar bir xil tartiblangan bo'ladi.

(2) tengsizlikda

$$(a_1,\ a_2,\ a_3) = \left(\frac{1}{c(ac+bc)}, \frac{1}{b(ab+bc)}, \frac{1}{a(ab+ac)}\right),$$

$$(b_1,\ b_2,\ b_3) = (ab, ac, bc),\ (x_1,\ x_2,\ x_3) = (ac, bc, ab)$$

deb olsak,

$$\frac{ab}{c(ac+bc)} + \frac{ac}{b(ab+bc)} + \frac{bc}{a(ab+ac)} \ge \frac{ac}{c(ac+bc)} + \frac{bc}{b(ab+bc)} + \frac{ab}{a(ab+ac)}$$

tengsizlikka,

$(x_1,\ x_2,\ x_3) = (bc, ab, ac)$ deb olsak,

$$\frac{ab}{c(ac+bc)} + \frac{ac}{b(ab+bc)} + \frac{bc}{a(ab+ac)} \ge \frac{bc}{c(ac+bc)} + \frac{ab}{b(ab+bc)} + \frac{ac}{a(ab+ac)}$$

tengsizlikka ega bo'lamiz.

Ularni hadma-had qo'shib

$$2\left(\frac{ab}{c(ac+bc)} + \frac{ac}{b(ab+bc)} + \frac{bc}{a(ab+ac)}\right) \ge \frac{1}{c} + \frac{1}{b} + \frac{1}{a}$$

ga ega bo'lamiz. $abc = 1$ shartni hisobga olib, o'rta qiymatlar haqidagi Koshi tengsizligiga ko'ra

$$\frac{1}{c} + \frac{1}{b} + \frac{1}{a} \ge 3\sqrt[3]{\frac{1}{c} \cdot \frac{1}{b} \cdot \frac{1}{a}} = 3.$$

Demak,

$$\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)}=\frac{ab}{c(ac+bc)}+\frac{ac}{b(ab+bc)}+\frac{bc}{a(ab+ac)}\geq\frac{3}{2}.$$

8-masala. (XMO –1978**)**. $\{a_1, a_2, ..., a_n\}$ – turli natural sonlardan iborat ketma-ketlik bo'lsin.

$$\sum_{k=1}^{n}\frac{a_k}{k^2}\geq\sum_{k=1}^{n}\frac{1}{k}$$

tengsizlik bajarilishini isbotlang.

Yechilishi. $(i_1, i_2, ..., i_n)$ – 1, 2, ..., n sonlarining shunday o'rin almashtirishi bo'lsinki, ular uchun $a_{i_1}<a_{i_2}<...<a_{i_n}$ bajarilsin.

$$\frac{1}{n^2}<\frac{1}{(n-1)^2}<\ldots<\frac{1}{1^2}$$

bo'lgani uchun, (1) tengsizlikka ko'ra

$$\sum_{k=1}^{n}\frac{a_k}{k^2}\geq\sum_{k=1}^{n}\frac{a_{i_k}}{k^2}.$$

Ravshanki, $a_{i_k}\geq k,\ k=1,2,...,n$.

Bundan $\sum_{k=1}^{n}\frac{a_k}{k^2}\geq\sum_{k=1}^{n}\frac{a_{i_k}}{k^2}\geq\sum_{k=1}^{n}\frac{k}{k^2}=\sum_{k=1}^{n}\frac{1}{k}$.

9-masala. (XMO-1964). a, b, c – biror uchburchakning tomonlari uzunliklari bo'lsin.

$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\leq 3abc$$

tengsizlikni isbotlang.

Yechilishi. $a\geq b\geq c$ deb faraz qilamiz. Dastlab quyidagini isbotlaymiz:

$$a(b+c-a)\leq b(c+a-b)\leq c(a+b-c).$$

Buning uchun

$$c(a+b-c)-b(c+a-b)=(b-c)(b+c-a)\geq 0,$$

$$b(c+a-b)-a(b+c-a)=(a-b)(a+b-c)\geq 0\,.$$

ekanligini eslatish kifoya.

Demak, (a,b,c) va $\big(a(b+c-a),b(c+a-b),c(a+b-c)\big)$ uchliklar turlicha tartiblangan bo'ladi .

(3) tengsizlikda

$(a_1,\ a_2,\ a_3)=\big(a(b+c-a),b(c+a-b),c(a+b-c)\big)$, $(b_1,\ b_2,\ b_3)=(a,b,c)$,

$(x_1,\ x_2,\ x_3)=(b,c,a)$ deb olsak,

$$\begin{aligned}&a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\leq\\&\leq ba(b+c-a)+cb(c+a-b)+ac(a+b-c)\end{aligned}$$

tengsizlikka,

$(x_1,\ x_2,\ x_3)=(c,a,b)$ deb olsak,

$$\begin{aligned}&a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c)\leq\\&\leq ca(b+c-a)+ab(c+a-b)+bc(a+b-c).\end{aligned}$$

tengsizlikka ega bo'lamiz.

Ohirgi ikkita tengsizliklarni qo'shib va soddalashtirib, berilgan tengsizlikni hosil qilamiz.

10-masala. (XMO-1983). a, b, c – biror uchburchakning tomonlari uzunliklari bo'lsin.

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0$$

tengsizlikni isbotlang.

Yechilishi. Umumiylikka putur etkazmagan holda $a\geq b$ deb olamiz.

Agar $a\geq b\geq c$ bo'lsa, u holda $\frac{1}{a}\leq\frac{1}{b}\leq\frac{1}{c}$ va oldingi masala yechimidan

$$c(a+b-c)\geq b(c+a-b)\geq a(b+c-a)\,.$$

ga ega bo'lamiz.

Ya'ni $\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right)$ va $\left(a(b+c-a),b(c+a-b),c(a+b-c)\right)$ uchliklar bir xil tartiblangan.

(2) tengsizlikda

$$(a_1,\ a_2,\ a_3)=\left(a(b+c-a),b(c+a-b),c(a+b-c)\right),\ (b_1,\ b_2,\ b_3)=\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right),$$

$$(x_1,\ x_2,\ x_3)=\left(\frac{1}{c},\frac{1}{a},\frac{1}{b}\right)$$

deb olsak,

$$a+b+c=\frac{1}{a}a(b+c-a)+\frac{1}{b}b(c+a-b)+\frac{1}{c}c(a+b-c)\geq$$
$$\geq\frac{1}{c}a(b+c-a)+\frac{1}{a}b(c+a-b)+\frac{1}{b}c(a+b-c)$$

tengsizlikni hosil qilamiz.

Soddalashtirishlardan so'ng bu tengsizlik berilgan tengsizlikka tengkuchli bo'lgan ushbu

$$\frac{1}{c}a(b-a)+\frac{1}{a}b(c-b)+\frac{1}{b}c(a-c)\leq 0$$

tengsizlikka keladi.

$a\geq c\geq b$ holni tahlil qilishni o'quvchilarga qoldiramiz.

11-masala. (4-Xalqaro Jautikov olimpiadasi, Almati, 2008 yil)

$abc=1$ shartni qanoatlantiruvchi ixtiyoriy $a,\ b,\ c$ musbat sonlar uchun

$$\frac{1}{(a+b)b}+\frac{1}{(b+c)c}+\frac{1}{(c+a)a}\geq\frac{3}{2}$$

tengsizlikni isbotlang.

Yechilishi. Tengsizlikning chap tomonini S orqali belgilaymiz.

$a \geq b \geq c$ deb faraz qilamiz. U holda $\frac{1}{a} \leq \frac{1}{b} \leq \frac{1}{c}$ va $\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$ tengsizliklar o'rinli, ya'ni $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ va $\left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right)$ uchliklar turlicha tartiblangan bo'ladi.

(3) tengsizlikda

$(a_1,\ a_2,\ a_3) = \left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right)$, $(b_1,\ b_2,\ b_3) = \left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, $(x_1,\ x_2,\ x_3) = \left(\frac{1}{c}, \frac{1}{a}, \frac{1}{b}\right)$

deb olsak

$$T = \frac{1}{a(b+c)} + \frac{1}{b(c+a)} + \frac{1}{c(a+b)} \leq \frac{1}{c(b+c)} + \frac{1}{a(c+a)} + \frac{1}{b(a+b)} = S$$

tengsizlikga ega bo'lamiz.

O'rta qiymat haqidagi Koshi tengsizligini va $abc = 1$ shartni hisobga olib, quyidagilarga ega bo'lamiz:

$$2S \geq S + T = \left(\frac{1}{(a+b)b} + \frac{1}{(a+b)c}\right) + \left(\frac{1}{(b+c)c} + \frac{1}{(b+c)a}\right)\left(\frac{1}{(c+a)a} + \frac{1}{(c+a)b}\right) =$$

$$= \frac{b+c}{(a+b)bc} + \frac{c+a}{(b+c)ca} + \frac{a+b}{(c+a)ab} \geq 3\sqrt[3]{\frac{b+c}{(a+b)bc} \cdot \frac{c+a}{(b+c)ca} \cdot \frac{a+b}{(c+a)ab}} = 3$$

Bundan $\frac{1}{(a+b)b} + \frac{1}{(b+c)c} + \frac{1}{(c+a)a} \geq \frac{3}{2}$ kelib chiqadi.

3-§. Klassik tengsizliklarni isbotlashda trans-tengsizlikni qo'llash.

Barcha $a_1, ..., a_n$ sonlar uchun (1) tengsizlikning muhim xususiy hollarini ta'kidlab o'tamiz:

$$\frac{b_1}{a_1}+\frac{b_2}{a_2}+...+\frac{b_n}{a_n}\geq n \qquad (4)$$

$$a_1^2+a_2^2+...+a_n^2\geq a_1b_1+a_2b_2+...+a_nb_n \qquad (5)$$

bu yerda n - ixtiyoriy natural son , $(b_1, ..., b_n)$ – $a_1, a_2, \dots, a_n$ sonlarning ixtiyoriy o'rin almashtirishi.

1-misol (O'rta qiymatlar haqidagi Koshi tengsizligi).

$x_1, x_2, ..., x_n$ musbat sonlar uchun

$$\frac{x_1+x_2+...+x_n}{n}\geq \sqrt[n]{x_1x_2...x_n},$$

tengsizlik o'rinli, shu bilan birga tenglik $x_1 = x_2 = ... = x_n$ bo'lgandagina bajariladi.

Yechilishi. $G=\sqrt[n]{x_1x_2...x_n}$, $a_1=\frac{x_1}{G}$, $a_2=\frac{x_1x_2}{G^2}$, ..., $a_n=\frac{x_1x_2...x_n}{G^n}=1$ bo'lsin.

(4) tengsizlikka binoan $\frac{x_1+x_2+...+x_n}{n}\geq G$ tengsizlikka teng ekvivalent bo'lgan ushbu

$$n\leq\frac{a_1}{a_n}+\frac{a_2}{a_1}+...+\frac{a_n}{a_{n-1}}=\frac{x_1}{G}+\frac{x_2}{G}+...+\frac{x_n}{G}$$

tengsizlikka egamiz. Tenglik bajarilishi uchun $a_1 = a_2 = ... = a_n$ ya'ni $x_1 = x_2 = ... = x_n$ bo'lishi zarur va etarli.

2-misol.(O'rta geometrik va o'rta garmonik qiymatlar orasidagi tengsizlik)

$x_1, x_2, ..., x_n$ musbat sonlar uchun

$$\sqrt[n]{x_1 x_2 \ldots x_n} \geq \frac{n}{x_1^{-1} + x_2^{-1} + \ldots + x_n^{-1}}$$

tengsizlik o'rinli, shu bilan birga tenglik $x_1 = x_2 = \ldots = x_n$ bo'lgandagina bajariladi.

Yechilishi. Oldingi misoldagi $G, a_1, a_2, \ldots, a_n$ sonlarni qaraymiz.

(4) tengsizlikka binoan

$$\frac{n}{x_1^{-1} + x_2^{-1} + \ldots + x_n^{-1}} \leq G$$

tengsizlikka teng ekvivalent bo'lgan ushbu

$$n \leq \frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_n}{a_1} = \frac{G}{x_1} + \frac{G}{x_2} + \ldots + \frac{G}{x_n}$$

tengsizlikka egamiz.

Tenglik bajarilishi uchun $a_1 = a_2 = \ldots = a_n$ ya'ni

$$x_1 = x_2 = \ldots = x_n$$

bo'lishi zarur va etarli.

3-misol. (O'rta kvadratik va o'rta arifmetik qiymatlar orasidagi tengsizlik)

Ixtiyoriy $x_1, x_2, \ldots, x_n$ sonlar uchun

$$\sqrt{\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{n}} \geq \frac{x_1 + x_2 + \ldots + x_n}{n} \quad ,$$

tengsizlik o'rinli, shu bilan birga tenglik $x_1 = x_2 = \ldots = x_n$ bo'lgandagina bajariladi.

Yechilishi.

(5) tengsizlikka ko'ra

$$x_1^2 + x_2^2 + \ldots + x_n^2 \geq x_1 x_2 + x_2 x_3 + \ldots + x_n x_1$$

$$x_1^2 + x_2^2 + \ldots + x_n^2 \geq x_1 x_3 + x_2 x_4 + \ldots + x_n x_2$$

……………………………………………

$$x_1^2 + x_2^2 + \ldots + x_n^2 \geq x_1 x_n + x_2 x_1 + \ldots + x_n x_{n-1}$$

munosabatlarga ega bo'lamiz.

Bu tengsizliklarni barchasini

$x_1^2 + x_2^2 + ... + x_n^2 = x_1^2 + x_2^2 + ... + x_n^2$ tenglik bilan qo'shib, natijada

$$n(x_1^2 + x_2^2 + ... + x_n^2) \geq (x_1 + x_2 + ... + x_n)^2$$

tengsizlikni hosil qilamiz.

4-misol. (Koshi-Bunyakovskiy-Shvarts tengsizligi)

n sondan iborat ikkita a_1, a_2, …, a_n, b_1, b_2, ... , b_n ketma-ketlik berilgan bo'lsin. U holda

$$(a_1b_1 + a_2b_2 + ... + a_nb_n)^2 \leq (a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2)$$

tengsizlik o'rinli. Tenglik biror o'zgarmas k son uchun $a_i = kb_i, i = 1, 2, ..., n$, bo'lgandagina bajariladi.

Yechilishi. Agar $a_1 = a_2 = … = a_n = 0$ yoki $b_1 = b_2 = … = b_n = 0$ bo'lsa, u holda tengsizlik bajariladi. Shuning uchun

$$P = \sqrt{a_1^2 + a_2^2 + ... + a_n^2}, \; Q = \sqrt{b_1^2 + b_2^2 + ... + b_n^2}$$

sonlarni noldan farqli deb hisoblaymiz.

Quyidagicha aniqlangan $x_1, x_2, ..., x_{2n}$ ketma-ketlikni qaraymiz:

$$x_i = \frac{a_i}{P}, x_{n+i} = \frac{b_i}{Q}, i = 1, 2, ..., n.$$

U holda

$$2 = \frac{a_1^2 + a_2^2 + ... + a_n^2}{P^2} + \frac{b_1^2 + b_2^2 + ... + b_n^2}{Q^2} = x_1^2 + x_2^2 + ... + x_{2n}^2$$

ga egamiz.

(5) tengsizlikka ko'ra

$$x_1^2 + x_2^2 + ... + x_{2n}^2 \geq x_1x_{n+1} + x_2x_{n+2} + ... + x_nx_{2n} + x_{n+1}x_1 + x_{n+2}x_2 + ... + x_{2n}x_n =$$

$$= \frac{2(a_1b_1 + a_2b_2 + ... + a_nb_n)}{PQ}$$

ga egamiz. Natijada

$$1 \geq \frac{a_1 b_1 + a_2 b_2 + ... + a_n b_n}{PQ}$$

tengsizlikni hosil qilamiz.

Eslatib o'tamiz, tenglik $a_i = \frac{P}{Q} b_i, i = 1, 2, ..., n$, shart bajarilganda bo'ladi. Bu shart esa $x_i = x_{n+i}$, $i = 1, 2, ..., n$ shartiga ekvivalent.

5-misol. (Chebishev tengsizligi).

n sondan iborat ikkita a_1, a_2, …, a_n, b_1, b_2, ... , b_n ketma-ketliklar berilgan bo'lsin. Faraz qilamiz $a_1 \geq a_2 \geq ... \geq a_n$ shart bajarilsin.

U holda

a) $\frac{a_1 + a_2 + ... + a_n}{n} \cdot \frac{b_1 + b_2 + ... + b_n}{n} \leq \frac{a_1 b_1 + a_2 b_2 + ... + a_n b_n}{n}$, agar $b_1 \geq b_2 \geq ... \geq b_n$

b) $\frac{a_1 + a_2 + ... + a_n}{n} \cdot \frac{b_1 + b_2 + ... + b_n}{n} \geq \frac{a_1 b_1 + a_2 b_2 + ... + a_n b_n}{n}$, agar $b_1 \leq b_2 \leq ... \leq b_n$

Isbot.

a) (5) tengsizlikka ko'ra

$$a_1 b_1 + a_2 b_2 + ... + a_n b_n = a_1 b_1 + a_2 b_2 + ... + a_n b_n$$

$$a_1 b_1 + a_2 b_2 + ... + a_n b_n \geq a_1 b_2 + a_2 b_3 + ... + a_n b_1$$

$$a_1 b_1 + a_2 b_2 + ... + a_n b_n \geq a_1 b_3 + a_2 b_4 + ... + a_n b_2$$

……………………………………………

$$a_1 b_1 + a_2 b_2 + ... + a_n b_n \geq a_1 b_n + a_2 b_1 + ... + a_n b_{n-1}$$

munosabatlarga egamiz, ularni qo'shib

$n(a_1 b_1 + a_2 b_2 + ... + a_n b_n) \geq (a_1 + a_2 + ... + a_n) \cdot (b_1 + b_2 + ... + b_n)$ yoki

$$\frac{a_1 + a_2 + ... + a_n}{n} \cdot \frac{b_1 + b_2 + ... + b_n}{n} \leq \frac{a_1 b_1 + a_2 b_2 + ... + a_n b_n}{n}$$

ni hosil qilamiz.

b) holi shunga o’xshash isbotlanadi.

3-BOB. KARAMATA TENGSIZLIGI.

Ta'rif: $x=(x_1,x_2,...,x_n)$ va $y=(y_1,y_2,...,y_n)$ n-liklar quyidagi shartlarni qanoatlantirsin:

1. $x_1 \ge x_2 \ge ... \ge x_n$ va $y_1 \ge y_2 \ge ... \ge y_n$

2. $\sum_{i=1}^{k} x_i \ge \sum_{i-1}^{k} y_i, \quad k=1,...,n-1$, $\sum_{i=i}^{n} x_i = \sum_{i=i}^{n} y_i$, ya'ni

$$x_1 \ge y_1$$
$$x_1 + x_2 \ge y_1 + y_2$$
....
....
$$x_1 + x_2 + ... + x_{n-1} \ge y_1 + y_2 + ... + y_{n-1}$$
$$x_1 + x_2 + ... + x_n \quad = \quad y_1 + y_2 + ... + y_n$$

Bu holda $x=(x_1,x_2,...,x_n)$ n-lik $y=(y_1,y_2,...,y_n)$ n-likni *majorlaydi* deyiladi va bu munosabat $x \succ y$ yoki $y \prec x$ kabi yoziladi.

Misollar:

$$1.\ \left(\frac{1}{n},...\frac{1}{n}\right) \prec \left(\frac{1}{n-1},...\frac{1}{n-1},0\right) \prec ... \prec \left(\frac{1}{2},\frac{1}{2},0,...0\right) \prec \left(1,0,...0\right).$$

2. Agar $m \ge l$ va $c \ge 0$ bo'lsa, u holda

$$\left(\underbrace{\frac{l}{m}c,...,\frac{l}{m}c}_{m\ \text{марта}},0,...,0\right) \prec \left(\underbrace{c,...,c}_{l\ \text{марта}},0,...,0\right)$$

munosabat o'rinli.

3. Agar $a_i \ge 0$ va $\sum_{i=1}^{n} a_i = 1$ bo'lsa, u holda

$$\left(\frac{1}{n},...,\frac{1}{n}\right) \prec \left(a_1,...,a_n\right) \prec \left(1,0,...,0\right)$$

munosabat o'rinli.

4. Agar $c \geq 0$ bo'lsa, u holda

$$\frac{1}{\sum_{i=1}^{n} x_i + nc}(x_1 + c, ..., x_n + c) \prec \frac{1}{\sum_{i=1}^{n} x_i}(x_1, ..., x_n)$$

munosabat o'rinli.

5. Agar $y_1 = y_2 = ... = y_n = \frac{x_1 + x_2 + ... + x_n}{n}$ bo'lsa $(x_1, x_2, ..., x_n) \succ (y_1, y_2, ..., y_n)$ bo'ladi.

6. α, β, γ - uchburchak burchaklari bo'lsin, u holda

A) barcha uchburchaklar uchun

$$\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) \prec (\alpha, \beta, \gamma) \prec (\pi, 0, 0)$$

munosabat;

B) o'tkir burchakli uchburchaklar uchun

$$\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) \prec (\alpha, \beta, \gamma) \prec \left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right)$$

munosabat;

C) o'tmas burchakli uchburchaklar uchun

$$\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) \prec (\alpha, \beta, \gamma) \prec (\pi, \pi, 0)$$

munosabat o'rinli.

Lemma (uch vatar haqida). f - qavariq funksiya bo'lsin. U holda uchun har qanday $z < y < z$ uchun

$$\frac{f(y) - f(z)}{y - z} \leq \frac{f(x) - f(x)}{x - z} \leq \frac{f(x) - f(y)}{x - y}$$

qo'shtengsizlik o'rinli.

Isbot: f - qavariq funksiya bo'lganligi uchun

$$f(\lambda x + (1-\lambda)z) \leq \lambda(x) + (1-\lambda)f(z)$$

tengsizlik bajariladi, bu yerda $\lambda \in (0,1)$.

$\lambda = \frac{y-z}{x-z}$ deb olamiz va soddalashtirishlardan so'ng yuqoridagi tengsizlik

$$(x-z)f(y) \le (x-y)f(z) + (y-z)f(x)$$

tengsizlikka olib kelinadi.

Bu tengsizlik esa

$$\frac{f(y)-f(z)}{y-z} \le \frac{f(x)-f(x)}{x-z} \le \frac{f(x)-f(y)}{x-y}$$

ikkala ham tengsizlikka tengkuchli.

Natija. Qavariq f funksiya berilgan bo'lsin. U holda uchun har qanday

$x_1 \ge x_2$, $y_1 \ge y_2$, $x_1 \ne y_1$, $x_2 \ne y_2$ uchun

$$\frac{f(x_1)-f(y_1)}{x_1-y_1} \ge \frac{f(x_2)-f(y_2)}{x_2-y_2}$$

tengsizlik bajariladi.

Lemma (Abel' almashtirishi). $A_k = \sum_{i=1}^{k} a_i$ bo'lsa, u holda

$\sum_{k=1}^{n} a_k b_k = \sum_{k-1}^{n-1} A_k (b_k - b_{k+1}) + A_n b_n$ tenglik o'rinli.

Isbot.

$$a_1b_1 + a_2b_2 + ... + a_{n-1}b_{n-1} + a_nb_n = A_1b_1 + (A_2 - A_1)b_2 + ... + (A_{n-1} - A_{n-2})b_{n-1} + (A_n - A_{n-1})b_n =$$
$$= A_1(b_1 - b_2) + A_2(b_2 - b_3) + ... + A_{n-1}(b_{n-1} - b_n) + A_nb_n.$$

Teorema (Karamata tengsizligi)**.** Qavariq (mos ravishda botiq) f funksiya berilgan bo'lsin. Agar $x \prec y$ bo'lsa

$$\sum_{i=1}^{n} f(x_i) \le \sum_{i=1}^{n} f(y_i) \ (1)$$

$$(\sum_{i=1}^{n} f(x_i) \ge \sum_{i=1}^{n} f(y_i) \ ). (1')$$

tengsizlik bajariladi.

Isbot: Qavariq f funksiya holini qarash etarli. Umumiylikka putur etkazmasdan $x_k \neq y_k$ deb hisoblashimiz mumkin.

$D_k = \dfrac{f(y_k) - f(x_k)}{y_k - x_k}$, $X_k = \sum_{i=1}^{k} x_i$, $Y_k = \sum_{i=1}^{k} y_i$ belgilashlarni kiritamiz.

U holda $Y_k \geq X_k, Y_n = X_n$.

Uch vatar haqida lemma natijasiga ko'ra $D_k \geq D_{k+1}$.

Demak, $\sum_{k=1}^{n-1} (Y_k - X_k) \cdot (D_k - D_{k+1}) + (X_n - Y_n) D \geq 0$.

Abel' almashtirishini qo'llab, $\sum_{k=1}^{n} (y_k - x_k) \cdot D_k \geq 0$ ni hosil qilamiz. Teorema isbot bo'ldi.

<u>*Eslatma*</u> *1.* Isbot qilingan tengsizlikka Karamata nomi berilishi unchalik to'g'ri emas. 1923 yilda Shur bu tengsizlikni majorlash shartini boshqacharok ifodalab isbotladi. 1920 yilda Xardi, Littlvud va Polia bu tengsizlikni ifodaladilar va uning uzluksiz analogini isbotladilar. 3 yildan keyin Karamata bu tengsizlikni umumiy holda isbotladi.

Karamata tengsizligidan foydalangan holda isbotlash mumkin bo'lgan ikkita tengsizliklarni ko'rib chiqamiz.

Misollar. 1. (Yensen tengsizligi). Agar f -qavariq funksiya bo'lsa,

$$\frac{\sum_{i=1}^{n} f(x_i)}{n} \geq f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)$$

tengsizlik o'rinli bo'ladi.

Isbot. $y_1 = y_2 = ... = y_n = \dfrac{x_1 + x_2 + ... + x_n}{n}$ deb olamiz. $(x_1, x_2, ..., x_n) \succ (y_1, y_2, ..., y_n)$ bo'lgani uchun Karamata tengsizligidan bevosita Yensen tengsizligi kelib chiqadi.

2. Ixtiyoriy musbat a,b,c lar uchun $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\le\frac{1}{2a}+\frac{1}{2b}+\frac{1}{2c}$.

Isbot. $(2a,2b,2c)\succ(a+b,a+c,b+c)$ ga egamiz. Karamata tengsizligini $f(x)=\frac{1}{x}$ funksiya uchun qo'llash etarli.

4-BOB. TENGSIZLIKLARNI TRIGONOMETRIK ALMASHTIRISHLAR YORDAMIDA ISBOTLASH

1-§. Trigonometrik almashtirishlar

Ba’zida tengsizlikni isbotlashda trigonometrik almashtirish olish yaxshi foyda beradi. Almashtirish qulay olinganda tengsizlik darhol isbotlanadigan, oddiy shaklga kelib qoladi. Shuningdek trigonometrik funksiyalarning yaxshi ma’lum bo’lgan xossalari yordam berishi mumkin.

Biz dastlab bunday almashtirishlarni kiritamiz, so’ng ma’lum bo’lgan trigonometrik ayniyatlar va tengsizliklarni keltiramiz va nihoyat bir nechta olimpiada masalalarini muhokama qilamiz.

Teorema 1. Faraz qilaylik α,β,γ burchaklar $(0;\pi)$ dan olingan. U holda bu α,β,γ burchaklar biror uchburchakning ichki burchaklari bo’lishi uchun quyidagi tenglikning bajarilishi zarur va etarli

$$tg\frac{\alpha}{2}tg\frac{\beta}{2}+tg\frac{\beta}{2}tg\frac{\gamma}{2}+tg\frac{\gamma}{2}tg\frac{\alpha}{2}=1$$

Isbot. Dastlab shuni ta’kidlash joizki $\alpha=\beta=\gamma$ bo’lgan holda teoremaning tasdig’i o’rinlidir. Umumiylikka ziyon etkazmasdan $\alpha\neq\beta$ deb faraz qilaylik. $0<\alpha+\beta<2\pi$ bo’lganligi uchun $(-\pi;\pi)$ intervalda $\alpha+\beta+\gamma^1=\pi$ shartni qanoatlantiruvchi γ^1 mavjud.

Qo’shish formulalari va $tgx=ctg\left(\frac{\pi}{2}-x\right)$ formulaga ko’ra

$$tg\frac{\gamma^1}{2}=ctg\frac{\alpha+\beta}{2}=\frac{1-tg\frac{\alpha}{2}tg\frac{\beta}{2}}{tg\frac{\alpha}{2}+tg\frac{\beta}{2}},$$

ya’ni

$$tg\frac{\alpha}{2}tg\frac{\beta}{2}+tg\frac{\beta}{2}tg\frac{\gamma^1}{2}+tg\frac{\gamma^1}{2}tg\frac{\alpha}{2}=1 \qquad (1)$$

tenglik o'rinli bo'ladi. Faraz qilaylik biror $\alpha,\beta,\gamma\in(0;\pi)$ burchaklar uchun

$$tg\frac{\alpha}{2}tg\frac{\beta}{2}+tg\frac{\beta}{2}tg\frac{\gamma}{2}+tg\frac{\gamma}{2}tg\frac{\alpha}{2}=1 \qquad (2)$$

tenglik o'rinli bo'lsin.

Biz isbotlaymizki $\gamma=\gamma^1$ va bu bizga α,β,γ lar biror uchburchak burchaklari ekanligini beradi. (1) dan (2) ni ayirib $tg\frac{\gamma}{2}=tg\frac{\gamma^1}{2}$ ni hosil qilamiz. Shuning uchun $\left|\frac{\gamma-\gamma^1}{2}\right|=k\pi, k\geq 0, k\in Z$. Ammo $\left|\frac{\gamma-\gamma^1}{2}\right|\leq\left|\frac{\gamma}{2}\right|+\left|\frac{\gamma^1}{2}\right|<\pi$ tengsizlik o'rinli. Demak, $k=0$, shuning uchun $\gamma=\gamma^1$. Tasdiq isbotlandi.

Teorema 2. Faraz qilaylik α,β,γ burchaklar $(0;\pi)$ dan olingan. U holda bu α,β,γ burchaklar biror uchburchakning ichki burchaklari bo'lishi uchun quyidagi tenglikning bajarilishi zarur va etarli

$$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=1$$

Isbot. $0<\alpha+\beta<2\pi$ bo'lganligi uchun shunday $\gamma^1\in(-\pi;\pi)$ mavjudki $\alpha+\beta+\gamma^1=\pi$ tenglik o'rinli bo'ladi. Ko'paytmani yig'indiga keltirish va ikkilangan burchak formulalariga asosan quyidagi munosabatlar o'rinli

$$\sin^2\frac{\gamma^1}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma^1}{2}=\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha+\beta}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)=\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}=$$

$$=\frac{\cos\alpha+\cos\beta}{2}=\frac{\left(1-2\sin^2\frac{\alpha}{2}\right)+\left(1-2\sin^2\frac{\beta}{2}\right)}{2}=1-\sin^2\frac{\alpha}{2}-\sin^2\frac{\beta}{2}.$$

Shunday qilib,

$$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma^1}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma^1}{2}=1 \qquad (1)$$

Faraz qilaylik biror $\alpha,\beta,\gamma\in(0;\pi)$ burchaklar uchun

$$\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=1 \qquad (2)$$

tenglik o'rinli bo'lsin. (1)dan (2) ni ayirib,

$$\sin^2\frac{\gamma}{2}-\sin^2\frac{\gamma^1}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\left(\sin\frac{\gamma}{2}-\sin\frac{\gamma^1}{2}\right)=0,$$

ya'ni

$$\left(\sin\frac{\gamma}{2}-\sin\frac{\gamma^1}{2}\right)\left(\sin\frac{\gamma}{2}+\sin\frac{\gamma^1}{2}+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right)=0.$$

munosabatni hosil qilamiz.

Ikkinchi qavs ichidagi ifodani quyidagicha ifodalaymiz

$$\sin\frac{\gamma}{2}+\sin\frac{\gamma^1}{2}+\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta}{2}=\sin\frac{\gamma}{2}+\cos\frac{\alpha-\beta}{2}$$

Ravshanki bu ifoda musbat qiymatlar qabul qiladi. Shuning uchun $\sin\frac{\gamma}{2}=\sin\frac{\gamma^1}{2}$, ya'ni $\gamma=\gamma^1$ bo'ladi. Demak, $\alpha+\beta+\gamma=\pi$. Tasdiq isbotlandi.

Almashtirishlar

A1. Faraz qilaylik α,β,γ lar uchburchakning ichki burchaklari bo'lsin.Quyidagicha almashtirishni qaraylik

$$A=\frac{\pi-\alpha}{2},\ B=\frac{\pi-\beta}{2},\ C=\frac{\pi-\gamma}{2}.$$

Ravshanki $A+B+C=\pi$ va $0\le A,B,C<\frac{\pi}{2}$. Bu almashtirish bizga biror masalani hal qilishda istalgan uchburchak o'rniga o'tkir burchakli uchburchakni qarash imkonini beradi. Quyidagi munosabatlar o'rinli ekanligini ta'kidlash joiz:

$$\sin\frac{\alpha}{2}=\cos A,\ \cos\frac{\alpha}{2}=\sin A,\ tg\frac{\alpha}{2}=ctgA,\ ctg\frac{\alpha}{2}=tgA$$

A2. Faraz qilaylik x, y, z lar musbat haqiqiy sonlar bo'lsin. U holda tomonlari uzunliklari $a=x+y, b=y+z, c=z+x$ lardan iborat bo'lgan uchburchak mavjud. $s=x+y+z$ bo'lsa, $(x,y,z)=(s-a,s-b,s-c)$. Shartga ko'ra x,y,z lar musbatligi uchun $s-a,\ s-b,\ s-c$ lar uchburchak tengsizligini qanoatlantiradi.

A3. Faraz qilaylik musbat a,b,c sonlar $ab+bc+ca=1$ shartni qanoatlantirsin. Biz ushbu $f:\left(0;\frac{\pi}{2}\right)\to\left(0;+\infty\right)$, $f\left(x\right)=tgx$ funksiya yordamida quyidagicha almashtirish kiritishimiz mumkin

$$a=tg\frac{\alpha}{2}, b=tg\frac{\beta}{2}, c=tg\frac{\gamma}{2}$$

bunda α,β,γ lar biror uchburchakning burchaklari.

A4. Faraz qilaylik musbat a,b,c sonlar $ab+bc+ca=1$ shartni qanoatlantirsin. A1 va A3 larga ko'ra quyidagilarga egamiz

$$a=ctgA,\ b=ctgB,\ c=ctgC,$$

bunda A,B,C lar o'tkir burchakli uchburchakning burchaklari.

A5. Faraz qilaylik musbat a,b,c sonlar $ab+bc+ca=abc$ shartni qanoatlantirsin. Bu tenglikning ikkala tarafini a,b,c sonlarning ko'paytmasiga bo'lib, quyidagiga ega bo'lamiz $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=1$. A3 ga ko'ra quyidagicha almashtirish olamiz

$$\frac{1}{a}=tg\frac{\alpha}{2},\ \frac{1}{b}=tg\frac{\beta}{2},\ \frac{1}{c}=tg\frac{\gamma}{2}$$

ya'ni

$$a=ctg\frac{\alpha}{2}, b=ctg\frac{\beta}{2}, c=ctg\frac{\gamma}{2}$$

bunda α,β,γ lar biror uchburchakning burchaklari.

A6. Faraz qilaylik musbat a,b,c sonlar $ab+bc+ca=abc$ shartni qanoatlantirsin.

A1 va A5 ga ko'ra

bunda A,B,C lar o'tkir burchakli uchburchakning burchaklari.

A7. Faraz qilaylik musbat a,b,c sonlar $a^2+b^2+c^2+2abc=1$ shartni qanoatlantirsin. Shartga ko'ra uchta son ham musbatligi uchun $a,b,c<1$ bo'ladi. Ushbu $f:(0;\pi)\to(0;1)$, $f(x)=\sin\frac{x}{2}$ funksiya hamda 2-teorema yordamida quyidagicha almashtirish olishimiz mumkin

$$a=\sin\frac{\alpha}{2},\ b=\sin\frac{\beta}{2},\ c=\sin\frac{\gamma}{2}$$

bunda α,β,γ lar biror uchburchakning burchaklari.

A8. Faraz qilaylik musbat a,b,c sonlar $a^2+b^2+c^2+2abc=1$ shartni qanoatlantirsin. A1 va A7 larga ko'ra quyidagicha almashtirish olishimiz mumkin

$$a=\cos A,\ b=\cos B,\ c=\cos C,$$

bunda A,B,C lar o'tkir burchakli uchburchakning burchaklari.

A9. Faraz qilaylik x,y,z lar musbat sonlar bo'lsin. A2 yordamida quyidagi

$$\sqrt{\frac{yz}{(x+y)(x+z)}},\ \sqrt{\frac{zx}{(y+z)(y+x)}},\sqrt{\frac{xy}{(z+x)(z+y)}}$$

ifodalarni ushbu

$$\sqrt{\frac{(s-b)(s-c)}{bc}},\ \sqrt{\frac{(s-c)(s-a)}{ca}},\sqrt{\frac{(s-a)(s-b)}{ab}}$$

ifodalarga almashtiramiz. Quyidagi ayniyatlarga ko'ra

$$\sin\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},\ \cos\frac{\alpha}{2}=\sqrt{\frac{s(s-a)}{bc}},$$

bizning dastlabki ifodalarimiz mos ravishda quyidagi shaklga keladi

$\sin\frac{\alpha}{2},\ \sin\frac{\beta}{2},\ \sin\frac{\gamma}{2},$

bunda α, β, γ lar biror uchburchakning burchaklari.

A10. Xuddi A9 dagi kabi quyidagi

$$\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}},\ \sqrt{\frac{y(x+y+z)}{(y+z)(y+x)}},\ \sqrt{\frac{z(x+y+z)}{(z+x)(z+y)}},$$

ifodalarni mos ravishda ushbu

$$\cos\frac{\alpha}{2},\ \cos\frac{\beta}{2},\ \cos\frac{\gamma}{2},$$

bunda α, β, γ lar biror uchburchakning burchaklari.

Mashq 1. Faraz qilaylik musbat p, q, r sonlar $p^2+q^2+r^2+2pqr=1$ shartni qanoatlantirsin.U holda $p=\cos A, q=\cos B, r=\cos C$ shartni qanoatlantiruvchi o'tkir burchakli ABC uchburchak mavjudligini ko'rsating.

Mashq 2. Faraz qilaylik nomanfiy p, q, r sonlar $p^2+q^2+r^2+2pqr=1$ shartni qanoatlantirsin. U holda $p=\cos A, q=\cos B, r=\cos C$ va $A+B+C=\pi$ shartlarni qanoatlantiruvchi $A, B, C \in \left[0; \frac{\pi}{2}\right]$ burchaklar mavjudligini ko'rsating.

Quyida biz ko'plab masalalarni yechishda yordam beradigan bir qator tengsizliklar va ayniyatlar keltiramiz. Bularning deyarli barchasi yaxshi-ma'lum munosabatlar bo'lib isbotlari qiyin emas. Bu munosabatlarning ko'pchiligining isbotini adabiyotlardan topish mumkin.

Tengsizliklar

Faraz qilaylik α, β, γ lar ABC uchburchakning burchaklari bo'lsin. Quyidagi tengsizliklar o'rinli

1. $\cos\alpha+\cos\beta+\cos\gamma \le \sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2} \le \frac{3}{2}$

2. $\sin\alpha+\sin\beta+\sin\gamma\le\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\le\frac{3\sqrt{3}}{2}$

3. $\cos\alpha\cos\beta\cos\gamma\le\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\le\frac{1}{8}$

4. $\sin\alpha\sin\beta\sin\gamma\le\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\le\frac{3\sqrt{3}}{8}$

5. $ctg\frac{\alpha}{2}+ctg\frac{\beta}{2}+ctg\frac{\gamma}{2}\ge3\sqrt{3}$

6. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma\ge\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}\ge\frac{3}{4}$

7. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma\le\cos^2\frac{\alpha}{2}+\cos^2\frac{\beta}{2}+\cos^2\frac{\gamma}{2}\le\frac{9}{4}$

8. $ctg\alpha+ctg\beta+ctg\gamma\ge tg\frac{\alpha}{2}+tg\frac{\beta}{2}+tg\frac{\gamma}{2}\ge\sqrt{3}$

Ayniyatlar

Faraz qilaylik α,β,γ lar ABC uchburchakning burchaklari bo'lsin. Quyidagi ayniyatlar o'rinli

1. $\cos\alpha+\cos\beta+\cos\gamma=1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

2. $\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

3. $\sin2\alpha+\sin2\beta+\sin2\gamma=4\sin\alpha\sin\beta\sin\gamma$

4. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2+2\cos\alpha\cos\beta\cos\gamma$

Istalgan α,β,γ burchaklar (uchburchak burchaklari bo'lishi shart emas) uchun quyidagi ayniyatlar o'rinli

$$\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)=4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$$

$$\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)=4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$$

2-§. Trigonometrik almashtirishlarning tadbiqlari

1-masala. (Janubiy Koreya, 1998) Faraz qilaylik musbat x,y,z sonlar $x+y+z=xyz$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+y^2}}+\frac{1}{\sqrt{1+z^2}}\le\frac{3}{2}.$$

Bu masalani yechishda o'quvchini xayoliga eng birinchi $f(t)=\frac{1}{\sqrt{1+t^2}}$ funksiya uchun Iensen tengsizligini qo'llash kelishi mumkin. Ammo bu f funksiya R^+ to'plamda yuqoriga qavariq emas. Ammo shunisi qiziqarliki $f(tg\theta)$ funksiya yuqoriga qavariq!

Isboti. Quyidagicha almashtirish olaylik

$$x=tgA,\ y=tgB,\ z=tgC,\ A,B,C\in(0;\frac{\pi}{2})$$

Ushbu $1+tg^2\alpha=\frac{1}{\cos^2\alpha}$, $\cos\alpha\neq0$ ayniyatga ko'ra berilgan tengsizlik quyidagicha ko'rinishni oladi

$$\cos A+\cos B+\cos C\le\frac{3}{2}$$

Quyidagi $tg(\pi-C)=-z=\frac{x+y}{1-xy}=tg(A+B)$ va $\pi-C, A+B\in(0;\pi)$ munosabatlardan $\pi-C=A+B$ yoki $A+B+C=\pi$ tenglikni olamiz. Demak, istalgan ABC uchburchak uchun $\cos a+\cos B+\cos C\le\frac{3}{2}$ tengsizlikni isbot qilsak etarli ekan.Bu esa quyidagi munosabatdan kelib chiqadi

$$3-2(\cos A+\cos B+\cos C)=(\sin A-\sin B)^2+(\cos A+\cos B-1)^2\ge0.$$

Isbot tugadi.

2-masala. (FML, ochiq olimpiada, Rossiya) Faraz qilaylik musbat x, y, z sonlar $x+y+z=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$\sqrt{\frac{xy}{z+xy}}+\sqrt{\frac{yz}{x+yz}}+\sqrt{\frac{zx}{y+zx}}\leq\frac{3}{2}$$

Isboti. Yuqoridagi tengsizlik ushbu tengsizlikka teng kuchli

$$\sqrt{\frac{yz}{(x+y)(x+z)}}+\sqrt{\frac{zx}{(y+z)(y+x)}}+\sqrt{\frac{xy}{(z+x)(z+y)}}\leq\frac{3}{2}$$

A9 ga ko'ra bu tengsizlikning uchta hadini $\sin\frac{\alpha}{2}, \sin\frac{\beta}{2}, \sin\frac{\gamma}{2}$ larga almashtiramiz va demak, ushbu $\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\leq\frac{3}{2}$ tengsizlikni isbotlashimiz kerak. Bu tengsizlikning o'rinli ekanligi ravshan.(Iensen tengsizligidan osongina kelib chiqadi)

Isbot tugadi.

3-masala. (Eron, 1997) Faraz qilaylik x, y, z sonlar $x, y, z>1, \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$ shartlarni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\leq\sqrt{x+y+z}$$

Isboti. Quyidagicha $(x, y, z)=(a+1, b+1, c+1)$ almashtirish olaylik, bunda $a, b, c>0$ va shartga ko'ra $ab+bc+ca+2abc=1$ tenglik o'rinli.U holda quyidagi tengsizlikni isbotlash etarli

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\leq\sqrt{a+b+c+3}.$$

Ikkala tarafni kvadratga oshirib va ayrim xadlarni yo'qotib quyidagi tegsizlikka kelamiz

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\leq\frac{3}{2}.$$

A7 dan foydalanib $(ab,bc,ca)=(\sin^2\frac{\alpha}{2},\sin^2\frac{\beta}{2},\sin^2\frac{\gamma}{2})$ ni olamiz, bunda ABC ixtiyoriy uchburchak. Demak quyidagi tengsizlikni isbotlashimiz kerak

$$\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\le\frac{3}{2}$$

Bu tengsizlikning o'rinli ekanligi ma'lum.Isbot tugadi.

4-masala. (Crux Mathematicorum and Mathematical Mayhem) Faraz qilaylik x,y,z lar musbat sonlar bo'lsin. Quyidagi tengsizlikni isbotlang

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\le 1$$

Isboti. Bu tengsizlik quyidagi tengsizlikka teng kuchli

$$\sum\frac{1}{1+\sqrt{\frac{(x+y)(x+z)}{x^2}}}\le 1$$

Berilgan tengsizlik bir jinsli bo'lganligi uchun umumiylikka ziyon etkazmasdan $xy+yz+zx=1$ deb faraz qilishimiz mumkin. A3 almashtirishdan foydalanamiz

$$\frac{(x+y)(x+z)}{x^2}=\frac{(tg\frac{\alpha}{2}+tg\frac{\beta}{2})(tg\frac{\alpha}{2}+tg\frac{\gamma}{2})}{tg^2\frac{\alpha}{2}}=\frac{1}{\sin^2\frac{\alpha}{2}},$$

qolgan xadlar ham shunga o'xshash ifodalanadi. Tengsizlik quyidagi shaklga keladi

$$\frac{\sin\frac{\alpha}{2}}{1+\sin\frac{\alpha}{2}}+\frac{\sin\frac{\beta}{2}}{1+\sin\frac{\beta}{2}}+\frac{\sin\frac{\gamma}{2}}{1+\sin\frac{\gamma}{2}}\le 1,$$

ya'ni

$$2\le\frac{1}{1+\sin\frac{\alpha}{2}}+\frac{1}{1+\sin\frac{\beta}{2}}+\frac{1}{1+\sin\frac{\gamma}{2}}.$$

Boshqa tomondan yaxshi tanish bo'lgan $\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\leq\frac{3}{2}$ tengsizlik va Koshi-Bunyakovskiy-Shvarts tengsizligidan foydalanib quyidagi tengsizlikka ega bo'lamiz

$$2\leq\frac{9}{\left(1+\sin\frac{\alpha}{2}\right)+\left(1+\sin\frac{\beta}{2}\right)+\left(1+\sin\frac{\gamma}{2}\right)}\leq\sum\frac{1}{1+\sin\frac{\alpha}{2}}$$

Isbot tugadi.

5-masala. (Ruminiya, 2005) Faraz qilaylik musbat a,b,c sonlar $(a+b)(b+c)(c+a)=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$ab+bc+ca\leq\frac{3}{4}$$

Isboti. Bu tengsizlik quyidagi tengsizlikka teng kuchli

$$(ab+bc+ca)^3\leq\left(\frac{3}{4}\right)^3(a+b)^2(b+c)^2(c+a)^2$$

Bu tengsizlik bir jinsli bo'lganligi uchun umumiylikka ziyon etkazmasdan $ab+bc+ca=1$ deb faraz qilishimiz mumkin. A3 almashtirishdan foydalanamiz

$$(a+b)(b+c)(c+a)=\prod\left(\frac{\cos\frac{\gamma}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}\right)=\frac{1}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}.$$

Demak, ushbu

$$\left(\frac{4}{3}\right)^3\leq\frac{1}{\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}}$$

yoki

$$4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\leq\frac{3\sqrt{3}}{2}$$

tengsizlikni isbotlash etarli. Ushbu

$$\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

ayniyatga asosan quyidagi tengsizlikni isbotlashimiz kerak

$$\sin\alpha+\sin\beta+\sin\gamma\le\frac{3\sqrt{3}}{2}$$

Bu tengsizlik esa $f(x)=\sin x$ funksiya $(0;\pi)$ intervalda yuqoriga qavariqligi uchun Iensen tengsizligidan kelib chiqadi. Isbot tugadi.

6-masala. (Polsha, 1999) Faraz qilaylik musbat a,b,c sonlar $a+b+c=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$a^2+b^2+c^2+2\sqrt{3abc}\le 1.$$

Isboti. Ushbu $a=xy, b=yz, c=zx$ almashtirish bilan tengsizlik quyidagi shaklga keladi

$$x^2y^2+y^2z^2+z^2x^2+2\sqrt{3}xyz\le 1$$

bunda $x,y,z>0$ va $xy+yz+zx=1$. Yuqoridagi tengsizlik quyidagi tengsizlikka teng kuchli

$$(xy+yz+zx)^2+2\sqrt{3}xyz\le 1+2xyz(x+y+z),$$

yoki

$$\sqrt{3}\le x+y+z$$

A3 almashtirishga ko'ra

$$tg\frac{\alpha}{2}+tg\frac{\beta}{2}+tg\frac{\gamma}{2}\ge\sqrt{3}$$

tengsizlikni isbotlash etarli. Bu tengsizlik esa $f(x)=tg\frac{x}{2}$ funksiya $(0;\pi)$ intervalda qavariqligi uchun Iensen tengsizligidan kelib chiqadi. Isbot tugadi.

7-masala. Faraz qilaylik x,y,z lar musbat sonlar bo'lsin.Quyidagi tengsizlikni isbotlang

$$\sqrt{x(y+z)}+\sqrt{y\left(z+x\right)}+\sqrt{z(x+y)}\geq 2\sqrt{\frac{(x+y)(y+z)(z+x)}{x+y+z}}$$

Isboti. Tengsizlikni quyidagicha yozib olamiz

$$\sqrt{\frac{x(x+y+z)}{(x+y)(x+z)}}+\sqrt{\frac{y(x+y+z)}{(y+z)(y+x)}}+\sqrt{\frac{z(x+y+z)}{(z+x)(z+y)}}\geq 2$$

A10 almashtirishga ko'ra

$$\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\geq 2$$

tengsizlikni isbot qilish etarli. A1 ga ko'ra o'tkir burchakli ABC uchburchak uchun

$$\sin A+\sin B+\sin C\geq 2$$

tengsizlikni isbotlash etarli. Bu tengsizlikni isbotlashning juda ko'p usullari mavjud. Biz Jordan tengsizligidan foydalanishni tavsiya qilamiz.

Jordan tengsizligi. Barcha $\alpha\in\left(0;\frac{\pi}{2}\right)$ lar uchun quyidagi tengsizlik o'rinli

$$\frac{2\alpha}{\pi}\leq\sin\alpha\leq\alpha .$$

U holda $\sin A+\sin B+\sin C\geq\frac{2A}{\pi}+\frac{2B}{\pi}+\frac{2C}{\pi}=2$

Isbot tugadi.

8-masala. Faraz qilaylik x,y,z lar musbat sonlar bo'lsin. Quyidagi tengsizlikni isbotlang

$$\sqrt{\frac{y+z}{x}}+\sqrt{\frac{z+x}{y}}+\sqrt{\frac{x+y}{z}}\geq\sqrt{\frac{16(x+y+z)^3}{3(x+y)(y+z)(z+x)}} .$$

Isboti. Qulaylik uchun quyidagicha belgilash olamiz:

$$\sum_{cyc}f(x,y,z)=f(x,y,z)+f(y,z,x)+f(z,x,y)$$

Berilgan tengsizlikni quyidagicha yozib olamiz

$$\sum_{cyc}(y+z)\sqrt{\frac{(x+y)(z+x)}{x(x+y+z)}}\geq\frac{4(x+y+z)}{\sqrt{3}}$$

A2 va A10 almashtirishlarga ko'ra

$$(y+z)\sqrt{\frac{(x+y)(z+x)}{x(x+y+z)}}=\frac{a}{\cos\frac{\alpha}{2}}=4R\sin\frac{\alpha}{2},$$

boshqa xadlarni ham shunday ifodalab olamiz.

Shuningdek

$$\frac{4(x+y+z)}{\sqrt{3}}=\frac{4R(\sin\alpha+\sin\beta+\sin\gamma)}{\sqrt{3}}$$

munosabat o'rinli. Bu yerda α,β,γ lar tashqi chizilgan aylana radiusi R bo'lgan uchburchakning burchaklari.

Shunday qilib quyidagi tengsizlikni isbotlashimiz kerak

$$\frac{\sqrt{3}}{2}\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right)\geq\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\sin\frac{\beta}{2}\cos\frac{\beta}{2}+\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\quad.$$

Ushbu $f(x)=\cos\frac{x}{2}$ funksiya $[0;\pi]$ kesmada botiqligi uchun Iensen tengsizligiga ko'ra quyidagi tengsizlik o'rinli

$$\frac{\sqrt{3}}{2}\geq\frac{1}{3}\left(\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\right)$$

Ushbu $f(x)=\sin\frac{x}{2}$ funksiya $[0;\pi]$ kesmada o'suvchi, $f(x)=\cos\frac{x}{2}$ funksiya $[0;\pi]$ kesmada kamayuvchi bo'lganligi uchun Chebishev tengsizligiga ko'ra quyidagi tengsizlik o'rinli

$$\frac{1}{3}\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right)\left(\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\right)\geq$$

$$\geq\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\sin\frac{\beta}{2}\cos\frac{\beta}{2}+\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}$$

Bu va bundan oldingi tengsizliklarga ko'ra

$$\frac{\sqrt{3}}{2}\left(\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right)\geq \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\sin\frac{\beta}{2}\cos\frac{\beta}{2}+\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}$$

tengsizlikning o’rinli ekanligi ko’rinib turibdi. Isbot tugadi.

Mashqlar

1. (Ruminiya, 2005) Faraz qilaylik musbat a,b,c sonlar $a+b+c=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}}\geq\sqrt{\frac{3}{2}}$$

2. (Ukraina, 2005) Faraz qilaylik musbat a,b,c sonlar $a+b+c=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$\sqrt{\frac{1}{a}-1}\sqrt{\frac{1}{b}-1}+\sqrt{\frac{1}{b}-1}\sqrt{\frac{1}{c}-1}+\sqrt{\frac{1}{c}-1}\sqrt{\frac{1}{a}-1}\geq 6$$

3. Faraz qilaylik musbat a,b,c sonlar $ab+bc+ca=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$$\frac{1}{\sqrt{b+c}}+\frac{1}{\sqrt{c+a}}+\frac{1}{\sqrt{a+b}}\geq 2+\frac{1}{\sqrt{2}}$$

4. (APMO, 2004) Musbat a,b,c sonlar uchun quyidagi tengsizlikni isbotlang

$$(a^2+2)(b^2+2)(c^2+2)\geq 9(ab+bc+ca)$$

5. (APMO, 2002) Faraz qilaylik musbat a,b,c sonlar $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ shartni qanoatlantirsin. Quyidagi tengsizlikni isbotlang

$\sqrt{a+bc}+\sqrt{b+ca}+\sqrt{c+ab}\geq\sqrt{abc}+\sqrt{a}+\sqrt{b}+\sqrt{c}$.

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Mundarija

O'ZBEKISTON RESPUBLIKASI XALQ TA'LIMI VAZIRLIGI

A. QO'CHQOROV, J. RASULOV

TENGSIZLIKLAR-III.

MASALALAR TO'PLAMI

Toshkent–2008

A. Qo’chqorov, J. Rasulov. **Tengsizliklar-III. Masalalar to’plami** / Toshkent, 2008 y.

Fizika –matematika fanlari doktori, professor A. A’zamov umumiy tahriri ostida.

Qo’llanmada oxirgi yillar mobaynida turli davlatlarda bo’lib o’tgan matematika olimpiadalarida taqdim qilingan tengsizliklar yechimlar bilan keltirilgan.

Qo’llanma umumiy o’rta ta’lim maktablari, akademik litseylar va kasb–hunar kollejlarining iqtidorli o’quvchilari, matematika fani o’qituvchilari hamda pedagogika oliy o’quv yurtlari talabalari uchun mo’ljallangan.

Qo’llanmadan sinfdan tashqari mashg’ulotlarda, o’quvchilarni turli matematik musobaqalarga tayyorlash jarayonida foydalanish mumkin.

Taqrizchilar: TVDPI matematika kafedrasi mudiri, f.–m.f.n., dotsent Sh.B. Bekmatov
TVDPI boshlang’ich ta’lim metodikasi kafedrasi dotsenti, ped. f.n. Z. S. Dadanov

Ushbu qo’llanma Respublika ta’lim markazi qoshidagi matematika fanidan ilmiy-metodik kengash tomonidan nashrga tavsiya etilgan. (15 iyun 2008 y., 8 -sonli bayyonnoma)

Qo’llanmaning yaratilishi Vazirlar Mahkamasi huzuridagi Fan va texnologiyalarni rivojlantirishni muvofiqlashtirish Q’omitasi tomonidan moliyalashtirilgan (ХИД 1-16 – sonli innovatsiya loyihasi)

Masalalar.

1. Agar a,b,c musbat sonlar va $n,k\in N$ bo'lsa, u holda quyidagi

$$\frac{a^{n+k}}{b^k}+\frac{b^{n+k}}{c^k}+\frac{c^{n+k}}{a^k}\geq a^n+b^n+c^n$$

tengsizlikni isbotlang.

2. (Rossiya -2003) Musbat a,b,c sonlar $a+b+c=1$ tenglikni qanoatlantirilsa, u holda

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\geq\frac{2}{1+a}+\frac{2}{1+b}+\frac{2}{1+c}$$

tengsizlikni isbotlang.

3. (Moldova -2005) Musbat a,b,c sonlar $a^4+b^4+c^4=3$ tenglikni qanoatlantirsa,

$$\frac{1}{4-ab}+\frac{1}{4-ac}+\frac{1}{4-bc}\leq 1$$

tengsizlikni isbotlang.

4. (Koreya -2000) Aytaylik, a,b,c, x,y,z haqiqiy sonlar quyidagi

$a\geq b\geq c>0$, $x\geq y\geq z>0$ shartlarni qanoatlantirsin, u holda

$$\frac{a^2x^2}{(by+cz)(bz+cy)}+\frac{b^2y^2}{(cz+ax)(cx+az)}+\frac{c^2z^2}{(ax+by)(ay+bx)}\geq\frac{3}{4}$$

tengsizlikni isbotlang.

5. (Yaponiya -2002) Aytaylik, $n\geq 3$ va $n\in N$ da musbat $a_1,a_2,...,a_n,b_1,b_2,....,b_n$ sonlar quyidagi $a_1+a_2+...+a_n=1$ *ва* $b_1^2+b_2^2+...+b_n^2=1$ shartlarni qanoatlantirsin. U holda

$$a_1(b_1+a_2)+a_2(b_2+a_3)+...+a_n(b_n+a_1)<1$$

tengsizlikni isbotlang.

6. Musbat a,b,c sonlar $a+b+c=1$ shartni qanoatlantirsa, u holda

$$\sqrt{\frac{ab}{ab+c}}+\sqrt{\frac{bc}{bc+a}}+\sqrt{\frac{ac}{ac+b}}\le\frac{3}{2}$$

tengsizlikni isbotlang.

7. (Eron -2005) Musbat a,b,c sonlar uchun

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2\ge(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

tengsizlikni isbotlang.

8. (Vengriya -1996) Musbat a,b sonlarning yig'indisi birga teng bo'lsa

$\frac{a^2}{a+1}+\frac{b^2}{b+1}\ge\frac{1}{3}$ tengsizlikni isbotlang.

9. Haqiqiy musbat x,y,z sonlar $xyz\ge1$ shartni qanoatlantirsa,

$$\frac{x^5}{x^5+y^2+z^2}+\frac{y^5}{y^5+z^2+x^2}+\frac{z^5}{z^5+x^2+y^2}\ge1$$

tengsizlikni isbotlang.

10. (XMO, Xalqaro Matematika olimpiadasi -2005) Musbat x,y,z sonlar $xyz\ge1$ shartni qanoatlantirsa,

$$\frac{x^5-x^2}{x^5+y^2+z^2}+\frac{y^5-y^2}{y^5+z^2+x^2}+\frac{z^5-z^2}{z^5+x^2+y^2}\ge0$$

tengsizlikni isbotlang.

11. (Bosniya -2002) Agar musbat x,y,z sonlar $xyz=x+y+z+2$ tenglikni qanoatlantirsa,

$$5(x+y+z)+18\ge8(\sqrt{xy}+\sqrt{yz}+\sqrt{zx})$$

tengsizlik o’rinli bo’lishini isbotlang.

12. (APMO -2005) Musbat a,b,c sonlar $abc=8$ shartni qanoatlantirsa, u holda.

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}}+\frac{b^2}{\sqrt{(1+b^3)(1+c^3)}}+\frac{c^2}{\sqrt{(1+c^3)\left(1+a^3\right)}}\geq\frac{4}{3}$$

tengsizlikni isbotlang.

13. (Rossiya -1999) Musbat haqiqiy x va y sonlar $x^2+y^3\geq x^3+y^4$ tengsizlikni qanoatlantirsa, u holda $x^3+y^3\leq 2$ tengsizlikni isbotlang.

14. (APMO -2003). Agar a,b,c sonlari uchburchak tomonlarining uzunliklari bo’lib, $a+b+c=1$ shartni qanoatlantirsa, $n\in N$, $n\geq 2$ uchun

$$\sqrt[n]{a^n+b^n}+\sqrt[n]{b^n+c^n}+\sqrt[n]{c^n+a^n}<1+\frac{\sqrt[n]{2}}{2}$$

tengsizlikni isbotlang.

15. Musbat $a_1,a_2,...,a_n$ sonlar uchun quyidagi

$$n\cdot\sqrt[n]{\frac{G_n}{A_n}}+\frac{g_n}{G_n}\leq n+1$$

tengsizlikni isbotlang. Bu yerda

$$g_n=\sqrt[n]{a_1a_2...a_n},\ A_n=\frac{a_1+a_2+...+a_n}{n},\ \ G_n=\sqrt[n]{A_1A_2...A_n}\ .$$

16. (Yaponiya -2005) Musbat a,b,c sonlar yig’indisi birga teng bo’lsa,

$$a\sqrt[3]{1+b-c}+b\sqrt[3]{1+c-a}+\sqrt[3]{1+a-b}\leq 1$$

tengsizlikni isbotlang.

17. (Kolmogorov kubogi, Rossiya -2004) Musbat haqiqiy a,b,c,d sonlar $abcd=1$ shartni qanoatlantirsa, u holda

$$\frac{1+ab}{1+a}+\frac{1+bc}{1+b}+\frac{1+cd}{1+c}+\frac{1+da}{1+d}\geq 4$$

tengsizlikni isbotlang.

18. (Kolmogorov kubogi, Rossiya -2004) Musbat a,b,c sonlarning yig'indisi birga teng bo'lsa,

$$(ab+bc+ca)\left(\frac{a}{b(b+1)}+\frac{b}{c(c+1)}+\frac{c}{a(a+1)}\right)\geq\frac{3}{4}$$

tengsizlik o'rinli bo'lishini isbotlang.

19. Musbat a,b,c sonlar $a+b+c=1$ shartni qanoatlantirsa,

$$\frac{a^2+b}{b+c}+\frac{b^2+c}{c+a}+\frac{c^2+a}{a+b}\geq 2$$

tengsizlikni o'rinli bo'lishini isbotlang

20. Musbat x,y,z sonlar uchun

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+x)(y+z)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\leq 1$$

tengsizlikni isbotlang.

21. (Xitoy -2004) Musbat a,b,c sonlar uchun

$$\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}}\leq\frac{3\sqrt{2}}{2}$$

tengsizlikni isbotlang.

22. (Turkiya -1998) Agar $0\leq a\leq b\leq c$ shart bajarilsa,

$$(a+3b)(b+4c)(c+2a)\geq 60abc$$

tengsizlikni isbotlang.

23. (Buyuk Ipak yo'li Xalqaro olimpiadasi -2006) Musbat a,b,c sonlar uchun $abc=1$ shart bajarilsa,

$$4\left(\sqrt[3]{\frac{a}{b}}+\sqrt[3]{\frac{b}{c}}+\sqrt[3]{\frac{c}{a}}\right)\le 3\left(2+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^{\frac{2}{3}}$$

tengsizlikni isbotlang.

24. (Albaniya -2002) Musbat a,b,c sonlar uchun

$$(a+b+c)+\sqrt{a^2+b^2+c^2}\le\frac{\sqrt{3}+1}{3\sqrt{3}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(a^2+b^2+c^2\right)$$

tengsizlikni isbotlang.

25. (AQSh -2004) Musbat a,b,c sonlar uchun

$$\left(a^5-a^2+3\right)(b^5-b^2+3)(c^5-c^2+3)\ge\left(a+b+c\right)^3$$

tengsizlikni isbotlang.

26. Musbat a,b,c sonlar uchun

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)\ge(ab+bc+ca)^3$$

tengsizlikni isbotlang.

27. (AQSh -2001) Agar musbat a,b,c sonlar $a^2+b^2+c^2+abc=4$ shartni qanoatlantirsa, $0<ab+bc+ca-abc\le 2$ tengsizlikni isbotlang.

28. Haqiqiy x,y sonlar $x\ne 0,\ xy(x^2-y^2)=x^2+y^2$ shartlarni qanoatlantirsa, $x^2+y^2\ge 4$ tengsizlikni isbotlang.

29. (Vyetnam -2001) Musbat a,b,c sonlar $21ab+2bc+8ac\le 12$ shartni qanoatlantirsa, $P(a,b,c)=\frac{1}{a}+\frac{2}{b}+\frac{3}{c}$ ifodaning eng kichik qiymatini toping.

30. Musbat a,b,c sonlar uchun $\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c+\frac{4(a-b)^2}{a+b+c}$ tengsizlikni isbotlang.

31. (Hindiston -2000) Haqiqiy $a_1,...,a_n$ conlar uchun $a_1\leq a_2\leq...\leq a_n$ va $a_1+a_2+...+a_n=0$ shartlar bajarilsa, $na_1a_n+\sum_{i=1}^{n}a_i^2\leq 0$ tengsizlikni isbotlang.

32. (Ruminiya -1999) Musbat $x_1,x_2,...x_n$ sonlar $\prod_{i=1}^{n}x_i=1$ shartni qanoatlantirsa,

$$\frac{1}{n-1+x_1}+\frac{1}{n-1+x_2}+.....+\frac{1}{n-1+x_n}\leq 1$$

tengsizlikni isbotlang.

33. (Polsha -1991) Haqiqiy x,y,z sonlar $x^2+y^2+z^2=2$ shartni qanoatlantirsa, $x+y+z-xyz\leq 2$ tengsizlikni isbotlang.

34. (Vyetnam -2001) Musbat x,y,z sonlar

$\frac{1}{\sqrt{2}}\leq z<\frac{1}{2}\min\left\{x\sqrt{2},y\sqrt{3}\right\}$, $x+z\sqrt{3}\geq\sqrt{6}$, $y\sqrt{3}+z\sqrt{10}\geq 2\sqrt{5}$ shartlarni qanoatlantirsa, $P(x,y,z)=\frac{1}{x^2}+\frac{2}{y^2}+\frac{3}{z^2}$ ifodaning eng katta qiymatini toping.

35. (AQSh -2003) Musbat a,b,c sonlar uchun

$$\frac{\left(2a+b+c\right)^2}{2a^2+(b+c)^2}+\frac{\left(2b+c+a\right)^2}{2b^2+(a+c)^2}+\frac{(2c+a+b)^2}{2c^2+(a+b)^2}\leq 8$$

tengsizlikni isbotlang.

36. (Albaniya -2004) Musbat a,b,c sonlarning ko'paytmasi birga teng bo'lsa,

$$\frac{1}{\sqrt{a+\frac{1}{b}+0,64}}+\frac{1}{\sqrt{b+\frac{1}{c}+0,64}}+\frac{1}{\sqrt{c+\frac{1}{a}+0,64}}\geq 1,2$$

tengsizlikni isbotlang.

37. Musbat x, y, z sonlar $xyz=1$ shartni qanoatlantirsa,

$$x+y+z-\left(\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\right)\geq\frac{1}{x(1+z)}+\frac{1}{y(1+x)}+\frac{1}{z(1+y)}$$

tengsizlikni isbotlang.

38. Musbat x, y, z sonlar uchun

$$3(x^2-x+1)(y^2-y+1)(z^2-z+1)\geq(xyz)^2+xyz+1$$

tengsizlikni isbotlang.

39. (Belorussiya -1997) $n\in N,\ n>1$ uchun

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}<n-n^{\frac{n-1}{n}}$$

tengsizlikni isbotlang.

40. Birdan katta haqiqiy x, y, z sonlar $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$ tenglikni kanoatlantirsa, $\sqrt{x+y+z}\geq\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$ tengsizlikni isbotlang.

41. (Hindiston -2002) Musbat a, b, c sonlar $a^2+b^2+c^2=3abc$ tenglikni qanoatlantirsa, $\frac{a}{b^2c^2}+\frac{b}{c^2a^2}+\frac{c}{a^2b^2}\geq\frac{9}{a+b+c}$ tengsizlikni isbotlang.

42. (Ukraina -2002) Musbat x, y, z sonlar uchun

$$\frac{1}{(x+y+z)^2}+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\geq\frac{28\sqrt{3}}{9\sqrt{xyz(x+y+z)}}$$ t

tengsizlikni isbotlang.

43. (Ukraina -2001) $a_1, a_2, ..., a_n$ haqiqiy sonlar

$a_1+a_2+...+a_n\geq n^2$, $a_1^2+a_2^2+...+a_n^2\leq n^2+1$ shartlarni qanoatlantirsa,

$n-1\leq a_k\leq n+1$ tengsizlikni isbotlang.

44. (Ukraina -2001) Musbat a,b,c *ва* α,β,γ $\alpha+\beta+\gamma=1$ sonlar uchun

$\alpha a+\beta b+\gamma c+2\sqrt{(\alpha\beta+\beta\gamma+\gamma\alpha)(ab+bc+ca)}\leq a+b+c$ tengsizlikni isbotlang.

45. (Ukraina -2000) Musbat a,b sonlar uchun

$$\frac{1}{a^3}+\frac{1}{3ab^2}+\frac{1}{3a^2b}+\frac{1}{b^3}\geq\frac{64}{3(a+b)^3}$$

tengsizlikni isbotlang.

46. (Ukraina -1999) Haqiqiy $x_1, x_2, ..., x_6\in[0;1]$ sonlar uchun

$$\frac{x_1^3}{x_2^5+x_3^5+x_4^5+x_5^5+x_6^5+5}+\frac{x_2^3}{x_1^5+x_3^5+x_4^5+x_5^5+x_6^5+5}+...+\frac{x_6^3}{x_1^5+x_2^5+x_3^5+x_4^5+x_5^5+5}\leq\frac{3}{5}$$

tengsizlikni isbotlang.

47. (Bosniya -2002) Musbat a,b,c sonlari $a^2+b^2+c^2=1$ shartni qanoatlantir-sa,

$\frac{a^2}{1+2bc}+\frac{b^2}{1+2ac}+\frac{c^2}{1+2ab}\geq\frac{3}{5}$ tengsizlikni isbotlang.

48. (Yugoslaviya -2002) Musbat $x_1x_2.....x_{2001}$ sonlar uchun

$$x_i^2\geq x_1^2+\frac{x_2^2}{2^3}+\frac{x_3^2}{3^3}+...+\frac{x_{i-1}^2}{(i-1)^3},\quad 2\leq i\leq 2001$$

shart bajarilsa, $\sum_{i=2}^{2001} \frac{x_i}{x_1 + x_2 + ... + x_{i-1}} > 1,999$ tengsizlikni isbotlang.

49. (Hindiston -2002) Musbat a,b,c sonlar uchun

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{c+a}{c+b} + \frac{a+b}{a+c} + \frac{b+c}{b+a}$$

ekanligini ko'rsating.

50. (XMO -1998) Agar $y_i \geq 1 \ (i = 1,2,...n)$ bo'lsa,

$$\frac{1}{1+y_1} + \frac{1}{1+y_2} + ... + \frac{1}{1+y_n} \geq \frac{n}{1+\sqrt[n]{y_1 y_2 ... y_n}}$$

tengsizlikni isbotlang.

51. (Hindiston -2004) Ixtiyoriy $x_i \in (0; \frac{1}{2}]$ sonlar uchun (i=1, 2,…, *n*)

$$\frac{\prod_{i=1}^{n} x_i}{\left(\sum_{i=1}^{n} x_i\right)^n} \leq \frac{\prod_{i=1}^{n}(1-x_i)}{\left(\sum_{i=1}^{n}(1-x_i)\right)^n}$$

tengsizlikni isbotlang.

52. (XMO -2001) $x_1, x_2, ...x_n$ haqiqiy sonlar uchun

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + ... + \frac{x_n}{1+x_1^2+x_2^2+...+x_n^2} < \sqrt{n}$$

tengsizlikni isbotlang.

53. (XMO -2004) Musbat a,b,c sonlar $ab + bc + ca = 1$ tenglikni qanoatlantirsa,

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}$$

tengsizlikni isbotlang.

54. (XMO -2004) Musbat x, y, z, a, b, c sonlar $ab + bc + ca = 1$ shartni qanoatlantirsa

$$3abc(x + y + z) \le \frac{2}{3} + ax^3 + by^3 + cz^3$$

tengsizlikni isbotlang.

55. (Moldova -2001) $a_1, a_2, ..., a_n$ haqiqiy musbat sonlar uchun

$$\frac{1}{\sum_{i=1}^{n} \frac{1}{1 + a_i}} - \frac{1}{\sum_{i=1}^{n} \frac{1}{a_i}} \ge \frac{1}{n}$$

tengsizlikni isbotlang.

56. (AQSh -1992) $a_0, a_1, a_2, ..., a_n$ musbat haqiqiy sonlari $a_{i-1} \cdot a_{i+1} \le a_i^2$ $(i = 1, 2, ..., n-1)$ shartni qanoatlantirsa

$$\frac{a_0 + a_1 + ... + a_n}{n+1} \cdot \frac{a_1 + a_2 + ... + a_{n-1}}{n-1} \ge \frac{a_0 + a_1 + ... + a_{n-1}}{n} \cdot \frac{a_0 + a_1 + ... + a_n}{n}$$

tengsizlikni isbotlang.

57. (Polsha -1996) a, b, c , x, y, z nomanfiy sonlar bo'lib,

$x + y + z = a + b + c = 1,\ \ 0 \le x, y, z \le \frac{1}{2}$ shartlarni qanoatlantirsa,

$$ax + by + cz \ge 8abc$$

tengsizlikni isbotlang.

58. (XMO -1998) Yig'indisi birdan kichik bo'lgan $x_1, x_2, ..., x_n$ musbat sonlar uchun

$$n^{n+1} x_1 x_2 .... x_n (1 - x_1 - x_2 - ... - x_n) \le (x_1 + x_2 + ... + x_n)(1 - x_1)(1 - x_2)...(1 - x_n)$$

tengsizlikni isbotlang.

59. $0 \le a,b,c \le 1$ sonlar uchun $\frac{a}{bc+1}+\frac{b}{ac+1}+\frac{c}{ab+1} \le 2$ tengsizlikni isbotlang.

60. a, b, c, d $\in [1;2]$ sonlar uchun $\frac{a+b}{b+c}+\frac{c+d}{d+a} \le \frac{4(a+c)}{b+d}$ tengsizlikni isbotlang.

61. (Vyetnam -2002) *a, b, c* uchburchak tomonlari va $0 \le t \le 1$ uchun

$\sqrt{\frac{a}{b+c-ta}}+\sqrt{\frac{b}{a+c-tb}}+\sqrt{\frac{c}{a+b-tc}} \ge 2\sqrt{1+t}$ tengsizlikni isbotlang.

62. Haqiqiy *a, b, c* sonlar $a+b+c=0$ shartni qanoatlantirsa,

$a^2b^2+b^2c^2+c^2a^2+3 \ge 6abc$ tengsizlikni isbotlang.

63. (O'zbekiston -2001) $x_1, x_2, ..., x_{2002}$ musbat sonlar $\sum_{i=1}^{2002}\frac{1}{1+x_i^2}=1$ shartni

qanoatlantirsa, $x_1x_2...x_{2002} \ge 2001^{1001}$ tengsizlikni isbotlang.

64. (Belorussiya -1999) Agar musbat *a, b, c* sonlar $a^2+b^2+c^2=3$ tenglikni

qanoatlantirsa, $\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac} \ge \frac{3}{2}$ tengsizlik o'rinli bo'lishini isbotlang.

65. Agar $a \ge b \ge c > 0$ va $x, y, z \in R$ bo'lsa,

$$(ax+by+cz)(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}) \le \frac{(a+c)^2}{4ac}(x+y+z)^2$$

tengsizlikni isbotlang.

66. (Singapur -2004) Agar *a, b, c* > 0 bo'lsa, u holda

$$\frac{ab}{a+b+2c}+\frac{bc}{b+c+2a}+\frac{ac}{a+c+2b} \le \frac{1}{4}(a+b+c)$$

tengsizlikni isbotlang.

67. (Polsha -2006) Agar musbat *a, b, c* sonlar *ab*+*bc*+*ca*=*abc* tenglikni qanoatlantirsa, u holda $\frac{a^4+b^4}{ab(a^3+b^3)}+\frac{b^4+c^4}{bc(b^3+c^3)}+\frac{c^4+a^4}{ac(a^3+c^3)}\geq 1$ tengsizlikni isbotlang.

68. (Sankt-Peterburg -2006) Agar *a, b, c, d* musbat sonlar $a^2+b^2+c^2+d^2=1$ tenglikni qanoatlantirsa, u holda $a+b+c+d+\frac{1}{abcd}\geq 18$ tengsizlikni isbotlang.

69. (APMO , Osiyo va Tinch Okean qirg'og'i davlatlari olimpiadasi -2004) Agar *x, y, z* musbat sonlar bo'lsa,

$$(x^2+2)(y^2+2)(z^2+2)\geq 9(xy+yz+zx)$$

tengsizlik o'rinli bo'lishini ko'rsating.

70. (Xitoy -2004) Agar natural $a_1\leq a_2\leq\ldots\leq a_n$ sonlar ketma-ketligi $\sum_{i=1}^{n}\frac{1}{a_i}<1$ shartni qanoatlantirsa $\left(\sum_{i=1}^{n}\frac{1}{a_i^2+x^2}\right)^2\leq\frac{1}{2a_1^2-2a_1+2x^2}$ tengsizlikni isbotlang.

($x\in R$).

71. (Rossiya -2002) Musbat x,y,z sonlar $x+y+z=3$ shartni qanoatlantirsa, $\sqrt{x}+\sqrt{y}+\sqrt{z}\geq xy+yz+zx$ tengsizlikni isbotlang.

72. (Belorussiya -2002) Natural a,b sonlar uchun $\left|a\sqrt{2}-b\right|>\frac{1}{2(a+b)}$ tengsizikni isbotlang.

73. (Belorussiya -2002) Musbat haqiqiy a,b,c,d sonlar uchun

$$\sqrt{(a+c)^2+(b+d)^2}\le\sqrt{a^2+b^2}+\sqrt{c^2+d^2}\le\sqrt{(a+c)^2+(b+d)^2}+\frac{2|ad-bc|}{\sqrt{(a+c)^2+(b+d)^2}}$$

tengsizlikni isbotlang.

74. (APMO -2002) Musbat a,b,c sonlar $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ shartlarni qanoatlantirsa, $\sqrt{a+bc}+\sqrt{b+ac}+\sqrt{c+ab}\ge\sqrt{abc}+\sqrt{a}+\sqrt{b}+\sqrt{c}$ ttengsizlikni isbotlang.

75. (APMO -1996) a,b,c uchburchak tomonlari bo'lsa,

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b}\le\sqrt{a}+\sqrt{b}+\sqrt{c}$$

tengsizlikni isbotlang.

76. Musbat a,b,c sonlar $a+b+c=3$ shartni qanoatlantirsa, $2(a^3b+b^3c+c^3a)\ge 3(a^2b+b^2c+c^2a-1)$ tengsizlikni isbotlang.

77. (APMO -1998) a,b,c musbat sonlar uchun

$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)\ge 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$ tengsizlikni isbotlang.

78. (Singapur -2001) $n\in N$ va $a_1,a_2,...,a_n$ sonlar $\sum_{i=1}^{n}a_i=1$ shartni qanoatlantirsa,

$\frac{a_1^4}{a_1^2+a_2^2}+\frac{a_2^4}{a_2^2+a_3^2}+...+\frac{a_n^4}{a_n^2+a_1^2}\ge\frac{1}{2n}$ tengsizlikni isbotlang.

79. (XMO -2000) Musbat a,b,c sonlar $abc=1$ shartni qanoatlantirsa,

$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right)\le 1$ tengsizlikni isbotlang.

80. (Qozog'iston -2000) Yig'indisi birga teng bo'lgan *a, b, c* sonlar uchun

$\frac{a^7+b^7}{a^5+b^5}+\frac{b^7+c^7}{b^5+c^5}+\frac{c^7+a^7}{c^5+a^5}\geq\frac{1}{3}$ tengsizlikni isbotlang.

81. (Yaponiya -2002) Musbat x, y sonlar uchun $x+y+\frac{2}{x+y}+\frac{1}{2xy}\geq\frac{7}{2}$

tengsizlikni isbotlang.

82. (Pol'sha -1996) Musbat a,b,c sonlarning yig'indisi birga teng bo'lsa,

$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\leq\frac{9}{10}$ tengsizlik o'rinli bo'lishini isbotlang.

83. (Gonkong -2005) Musbat a,b,c,d sonlarning yig'indisi birga teng bo'lsa,

$6\left(a^3+b^3+c^3+d^3\right)\geq\left(a^2+b^2+c^2+d^2\right)+\frac{1}{8}$ tengsizlikni isbotlang.

84. (Kanada -1998) Istalgan natural n soni $(n\geq 2)$ uchun

$\frac{1}{n+1}\left(1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n-1}\right)>\frac{1}{n}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+...+\frac{1}{2n}\right)$ tengsizlikni isbotlang.

85. (Bosniya -2002) Agar $a>0$ va $0<b<1$ bo'lsa, $\sqrt{1+a^2}+\sqrt{1-b^2}\leq\frac{a}{b}+\frac{b}{a}$

tengsizlikni isbotlang.

86. (Polsha -1995) Agar $x_1=\frac{1}{2}$ va $x_n=\frac{2n-3}{2n}x_{n-1},\ n>1$ bo'lsa, $\sum_{i=1}^{n}x_i<1$ tengsizlik

o'rinli bo'lishini isbotlang.

87. (Bosniya -2002) Agar $x_i \in \left(0; \dfrac{\pi}{2}\right)$ $(i = 1, 2, ..., n)$ sonlari $\sum\limits_{i=1}^{n} tgx_i \le n$ shartni qanoatlantirsa, $\sin x_1 \cdot \sin x_2 \cdot ... \cdot \sin x_n \le 2^{-\frac{n}{2}}$ tengsizlikni isbotlang.

88. (Belorussiya -2000) Musbat $a, b, c; x, y, z$ sonlar uchun

$\dfrac{a^6}{x} + \dfrac{b^6}{y} + \dfrac{c^6}{z} \ge \dfrac{\left(a^2 + b^2 + c^2\right)}{3\left(x + y + z\right)}$ tengsizlikni isbotlang.

89. (AQSh -1997) Istalgan musbat a, b, c sonlar uchun

$\dfrac{1}{a^3 + b^3 + abc} + \dfrac{1}{b^3 + c^3 + abc} + \dfrac{1}{c^3 + a^3 + abc} \le \dfrac{1}{abc}$ tengsizlikni isbotlang.

90. (Pol'sha -2000) Aytaylik $x_i > 0$ $(i = 1, 2, ..., n)$ va $n > 2$ bo'lsin.

$x_1 + 2x_2 + 3x_3 + ... + nx_n \le \dfrac{n(n-1)}{2} + x_1 + x_2^2 + x_3^3 + ... + x_n^n$ tengsizlikni isbotlang.

91. (Gretsiya -2002) Agar a, b, c musbat sonlar $a^2 + b^2 + c^2 = 1$ shartni qanoatlantirsa, $\dfrac{a}{b^2 + 1} + \dfrac{b}{c^2 + 1} + \dfrac{c}{a^2 + 1} \ge \dfrac{3}{4}\left(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}\right)^2$ tengsizlikni isbotlang.

92. (Ukraina -2002) $a_i \ge 1$ $(i = 1, 2, ..., n),$ $n \ge 1,$ $A = 1 + a_1 + a_2 + ... + a_n$

$x_k = \dfrac{1}{1 + a_k x_{k-1}}$, $1 \le k \le n$ deb belgilash kiritsak, u holda $x_1 + x_2 + ... + x_n > \dfrac{n^2 A}{n^2 + A^2}$ tengsizlikni isbotlang.

93. (Sankt Peterburg -2004) Musbat a, b, c sonlar uchun

$$\frac{ab}{3a + b} + \frac{bc}{b + 2c} + \frac{ac}{c + 2a} \le \frac{2a + 20b + 27c}{49}$$

tengsizlikni isbotlang.

94. (Irlandiya -1998) $x \neq 0$ son uchun

$$x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} \geq 0$$

tengsizlikni isbotlang.

95. (Eron -1998) Birdan katta x, y, z sonlar $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$ shartni qanoatlantirsa,

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$$

tengsizlikni isbotlang.

96. (Vyetnam-1998) $x_1, x_2, ..., x_n$ $(n \geq 2)$ musbat sonlar

$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + ... + \frac{1}{x_n + 1998} = \frac{1}{1998}$ tenglikni qanoatlantirsa,

$\frac{\sqrt[n]{x_1 x_2 ... x_n}}{n-1} \geq 1998$ tengsizlikni isbotlang.

Yechimlar.

1. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligidan munosabatga ko'ra,

$$\underbrace{\frac{a^{n+k}}{b^k}+....+\frac{a^{n+k}}{b^k}}_{n}+\underbrace{b^n+b^n+....+b^n}_{k}\geq (n+k)\sqrt[n+k]{\frac{\left(a^{n+k}\right)^n b^{nk}}{b^{nk}}}=(n+k)a^n$$

yoki

$$n\cdot\frac{a^{n+k}}{b^k}+k\cdot b^h\geq (n+k)a^n\,.$$

Xuddi shunday,

$$n\cdot\frac{b^{n+k}}{c^k}+kc^n\geq (n+k)b^n,$$

$$n\cdot\frac{c^{n+k}}{a^k}+k\cdot a^n\geq \left(n+k\right)c^n$$

tengsizliklarni hosil qilamiz. Bu tengsizliklarni hadma-had qo'shib,

$$\frac{a^{n+k}}{b^k}+\frac{b^{n+k}}{c^k}+\frac{c^{n+k}}{a^k}\geq a^n+b^n+c^n$$

ni hosil qilamiz.

2. Ushbu $\frac{1}{a}+\frac{1}{b}\geq\frac{4}{a+b}$ tengsizlikdan foydalanamiz:

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=\frac{1}{2}\left(\frac{1}{b+c}+\frac{1}{a+b}\right)+\frac{1}{2}\left(\frac{1}{b+c}+\frac{1}{a+c}\right)+\frac{1}{2}\left(\frac{1}{a+b}+\frac{1}{a+c}\right)\geq$$

$$\frac{1}{2}\left(\frac{4}{2b+a+c}+\frac{4}{b+a+2c}+\frac{4}{b+2a+c}\right)=\frac{2}{1+a}+\frac{2}{1+b}+\frac{2}{1+c}.$$

3. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligidan foydalanib,

$$\frac{2}{4-ab}=\frac{4}{8-2ab}\leq\frac{4}{8-a^2-b^2}\leq\frac{1}{4-a^2}+\frac{1}{4-b^2}$$

tengsizlikni hosil qilamiz. Xuddi shunday,

$$\frac{2}{4-bc}\le\frac{1}{4-b^2}+\frac{1}{4-c^2},\ \frac{2}{4-ac}\le\frac{1}{4-a^2}+\frac{1}{4-c^2}\ .$$

Bu tengsizliklarni hadma-had qo'shib,

$$\frac{1}{4-ab}+\frac{1}{4-bc}+\frac{1}{4-ac}\le\frac{1}{4-a^2}+\frac{1}{4-b^2}+\frac{1}{4-c^2}$$

tengsizlikni hosil qilamiz. Berilgan $a^4+b^4+c^4=3$ shartdan $a^2<2$ ekanligi kelib chiqadi.Bundan quyidagi

$$\left(a^2-1\right)^2(2-a^2)\ge 0\Leftrightarrow\frac{1}{4-a^2}\le\frac{a^4+5}{18}$$

tengsizlik o'rinli. Xuddi shunday,

$$\frac{1}{4-b^2}\le\frac{b^4+5}{18},\ \frac{1}{4-c^2}\le\frac{c^4+5}{18}$$

tengsizliklar o'rinli. Bu tengsizliklarni hadma-had qo'shib,

$$\frac{1}{4-ab}+\frac{1}{4-bc}+\frac{1}{4-ca}\le\frac{1}{4-a^2}+\frac{1}{4-b^2}+\frac{1}{4-c^2}\le\frac{a^4+5}{18}+\frac{b^4+5}{18}+\frac{c^4+5}{18}=1$$

ekanligini hosil qilamiz.

4. Berilgan tengsizlikni chap tomonida turgan qo'shiluvchilarni mos ravishda *A,B,C* deb belgilaymiz.

$$\left(by+cz\right)(bz+cy)=(b^2+c^2)yz+bc(y^2+z^2)\le\left(\frac{y^2+z^2}{2}\right)(b^2+c^2)+bc(y^2+z^2)=$$

$$=\frac{1}{2}\left(y^2+z^2\right)\left(b+c\right)^2\ \Rightarrow\ A\ge 2\left(\frac{a}{b+c}\right)^2\frac{x^2}{y^2+z^2}$$

Xuddi shunday , $B\ge 2\left(\frac{b}{a+c}\right)^2\frac{y^2}{t^2+x^2},\ C\ge 2\left(\frac{c}{a+b}\right)^2\frac{z^2}{x^2+y^2}$ tengsizliklarni hosil qilamiz.Berilgan shartlarga ko'ra

$$\frac{a}{b+c}\ge\frac{b}{c+a}\ge\frac{c}{a+b},\ \frac{x^2}{y^2+z^2}\ge\frac{y^2}{z^2+x^2}\ge\frac{z^2}{x^2+y^2}$$

munosabatlar o'rinli. Chebishev tengsizligini qo'llasak,

$$A+B+C \geq 2\cdot\frac{1}{3}\left\{\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{a+c}\right)^2+\left(\frac{c}{a+b}\right)^2\right\}\left\{\frac{x^2}{y^2+z^2}+\frac{y^2}{x^2+z^2}+\frac{z^2}{x^2+y^2}\right\} \geq$$

$$\geq 2\cdot\frac{1}{3}\cdot\frac{1}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2\left(\frac{x^2}{y^2+z^2}+\frac{y^2}{z^2+x^2}+\frac{z^2}{x^2+y^2}\right).$$

Musbat α,β,γ sonlar uchun $\frac{\alpha}{\beta+\gamma}+\frac{\beta}{\gamma+\alpha}+\frac{\gamma}{\alpha+\beta}\geq\frac{3}{2}$ tengsizlikni isbotlaymiz.

$\alpha+\beta=\tau, \quad \beta+\gamma=s,\ \gamma+\alpha=t$ belgilash kiritib,

$$\frac{\alpha}{\beta+\gamma}+\frac{\beta}{\gamma+\alpha}+\frac{\gamma}{\alpha+\beta}=\frac{\tau+t-s}{2s}+\frac{\tau+s-t}{2t}+\frac{s+t-\tau}{2\tau}=$$

$$=\frac{1}{2}\left(\frac{\tau}{s}+\frac{t}{s}+\frac{\tau}{t}+\frac{s}{t}+\frac{s}{\tau}+\frac{t}{\tau}-3\right)\geq\frac{1}{2}(2+2+2-3)=\frac{3}{2}$$

ni hosil qilamiz. Bundan $A+B+C\geq 2\cdot\frac{1}{3}\cdot\frac{1}{3}\cdot\left(\frac{3}{2}\right)^2\cdot\frac{3}{2}=\frac{3}{4}$

Tenglik $a=b=c$ va $x=y=z$ bo'lganda bajariladi.

5. $t=a_1^2+a_2^2+...+a_n^2$ deb belgilab,

$$\begin{aligned}
&a_1a_2+a_2a_3+...+a_na_{n-1}+a_na_1\leq\\
&\leq a_1a_2+a_1a_3+a_1a_4+...+a_1a_n+\\
&+a_2a_3+a_2a_4+...+a_2a_n+\\
&+a_3a_4+...+a_3a_n+\\
&+.....+\\
&+a_{n-1}a_n=\\
&=\frac{1}{2}\left\{(a_1+a_2+...a_n)^2-(a_1^2+a_2^2+...a_n^2)\right\}=\frac{1}{2}(1-t)
\end{aligned}$$

munosabatlarni hosil qilamiz. Bu yerdan $t<1$ kelib chiqadi.

Koshi-Bunyakovskiy-Shvarts tengsizligini qo'llab,

$$(a_1b_1+a_2b_2+...+a_nb_n)^2\leq(a_1^2+a_2^2+...+a_n^2)(b_1^2+b_2^2+...+b_2^2)=t$$

yoki $a_1b_1+a_2b_2+...+a_nb_n\leq\sqrt{t}$ tengsizlikni topamiz. Bundan

$$a_1(b_1+a_2)+a_2(b_2+a_3)+...+a_n(b_n+a_1)=(a_1b_1+a_2b_2+...+a_nb_n)+$$
$$+(a_1a_2+a_2a_3+....+a_na_1)\le\sqrt{t}+\frac{1}{2}(1-t)=-\frac{1}{2}(\sqrt{t}-1)^2+1<1$$

6. Berilgan tengsizlikni chap tomonini T bilan belgilab, o'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligidan foydalanamiz, ya'ni:

$$T=\sqrt{\frac{ab}{ab+1-a-b}}+\sqrt{\frac{bc}{bc+1-b-c}}+\sqrt{\frac{ac}{ac+1-a-c}}=$$
$$=\sqrt{\frac{ab}{(1-a)(1-b)}}+\sqrt{\frac{bc}{(1-b)(1-c)}}+\sqrt{\frac{ac}{(1-a)(1-c)}}\le$$
$$\le\frac{1}{2}\left(\frac{a}{1-b}+\frac{b}{1-a}\right)+\frac{1}{2}\left(\frac{b}{1-c}+\frac{c}{1-b}\right)+\frac{1}{2}\left(\frac{a}{1-c}+\frac{c}{1-a}\right)\le$$
$$\le\frac{1}{2}\left(\frac{a}{a+c}+\frac{b}{b+c}+\frac{b}{b+a}+\frac{c}{c+a}+\frac{a}{b+a}+\frac{c}{b+c}\right)=\frac{3}{2}$$

7. Tengsizlikni ikkala qismidagi qavslarni ochib ixchamlasak

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\ge\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3 \ (*)$$

ifoda hosil bo'ladi.Endi $\frac{a}{b}=x, \frac{b}{c}=y, \frac{c}{a}=z$ deb belgilash kiritsak, u holda (*) tengsizlik

$$x^2+y^2+z^2+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ge x+y+z+3 \ (**)$$

ko'rinishga keladi. $xyz=1$ ekanligidan quyidagi

$$x^2+y^2+z^2\ge\frac{(x+y+z)^2}{3}\ge x+y+z \ (1)$$

va $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ge 3$ (2) tengsizliklar o'rinli. (1) va (2) larni hadma-had qo'shib (**) ni hosil qilamiz. Bundan (*) isbotlandi.

8. $a+b=1$ ekanligidan foydalanib yuqoridagi tengsizlikni quyidagi shakldda yozamiz:

$$\frac{1}{3}\le\frac{a^2}{(a+b)(a+(a+b))}+\frac{b^2}{(a+b)(b+(a+b))}$$

Bundan $a^2b+ab^2\le a^3+b^3$ yoki $\left(a^3+b^3\right)-\left(a^2b+ab^2\right)=(a-b)^2(a+b)\ge 0$

9. $yz(y^2+z^2)=y^3z+yz^3\le y^4+z^4$ tengsizlik o'rinli, chunki

$$y^4-y^3z-yz^3+z^4=\left(y^3-z^3\right)(y-z)\ge 0\ \Rightarrow x(y^4+z^4)\ge xyz\left(y^2+z^2\right)\ge y^2+z^2$$

yoki

$$\frac{x^5}{x^5+y^2+z^2}\ge\frac{x^5}{x^5+x(y^4+z^4)}=\frac{x^4}{x^4+y^4+z^4}\ .$$

Xuddi shunday,

$$\frac{y^5}{y^5+x^2+z^2}\ge\frac{y^4}{x^4+y^4+z^4},\ \frac{z^5}{z^5+x^2+y^2}\ge\frac{z^4}{x^4+y^4+z^4}\ .$$

Bu tengsizliklarni hadma-had ko'shib, isboti talab qilingan tengsizlikni hosil qilamiz.

10. Yuqoridagi tengsizlikni quyidagicha yozib olamiz:

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{y^2+x^2+z^2}{x^5+z^2+x^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2}\le 3$$

va Koshi-Bunyakovskiy-Shvarts tengsizligini qo'llab,

$$(x^5+y^2+z^2)(yz+y^2+z^2)\ge(x^{\frac{5}{2}}(yz)^{\frac{1}{2}}+y^2+z^2)^{\ 2}\ge(x^2+y^2+z^2)^2$$

yoki $\dfrac{x^2+y^2+z^2}{x^5+y^2+z^2}\le\dfrac{yz+y^2+z^2}{x^2+y^2+z^2}$. Xuddi shunday,

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}\le\frac{xz+x^2+z^2}{x^2+y^2+z^2},\quad \frac{x^2+y^2+z^2}{z^5+x^2+y^2}\le\frac{xy+x^2+y^2}{x^2+y^2+z^2}$$

munosabatlarni hosil qilamiz. Bu tengsizliklarni hadma-had qo'shsak,

$$\frac{x^2+y^2+z^2}{x^5+y^2+z^2}+\frac{x^2+y^2+z^2}{y^5+x^2+z^2}+\frac{x^2+y^2+z^2}{z^5+x^2+y^2}\le 2+\frac{xy+yz+zx}{x^2+y^2+z^2}\le 3$$

11. α,β,γ uchburchak burchaklari uchun

$1=\cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$ tenglikdan foydalanib,

$$\frac{\cos\alpha}{\cos\beta\cos\gamma}\cdot\frac{\cos\beta}{\cos\gamma\cos\alpha}\cdot\frac{\cos\gamma}{\cos\alpha\cos\beta}=\frac{\cos\alpha}{\cos\beta\cos\gamma}+\frac{\cos\beta}{\cos\gamma\cos\alpha}+\frac{\cos\gamma}{\cos\alpha\cos\beta}+2$$

ifodani hosil qilamiz .

$$x=\frac{\cos\alpha}{\cos\beta\cos\gamma},\ y=\frac{\cos\beta}{\cos\gamma\cos\alpha},\ z=\frac{\cos\gamma}{\cos\alpha\cos\beta}$$

deb belgilash kiritib, $\cos\alpha+\cos\beta+\cos\gamma\le\frac{3}{2}$ tengsizlikdan foydalansak,

$$\frac{1}{\sqrt{xy}}+\frac{1}{\sqrt{yz}+\sqrt{zx}}\frac{1}{}\le\frac{3}{2}\Leftrightarrow\ 2(\sqrt{x}+\sqrt{y}+\sqrt{z})\le 3\sqrt{xyz}\Leftrightarrow$$

$$4(x+y+z+2(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}))\le 9xyz\Leftrightarrow$$

$$8(\sqrt{xy}+\sqrt{yz}+\sqrt{zx})\le 9(x+y+z+2)-4(x+y+z)=5(x+y+z)+18$$

12. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligidan foydalanib,

$$\sqrt{1+a^3}=\sqrt{(1+a)(1+a^2-a)}\le\frac{2+a^2}{2},\ \sqrt{1+b^3}=\sqrt{(1+b)(1-b+b^2)}\le\frac{2+b^2}{2},$$

$$\sqrt{1+c^3}=\sqrt{(1+c)(1-c+c^3)}\le\frac{2+c^2}{2}$$

munosabatlarni topamiz. Endi quyidagi tengsizlikni isbotlasak yetarli:

$$\frac{4a^2}{(2+a^2)(2+b^2)}+\frac{4b^2}{(2+b^2)(2+c^2)}+\frac{4c^2}{(2+c^2)(2+a^2)}\ge\frac{4}{3}\Leftrightarrow$$

$$3\left(a^2(2+c^2)+b^2(2+a^2)+c^2(2+b^2)\right)\ge(2+a^2)(2+b^2)(2+c^2)\Leftrightarrow$$

$$a^2b^2+b^2c^2+c^2a^2+2(a^2+b^2+c^2)\ge a^2b^2c^2+8=72$$

Bu tengsizlik quyidagi tengsizliklarni hadma-had qo'shishdan hosil qilinadi:

$$a^2b^2+b^2c^2+c^2a^2\ge 3\sqrt[3]{(abc)^4}=48,\quad 2(a^2+b^2+c^2)\ge 6\sqrt[3]{a^2b^2c^2}=24$$

Bulardan isboti talab qilingan tengsizlikni hosil qilamiz.

13. Birinchi navbatda $x+y^2\geq x^2+y^3$ tengsizlikni isbotlaymiz. Faraz qilaylik, $x+y^2<x^2+y^3$ bo'lsin. $x^3+y^4\leq x^2+y^3$ tengsizlikdan foydalansak, farazimizga zid bo'lgan $2\left(x^2+y^3\right)\geq\left(x+x^3\right)+\left(y^2+y^4\right)\geq 2x^2+2y^3$ tengsizlik hosil bo'ladi. Shuning uchun

$$x+y^2\geq x^2+y^3\geq x^3+y^4\Rightarrow 2\left(x+y^2\right)\geq x^2+y^3+x^3+y^4\geq$$
$$\geq x^3+y^3+2x-1+2y^2-1\Rightarrow x^3+y^3\leq 2.$$

14. Umumiylikni chegaralamasdan $a\geq b\geq c$ deb olib, uchburchak tengsiz-ligini qo'llasak, $1=a+b+c>2a\Rightarrow b\leq a<\frac{1}{2}$ va bundan

$$a^n+b^n<\frac{1}{2^n}+\frac{1}{2^n}=\frac{2}{2^n}\Rightarrow\left(a^n+b^n\right)^{\frac{1}{n}}<\frac{2^{\frac{1}{n}}}{2}\quad(*)\ .$$

Endi qo'yidagini qaraymiz:

$$\left(b+\frac{c}{2}\right)^n=b^n+\frac{n}{2}c\,b^{n-1}+....+\frac{c^n}{2^n}>b^n+c^n\ \text{(chunki } \frac{n}{2}cb^{n-1}>c^n\text{)}.$$

Xuddi shunday,

$$\left(a+\frac{c}{2}\right)^n>a^n+c^n\ .$$

Demak,

$$\left(b^n+c^n\right)^{\frac{1}{n}}+\left(a^n+b^n\right)^{\frac{1}{n}}<b+\frac{c}{2}+a+\frac{c}{2}=1.\qquad(**)$$

(*) va (**) larni hadma-had qo'shib, isboti talab etilgan tengsizlikni hosil qilamiz.

15. $\frac{A_{k-1}}{A_k}=x_k$ va $x_1=1$ deb belgilash kiritsak,

$$n\sqrt[n]{\frac{G_n}{A_n}}=n\cdot\sqrt[n^2]{\frac{A_1A_2...A_n}{A_n^n}}=n\cdot\sqrt[n^2]{\frac{A_1}{A_2}\left(\frac{A_2}{A_3}\right)^2\left(\frac{A_3}{A_4}\right)^3...\left(\frac{A_{n-1}}{A_n}\right)^{n-1}}=$$

$$= n\sqrt[n^2]{x_2 x_3^2 x_4^3 \ldots x_n^{n-1}} = n\sqrt[n^2]{x_1^{\frac{n(n+1)}{2}} \cdot x_2\, x_3^2\, x_4^3 \ldots . x_n^{n-1}};$$

$$\frac{a_k}{A_k} = \frac{kA_k - (k-1)A_{k-1}}{A_k} = k - (k-1)\frac{A_{k-1}}{A_k} = k - (k-1)x_k;$$

$$\frac{g_n}{G_n} = \sqrt[n]{\frac{a_1}{A_1} \cdot \frac{a_2}{A_2} \ldots \frac{a_n}{A_n}} = \sqrt[n]{1.(2-x_2)(3-2x_3)\ldots(n-(n-1)x_k)}\,.$$

Umumlashgan Koshi tengsizligidan foydalansak

$$(\, a_i > 0,\ \alpha_i > 0,\ i = 1,2,\ldots,n \qquad \sqrt[\alpha_1+\alpha_2+\alpha_n]{a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n}} \le \frac{a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n}{\alpha_1 + \alpha_2 + \ldots + \alpha_n}),$$

$$n\sqrt[n]{\frac{G_n}{A_n}} + \frac{g_n}{G_n} = n\sqrt[n^2]{x_1^{\frac{n(n+1)}{2}} x_2 x_3^2 \ldots x_n^{n-1}} + \sqrt[n]{1 \cdot (2-x_2)(3-2x_3)\ldots(n-(n-1)x_n)} \le$$

$$\le \frac{1}{n}\left(\frac{n(n+1)}{2} + x_2 + 2x_3 + \ldots + (n-1)x_n\right) + \frac{1}{n}(1 + (2-x_2) + (3-2x_3) + \ldots + (n-1)x_n)$$

$$\frac{n+1}{2} + \frac{1}{n}(x_2 + 2x_3 + \ldots + (n-1)x_n) + \frac{(n+1)}{2} - \frac{1}{n}(x_2 + 2x_3 + \ldots + (n-1)x_n) = n+1.$$

16. Bu tengsizlikning chap tomonini S deb belgilab, quyidagi usulda o'rta arifmetik va o'rta geometrik miqdorlar o'rtasidagi munosabatni qo'llaymiz:

$$S \le a\left(\frac{1+1+(1+b-c)}{3}\right) + b\left(\frac{1+1+(1+c-a)}{3}\right) + c\left(\frac{1+1+(1+a-b)}{3}\right) =$$

$$\frac{3a+3b+3c+ab+ac+bc-ba-ca-cb}{3} = 1$$

17.

$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+cd}{1+c} + \frac{1+da}{1+d} = \left(\frac{1+ab}{1+a} + \frac{1+cd}{1+c}\right) + \left(\frac{1+bc}{1+b} + \frac{1+da}{1+d}\right) =$$

$$= \left(\frac{1+ab}{1+a} + \frac{1+ab}{ab(1+c)}\right) + \left(\frac{1+bc}{1+b} + \frac{1+bc}{bc(1+d)}\right) =$$

$$= (1+ab)\left(\frac{1}{1+a} + \frac{1}{ab(1+c)}\right) + (1+bc)\left(\frac{1}{1+b} + \frac{1}{bc(1+d)}\right) \ge$$

$$\ge \frac{4(1+ab)}{1+a+ab(1+c)} + \frac{4(1+bc)}{1+b+bc(1+d)} = 4$$

18. Bu tengsizlikni chap tomonini T bilan belgilab, umumlashgan Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagicha qo'llaymiz:

$$T\cdot\left(a\left(b+1\right)+b(c+1)+c(a+1)\right)\geq(a+b+c)^3=1$$

yoki $T\geq\dfrac{1}{ab+bc+ca+1}$ tengsizlikni va undan

$$T\geq\frac{1}{ab+bc+ca+1}\geq\frac{1}{\frac{(a+b+c)^2}{3}+1}=\frac{3}{4}$$

munosabatni hosil qilamiz.

-19.

$$\frac{a^2+b}{b+c}+\frac{b^2+c}{c+a}+\frac{c^2+a}{a+b}=\frac{a(1-b-c)+b}{b+c}+\frac{b(1-a-c)+c}{c+a}+\frac{c(1-a-b)+a}{a+b}=$$

$$=\frac{a+b}{b+c}-a+\frac{b+c}{c+a}-b+\frac{c+a}{c+b}-c=\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}-1\geq$$

$$\geq3\sqrt[3]{\frac{a+b}{b+c}\cdot\frac{b+c}{c+a}\cdot\frac{c+a}{a+b}}-1=2$$

20. Ixtiyoriy $x,y,z>0$ uchun $\left(x-\sqrt{yz}\right)^2\geq0\Leftrightarrow x^2+yz\geq2\sqrt{x^2yz}\Leftrightarrow$

$$x^2+xy+xz+yz\geq xy+2\sqrt{x^2yz}+xz\Leftrightarrow(x+y)(x+z)\geq\left(\sqrt{xy}+\sqrt{xz}\right)^2\Leftrightarrow$$

$$\sqrt{(x+y)(x+z)}\geq\sqrt{xy}+\sqrt{xz}$$

munosabatni topamiz. Bundan

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(x+y)(z+y)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\leq$$

$$\leq\frac{x}{x+\sqrt{xy}+\sqrt{xz}}+\frac{y}{y+\sqrt{yx}+\sqrt{yz}}+\frac{z}{z+\sqrt{zx}+\sqrt{zy}}=$$

$$=\frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}+\frac{\sqrt{y}}{\sqrt{y}+\sqrt{x}+\sqrt{z}}+\frac{\sqrt{z}}{\sqrt{z}+\sqrt{x}+\sqrt{y}}=1$$

21. Koshi-Bunyakovskiy-Shvarts tengsizligini qo'llab,

$$\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}}\le\sqrt{\left(\frac{2a(a+b+c)}{(a+b)(a+c)}+\frac{2b(a+b+c)}{(b+c)(b+a)}+\frac{2c(a+b+c)}{(c+a)(c+b)}\right)}\times$$

$$\times\sqrt{\left(\frac{a+c}{2(a+b+c)}+\frac{b+a}{2(a+b+c)}+\frac{c+b}{2(a+b+c)}\right)}=$$

$$=\sqrt{2\cdot(a+b+c)\left(\frac{a}{(a+b)(a+c)}+\frac{b}{(b+c)(b+a)}+\frac{c}{(c+a)(c+b)}\right)}$$

munosabatni hosil qilamiz. Endi

$$(a+b+c)\left(\frac{a}{(a+b)(a+c)}+\frac{b}{(b+c)(b+a)}+\frac{c}{(c+a)(c+b)}\right)=$$

$$=\frac{2(a+b+c)(ab+ac+bc)}{(a+b)(b+c)(c+a)}\le\frac{9}{4}$$

yoki

$$8\big(a+b+c\big)(ab+bc+ca)\le 9(a+b)(b+c)(c+a)\Leftrightarrow$$

$$6abc\le ab(a+b)+bc(b+c)+ac(a+c)$$

tengsizlikni isbotlash yetarli. Bu tengsizlik esa o'rta arifmetik va o'rta geometrik miqdorlar o'rtasidagi munosabatga ko'ra o'rinli. Bulardan yuqoridagi isboti talab etilgan tengsizlik isbotlandi.

22. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligidan quyidagicha foydalanamiz:

$$(a+3b)(b+4c)(c+2a)=(a+b+b+b)(b+c+c+c+c)(c+a+a)\ge$$

$$\ge 4\sqrt[4]{ab^3}\cdot 5\sqrt[5]{b\cdot c^4}\cdot 3\sqrt[3]{a^2c}=60\ a^{\frac{11}{12}}\cdot b^{\frac{19}{20}}\cdot c^{\frac{17}{15}}=$$

$$=60abc\frac{c^{\frac{2}{15}}}{a^{\frac{1}{12}}\cdot b^{\frac{1}{20}}}=60\ abc\frac{c^{\frac{1}{12}}\cdot c^{\frac{1}{20}}}{a^{\frac{1}{12}}b^{\frac{1}{20}}}=60abc\left(\frac{c}{a}\right)^{\frac{1}{12}}\cdot\left(\frac{c}{b}\right)^{\frac{1}{20}}\ge 60abc$$

23. $a=\frac{x}{y}, b=\frac{y}{t}, c=\frac{t}{x}$ deb belgilash kiritsak, u holda

$$4\left(\sqrt[3]{\frac{xt}{y^2}}+\sqrt[3]{\frac{yx}{t^2}}+\sqrt[3]{\frac{yt}{x^2}}\right)\le 3\left(2+\frac{x}{y}+\frac{y}{t}+\frac{t}{x}+\frac{y}{x}+\frac{t}{y}+\frac{x}{t}\right)^{\frac{2}{3}}$$

bundan

$$4\sqrt[3]{xyt}\left(\frac{1}{y}+\frac{1}{x}+\frac{1}{t}\right)\le 3\left(2+\frac{x}{y}+\frac{y}{t}+\frac{t}{x}+\frac{y}{t}+\frac{t}{y}+\frac{x}{t}\right)^{\frac{2}{3}},$$

$$\frac{4}{\sqrt[3]{(xyt)^2}}(xy+yt+tx)\le 3\left(2+\frac{x+t}{y}+\frac{t+y}{x}+\frac{t+x}{t}\right)^{\frac{2}{3}}.$$

Bundan $64(xy+yt+tx)^3\le 27((xy+yt+tx)(x+y+t)-xyt)^2$ tengsizlikni isbotlasak yetarli.

$$27((x+y+t)(xy+yt+tx)-xyt)^2\ge$$

$$\ge 27\left((x+y+t)(xy+yt+tx)-\frac{(x+y+t)(xy+yt+tx)}{9}\right)^2=$$

$$=27\left(\frac{8}{9}(x+y+t)(xy+yt+tx)\right)^2=64(xy+yt+tx)^2\frac{\left(x+y+t\right)^2}{3}\ge 64(xy+yt+tx)^3$$

24. O’rta arifmetik va o’rta geometrik miqdorlar haqidagi Koshi tengsizligiga ko’ra

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 3\left(\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}\right)^{\frac{1}{3}},\qquad a+b+c\ge 3\sqrt[3]{abc}$$

tengsizliklar o’rinli. Bulardan $(a+b+c)(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})\ge 9$ ekanligini topamiz. Bu tengsizlik va $a+b+c\le\sqrt{3(a^2+b^2+c^2)}$ tengsizliklarni hadma-had ko’paytirib,

$3\sqrt{3}\le\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\sqrt{a^2+b^2+c^2}\right)$ va bundan

$$\frac{\sqrt{3}+1}{3\sqrt{3}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a^2+b^2+c^2)\ge\sqrt{3(a^2+b^2+c^2)}+\sqrt{a^2+b^2+c^2}\ge$$

$$\ge(a+b+c)+\sqrt{a^2+b^2+c^2}.$$

25. Musbat x son uchun x^2-1 va x^3-1 ifodalar bir xil ishoraladir, ya’ni

$0 \le (x^2-1)(x^3-1) = x^5 - x^3 - x^2 + 1$ yoki $x^5 - x^2 + 3 \ge x^3 + 2$.Bu tengsizlikdan foydalansak, u holda $(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \ge (a^3 + 2)(b^3 + 2)(c^3 + 2)$. Bundan $\left(a^3 + 2\right)(b^3 + 2)(c^3 + 2) \ge (a+b+c)^3$ tengsizlikni isbotlash yetarli. Umumlashgan Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llaymiz:

$$(a^3+2)(b^3+2)(c^3+2) = (a^3+1+1)(1+b^3+1)(1+1+c^3) \ge (a+b+c)^3$$

26. Umumlashgan Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llaymiz:

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) =$$
$$= (a^2+ab+b^2)(c^2+b^2+bc)(ac+a^2+c^2) \ge (ac+ab+bc)^3$$

27. Agar $a,b,c>1$ bo'lsa, $a^2+b^2+c^2+abc>4$ bo'ladi. Agar $a \le 1$ bo'lsa, u holda $ab+bc+ca-abc \ge bc-abc = bc(1-a) \ge 0$.Endi $ab+bc+ca-abc \le 2$ tengsizlikni isbotlaymiz. $a = 2\cos A,\ b = 2\cos B,\ c = 2\cos C$ va $A,\ B,\ C \in \left[0, \frac{\pi}{2}\right]$ deb belgilash kiritsak, shartga ko'ra $A+B+C=\pi$ ekanligini topamiz va

$$\cos A\cos B + \cos B\cos C + \cos A\cos C - 2\cos A\cos B\cos C \le \frac{1}{2}$$

tengsizlikni isbotlasak yetarli bo'ladi. Faraz etaylik, $A \ge \frac{\pi}{3}$ yoki $1 - 2\cos A \ge 0$ .Bundan

$$\cos A\cos B + \cos B\cos C + \cos C\cos A - 2\cos A\cos B\cos C =$$
$$= \cos A(\cos B + \cos C) + \cos B\cos C(1 - 2\cos A)$$

Quyidagi $\cos B + \cos C \le \frac{3}{2} - \cos A$ va

$2\cos B\cos C = \cos(B-C) + \cos(B+C) \le 1 - \cos A$ tengsizliklardan foydalansak,

$$\cos A(\cos B + \cos C) + \cos B\cos C(1-2\cos A) \le \cos A\left(\frac{3}{2} - \cos A\right) + \left(\frac{1-\cos A}{2}\right)(1 - 2\cos A)$$

28. $x \neq 0$ bo'lgani uchun $x^2 + y^2 > 0$ bo'ladi. Bundan $\frac{(2xy)^2}{(x^2+y^2)^2} + \frac{\left(x^2-y^2\right)^2}{\left(x^2+y^2\right)^2} = 1$ ekanligini topamiz. Oxirgi tenglikda har bir qo'shiluvchi [-1;1] oraliqqa tegishli ekanligidan

$$\frac{(2xy)^2}{\left(x^2+y^2\right)^2} = \sin^2\alpha \ \text{ va } \ \frac{\left(x^2-y^2\right)^2}{\left(x^2+y^2\right)^2} = \cos^2\alpha$$

deb belgilash kiritish mumkin. Bundan

$$\cos^2\alpha \sin^2\alpha = \frac{4\left(xy\left(x^2-y^2\right)\right)^2}{\left(x^2+y^2\right)^4} = \frac{4}{\left(x^2+y^2\right)^2},\ (x^2+y^2)^2 = \frac{16}{\sin^2 2\alpha} \geq 16,$$

$$x^2 + y^2 \geq 4.$$

29. $a = \frac{x}{3}, b = \frac{4y}{5}, c = \frac{3z}{2}$ deb belgilash kiritsak, u holda masalaning sharti quyidagicha ko'rinishga ega bo'ladi: $7xy + 3yz + 5xz \leq 15$

O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini yuqoridagi tengsizlikka qo'llab $15 \geq 7xy + 3yz + 5xz \geq 15\sqrt[15]{x^{12}y^{10}z^8}$ yoki $x^6y^5z^4 \leq 1$ (*) tengsizlikni topamiz.

Endi (*) dan foydalansak:

$$P(a,b,c) = \frac{1}{a} + \frac{2}{b} + \frac{3}{c} = \frac{3}{x} + \frac{5}{2y} + \frac{2}{z} = \underbrace{\frac{1}{2x} + \ldots + \frac{1}{2x}}_{6\ ma} + \underbrace{\frac{1}{2y} + \ldots + \frac{1}{2y}}_{5\ ma} +$$

$$+ \underbrace{\frac{1}{2z} + \ldots + \frac{1}{2z}}_{4\ ma} \geq \frac{15}{2}\sqrt[15]{\frac{1}{x^6y^5z^4}} \geq \frac{15}{2}$$

Tenglik $x = y = z = 1$ yoki $a = \frac{1}{3}, b = \frac{4}{5},\ c = \frac{2}{3}$ bo'lganda bajariladi.

30. Koshi-Bunyakovskiy-Shvarts tengsizligini qo'llasak,

$$\frac{a^2}{b}-b+\frac{b^2}{c}-c+\frac{c^2}{a}-a=\frac{\left(a-b\right)^2}{a}+\frac{(b-c)^2}{b}+\frac{\left(a-c\right)^2}{c}=$$

$$=\frac{1}{a+b+c}\left(\frac{\left(a-b\right)^2}{a}+\frac{(b-c)^2}{b}+\frac{(a-c)^2}{c}\right)(a+b+c)\geq$$

$$\frac{1}{a+b+c}\left(|a-b|+|b-d|+|c-d|\right)^2=$$

$$=\frac{1}{a+b+c}\left(2\max\{a,b,c\}-2\min\{a,b,c\}\right)^2\geq\frac{4(a-b)^2}{a+b+c}$$

31. Umumiylikni chegaralamasdan $a_1\leq a_2\leq\ldots\leq a_{i-1}\leq 0\leq a_i\leq a_{i+1}\leq\ldots.\leq a_n$ deb olamiz va $a_k=-b_k\ (k=1,2,\ldots,i-1),\ b_k>0$ deb belgilaymiz, u holda $a_i\leq a_{i+1}\leq\ldots\leq a_n$ va $b_1\geq b_2\geq\ldots\geq b_{i-1}$, $b_1+b_2+..+b_{i-1}=a_i+a_{i+1}+..+a_n$ bo'ladi. $na_1a_n=-nb_1a_n$ ekanligidan $\sum_{i=1}^{n}a_i^2\leq nb_1a_n$ ko'rsatish yetarli.

$$nb_1a_n=\left(\underbrace{b_1a_n+b_1a_n+\ldots+b_1a_n}_{(i-1)\,-ma}\right)+\left(\underbrace{b_1a_n+b_1a_n+\ldots+b_1a_n}_{(n+1-i)\,-ma}\right)=$$

$$=b_1\left(a_n+a_n+\ldots+a_n\right)+a_n\left(b_1+b_1+\ldots+b_1\right)\geq b_1\left(a_i+a_{i+1}+\ldots+a_n\right)+$$
$$+a_n\left(b_1+b_2+\ldots+b_{i-1}\right)=b_i\left(b_1+b_2+\ldots+b_{i-1}\right)+a_n\left(a_i+a_{i+1}+\ldots+a_n\right)\geq$$
$$\geq b_1^2+b_2^2+\ldots+b_{i-1}^2+a_i^2+a_{i+1}^2+\ldots+a_n^2=\sum_{i=1}^{n}a_i^2.$$

32. **T**engsizlikni

$$\frac{n-1}{n-1+x_1}+\frac{n-1}{n-1+x_2}+\ldots+\frac{n-1}{n-1+x_n}\leq n-1$$

shaklda yozib, undan

$$\frac{x_1}{n-1+x_1}+\frac{x_2}{n-1+x_1}+\ldots+\frac{x_n}{n-1+x_1}\geq 1$$

tengsizlikni xosil kilamiz. $y_i=\dfrac{x_i}{n-1+x_i}$(*) belgilash kiritsak, u holda

$S = y_1 + y_2 + ... + y_n \geq 1$ tengsizlikni isbotlash yetarli. O'rta arifmetik va o'rta geometrik miqdorlar o'rtasidagi munosabatga ko'ra,

$$S - y_1 \geq (n-1)\sqrt[n-1]{\frac{y_1 y_2 \cdot ... \cdot y_n}{y_1}},$$

$$S - y_2 \geq (n-1)\sqrt[n-1]{\frac{y_1 y_2 \cdot ... \cdot y_n}{y_2}},$$

$$\dots\dots\dots\dots\dots\dots$$

$$S - y_n \geq (n-1)\sqrt[n-1]{\frac{y_1 y_2 \cdot ... \cdot y_n}{y_n}}$$

tengsizliklar o'rinli. Bu tengsizliklarning mos kismlarini kupaytirib

$$(S - y_1)(S - y_2)...(S - y_n) \geq (n-1)^n y_1 y_2 ... y_n;$$

tengsizlikni va (*) ko'ra $x_i = \dfrac{(n-1)y_i}{1 - y_i}$ yoki

$(n-1)^n y_1 y_2 ... y_n = (1 - y_1)(1 - y_2)...(1 - y_n)$ bulardan

$(S - y_1)(S - y_2)...(S - y_n) \geq (1 - y_1)(1 - y_2)...(1 - y_n)$ (**) munosabatni xosil kilamiz. Agar $0 < S < 1$ bo'lsa, $S - y_i < 1 - y_i$ yoki $\prod_{i=1}^{n}(S - y_i) \leq \prod_{i=1}^{n}(1 - y_i)$. Bu (**) ga ziddir. Demak $S \geq 1$ ekan.

33. Masalaning shartidan $|x| = \sqrt{2 - y^2 - z^2}$ va $yz \leq 1$ ekanligini ko'rish mumkin. Koshi-Bunyakovskiy-Shvarts tengsizliginidan quyidagi usulda foydalanib,

$$x + y + z - xyz = x(1 - yz) + y + z \leq |x| \cdot |1 - yz| + |y + z| =$$

$$= \sqrt{2 - y^2 - z^2}\,|1 - yz| + |y + z| \leq \sqrt{\left(\left(2 - y^2 - z^2\right) + \left(y + z\right)^2\right)\left(\left(1 - yz\right)^2 + 1\right)} =$$

$$= \sqrt{\left(2 + 2yz\right)\left(2 - 2yz + y^2 z^2\right)}$$

munosabatni hosil qilamiz. Endi $\left(1 + yz\right)(2 - 2yz + y^2 z^2) \leq 2$ ekanligini ko'rsatish yetarli. Bu tengsizlikning chap tomonidagi qavslarni ochib ixchamlash natijasida $y^3 z^3 \leq y^2 z^2$ yoki $yz \leq 1$ ni xosil qilamiz, bundan esa $x + y + z - xyz \leq 2$ tengsizlik isbotlandi.

34. $\frac{1}{x\sqrt{2}}=a,\ \frac{1}{y\sqrt{3}}=b,\ \frac{1}{2z}=c$ deb belgilash kiritsak, u holda

$Q(a,b,c)=2a^2+6b^2+12c^2$ ifodaning eng katta qiymatini topsak masala yechiladi.

a,b,c musbat sonlar quyidagi shartlarni qanoatlantiradi:

$$\max\{a,b,\}<c\le\frac{1}{2} \qquad (1)$$

$$c\sqrt{2}+a\sqrt{3}\ge 2\sqrt{6}ac \qquad (2)$$

$$c\sqrt{2}+b\sqrt{5}\ge 2\sqrt{10}bc \qquad (3)$$

(2) dan

$$\frac{\sqrt{2}}{a}+\frac{\sqrt{3}}{c}\ge 2\sqrt{6}\Rightarrow\frac{2}{a^2}+\frac{3}{c^2}\ge 12\Rightarrow\frac{1}{6}a^2\left(\frac{2}{a^2}+\frac{3}{c^2}\right)\ge 2a^2,$$

Bundan

$$a^2+c^2=2a^2+c^2-a^2\le\frac{1}{6}a^2\left(\frac{2}{a^2}+\frac{3}{c^2}\right)+c^2\left(1-\frac{a^2}{c^2}\right)\le\frac{1}{6}a^2\left(\frac{2}{a^2}+\frac{3}{c^2}\right)+\frac{1}{2}\left(1-\frac{a^2}{c^2}\right)=\frac{5}{6}$$

Xuddi shunday (1) va (3) dan $b^2+c^2\le\frac{7}{10}$ ekanligini topamiz.

Shunday qilib, $Q(a,b,c,)=2(a^2+c^2)+6(b^2+c^2)+4c^2\le\frac{118}{15}$

Tenglik $Q(a,b,c)=Q\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{5}},\frac{1}{\sqrt{2}}\right)=\frac{118}{15}$ bajariladi va $a=\frac{1}{\sqrt{3}},\ b=\frac{1}{\sqrt{5}},\ c=\frac{1}{\sqrt{2}}$

qiymatlar (1)-(2)-(3) shartlarni qanoatlantiradi.

Bundan $\max P(x,y,z)=\max Q(a,b,c)=\frac{118}{15}.$

35. $a+b=x,\ \ b+c=y,\ \ c+a=t$ belgilash kiritsak yuqoridagi tengsizlik

quyidagi ko'rinishga keladi:

$$\frac{2(x+t)^2}{(x+t-y)^2+2y^2}+\frac{2(y+t)^2}{(y+t-x)^2+2x^2}+\frac{2(x+y)^2}{(x+y-t)^2+2t^2}\le 8 \quad (*)$$

Ushbu $2(t^2+p^2)\ge(t+p)^2$ tengsizlikdan foydalansak,

$$\frac{4(x+t)^2}{(2(x+t-y)^2+2y^2)+2y^2}+\frac{4\left(y+t\right)^2}{(2(y+t-x)^2+2x^2)+2x^2}+\frac{4\left(x+y\right)^2}{(2(x+y-t)^2+2t^2)+2t^2}\le$$

$$\le\frac{4\left(x+t\right)^2}{(x+t)^2+2y^2}+\frac{4\left(y+t\right)^2}{(y+t)^2+2x^2}+\frac{4\left(x+y\right)^2}{(x+y)^2+2t^2}=$$

$$=\frac{4}{1+\dfrac{2y^2}{(x+t)^2}}+\frac{4}{1+2\dfrac{2x^2}{(y+t)^2}}+\frac{4}{1+2\dfrac{t^2}{(x+y)^2}}\le\frac{4}{1+\dfrac{y^2}{x^2+t^2}}+\frac{4}{1+\dfrac{x^2}{y^2+t^2}}+\frac{4}{1+\dfrac{t^2}{x^2+y^2}}=$$

$$=\frac{4\left(x^2+t^2\right)}{x^2+y^2+t^2}+\frac{4\left(y^2+t^2\right)}{x^2+y^2+t^2}+\frac{4\left(x^2+y^2\right)}{x^2+y^2+t^2}=8.$$

Bundan (*) isbotlandi.

36. O’rta arifmetik va o’rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo’llaymiz:

$$\frac{0,6}{\sqrt{0,36(a+\dfrac{1}{b}+0,64)}}+\frac{0,6}{\sqrt{0,36(b+\dfrac{1}{c}+0,64)}}+\frac{0,6}{\sqrt{0,36(c+\dfrac{1}{a}+0,64)}}\ge$$

$$\ge 1,2\left(\frac{1}{a+\dfrac{1}{b}+1}+\frac{1}{b+\dfrac{1}{c}+1}+\frac{1}{c+\dfrac{1}{a}+1}\right)=1,2$$

Chunki

$$\frac{1}{a+\dfrac{1}{b}+1}+\frac{1}{b+\dfrac{1}{c}+1}+\frac{1}{c+\dfrac{1}{a}+1}=\frac{1}{ab+b+1}+\frac{1}{bc+c+1}+\frac{1}{ac+a+1}=$$

$$=\frac{ac}{ac(ab+b+1)}+\frac{a}{a(bc+c+1)}+\frac{1}{ac+a+1}=\frac{ac}{a+ac+1}+\frac{a}{1+ac+a}+\frac{1}{ac+a+1}=1$$

37. Bu tengsizlikni shakl almashtirish natijasida

$$\left(z-\frac{y+1}{(x+1)y}\right)+\left(x-\frac{z+1}{(y+1)z}\right)+\left(y-\frac{x+1}{(z+1)x}\right)\geq 0 \quad \text{yoki}$$

$\frac{x+z}{x+1}+\frac{x+y}{y+1}+\frac{y+z}{z+1}\geq 3$ (*)ga teng kuchli tengsizlikka olib kelamiz.

Koshi-Bunyakovskiy-Shvarts tengsizligiga ko'ra $x+1\leq\sqrt{(x+xy)(x+z)}$

munosabat o'rinli. Bundan $x+z\geq\frac{(x+1)^2}{x(1+y)}$ yoki $\frac{x+z}{x+1}\geq\frac{x+1}{x(1+y)}$ munosabatni

hosil qilamiz. Demak

$$\frac{x+z}{x+1}+\frac{x+y}{y+1}+\frac{z+y}{z+1}\geq\frac{x+1}{x(1+y)}+\frac{1+y}{y(1+z)}+\frac{1+z}{z(1+x)}\geq$$
$$\geq 3\sqrt[3]{\frac{x+1}{x(1+y)}\cdot\frac{1+y}{y(1+z)}\cdot\frac{1+z}{z(1+x)}}=3$$

38. Avval $\forall t>0$ uchun $3(t^2-t+1)^3\geq t^6+t^3+1$ (*) tengsizlik o'rinli ekanligini ko'rsatamiz. (*) ni shakl almashtirib, quyidagi munosabatni hosil qilamiz: $(t-1)^4(2t^2-t+2)\geq 0$ bu tengsizlik $\forall t>0$ uchun o'rinli. Bundan

$3(x^2-x+1)^3 3(y^2-y+1)^3 3(z^2-z+1)^3\geq(x^6+x^3+1)(y^6+y^3+1)(z^6+z^3+1)$ (**)

munosabatni hosil qilamiz. Umumlashgan Koshi-Bunyakovskiy-Shvarts tengsizligini qo'llasak,

$(x^6+x^3+1)(y^6+y^3+1)(z^6+z^3+1)\geq(x^2y^2z^2+xyz+1)^3$ (***)

(**) va (***) larni hadma-had ko'paytirib, isboti talab etilgan tengsizlikni hosil qilamiz.

39. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini qo'llaymiz:

$$1+\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{3}\right)+...+\left(1-\frac{1}{n}\right)=1+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+...+\frac{n-1}{n}\geq n\sqrt[n]{1\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}...\frac{n-1}{n}}=$$

$$=n\sqrt[n]{\frac{1}{n}}=n^{\frac{n-1}{n}}.$$

40. Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llab,

$$\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\leq\sqrt{x+y+z}\cdot\sqrt{\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}}=\sqrt{x+y+z}$$

munosabatni hosil qilamiz.

41. Berilgan tengsizlikdan $(a^3+b^3+c^3)(a+b+c)\geq 9a^2b^2c^2=(a^2+b^2+c^2)^2$ tengsizlikni hosil qilamiz. Bu tengsizlik esa Koshi - Bunyakovskiy tengsizligiga ko'ra o'rinlidir.

42. O'rta arifmetik va o'rta geometrik miqdorlar o'rtasidagi munosabatni quyidagi usulda qo'llaymiz:

$$\frac{1}{(x+y+t)^2}+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{9}\left(\frac{9}{(x+y+z)^2}+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)+\frac{8}{9}\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)\geq$$

$$\geq\frac{4}{9}\sqrt[4]{\frac{9}{(x+y+z)^2x^2y^2z^2}}+\frac{8}{3}\sqrt[3]{\frac{1}{x^2y^2z^2}}\geq\frac{4}{9}\sqrt{\frac{3}{(x+y+z)xyz}}+\frac{8}{3}\sqrt{\frac{3}{xyz(x+y+z)}}=$$

$$=\frac{28\sqrt{3}}{9\sqrt{xyz(x+y+z)}}$$

43. $d_k=|n-a_k|$ belgilash kiritsak, u holda

$$\sum_{k=1}^{n}d_k^2=\sum_{k=1}^{n}\left(n-a_k\right)^2=n\cdot n^2-2n\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}a_k^2\leq n^3-2n\cdot n^2+n^3+1=1$$

munosabatni hosil bo'ladi. Bundan $d_k\leq 1$ yoki $n-1\leq a_k\leq n+1$.

44. Tengsizlikni ikkala qismini $a+b+c$ ga bo'lib va

$x=\dfrac{a}{a+b+c},\ y=\dfrac{b}{a+b+c},\ z=\dfrac{c}{a+b+c}$ belgilash kiritsak

$$\alpha x+\beta y+\gamma z+2\sqrt{(\alpha\beta+\beta\gamma+\gamma\alpha)(xy+yz+zx)}\le\frac{\alpha^2}{2}+\frac{x^2}{2}+\frac{\beta^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}+\frac{\gamma^2}{2}+$$
$$+\gamma\alpha+\alpha\beta+\beta\gamma+xy+yz+zx=\frac{1}{2}(\alpha+\beta+\gamma)^2+\frac{1}{2}(x+y+z)^2=\frac{1}{2}+\frac{1}{2}=1.$$

45. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llaymiz:

$$\frac{1}{a^3}+\frac{1}{3ab^2}+\frac{1}{3a^2b}+\frac{1}{b^3}=(\frac{1}{a^3}+\frac{1}{b^3})+\frac{1}{3ab}\left(\frac{1}{a}+\frac{1}{b}\right)\ge2\sqrt{\frac{1}{a^3b^3}}+\frac{2}{3ab}\sqrt{\frac{1}{ab}}=\frac{8}{3\sqrt{(ab)^3}}\ge$$
$$\ge\frac{64}{3(a+b)^3}$$

46. $x_1,x_2....x_6\in[0;1]$ ekanligidan tengsizlikning chap qismi quyidagi ifodadan kichik yoki teng

$$\frac{x_1^3}{x_1^5+x_2^5+.....+x_6^5+4}+\frac{x_2^3}{x_1^5+x_2^5+......+x_6^5+4}+...+\frac{x_6^3}{x_1^5+x_2^5+.....+x_6^5+4}=$$
$$=\frac{x_1^3+x_2^3+...+x_6^3}{x_1^5+x_2^5+...+x_6^5+4}\le\frac{3}{5}.(*)$$

Ixtiyoriy $t\ge0$ uchun $3t^5+2\ge5t^3\Leftrightarrow(t-1)^2(3t^3+6t^2+4t+2)\ge0$ munosabat o'rinli ekanligidan foydalansak (*) kelib chiqadi.

47. Tengsizlikni chap qismini S bilan belgilab, Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llab,

$S\cdot\left(a^2(1+2bc)+b^2(1+2ac)+c^2(1+2ab)\right)\ge(a^2+b^2+c^2)^2$ yoki

$$S\ge\frac{1}{1+2abc(a+b+c)}$$

munosabatni hosil qilamiz. $a^2+b^2+c^2 \ge \sqrt{3abc(a+b+c)}$ tengsizlikka ko'ra

$$S \ge \frac{1}{1+\frac{2}{3}3abc(a+b+c)} \ge \frac{1}{1+\frac{2}{3}\left(a^2+b^2+c^2\right)^2} = \frac{3}{5}$$

ekanligini hosil qilamiz.

48. Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llab,

$$\left(\frac{i(i-1)}{2}\right)^2 \cdot x_i^2 \ge \left(x_1^2+\frac{x_2^2}{2^3}+\frac{x_3^2}{3^3}+\ldots+\frac{x_{i-1}^2}{(i-1)^3}\right)\left(1+2^3+3^3+\ldots+(i-1)^3\right) \ge$$

$$\ge \left(x_1+x_2+x_3+\ldots+x_{i-1}\right)^2 \text{ ёки } \frac{x_i}{x_1+x_2+\ldots+x_{i-1}} \ge \frac{2}{i\left(i-1\right)}, \quad 2 \le i \le 2001$$

ekanligini topamiz. Bundan

$$\sum_{i=2}^{2001} \frac{x_i}{x_1+x_2+\ldots+x_{i-1}} \ge 2\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{2000\cdot 2001}\right) = 2\left(1-\frac{1}{2001}\right) > 1{,}999.$$

49. Tengsizlikni $\left(\frac{a}{b}-\frac{c+a}{c+b}+1\right)+\left(\frac{b}{c}-\frac{a+b}{a+c}+1\right)+\left(\frac{c}{a}-\frac{b+c}{b+a}+1\right) \ge 3$ yoki

$\frac{b^2+ac}{b(c+b)}+\frac{c^2+ab}{c(a+c)}+\frac{a^2+bc}{a(b+a)} \ge 3$ kurinishda yozamiz.Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llasak,

$$\frac{b^2+ac}{b(c+b)}+\frac{c^2+ab}{c(a+c)}+\frac{a^2+bc}{a(a+b)} \ge \frac{\left(b+c\right)a}{b(a+c)}+\frac{b(a+c)}{c(a+b)}+\frac{c\left(a+b\right)}{a(b+c)} \ge$$

$$\ge 3\sqrt[3]{\frac{(b+c)a}{b(a+c)}\cdot\frac{b(a+c)}{c(a+b)}\cdot\frac{c(a+b)}{a(b+c)}} = 3.$$

50. Avval tengsizlikni induktsiya metodi yordamida $n=2^k, k \in N$ sonlar uchun isbotlaymiz. $n=2$ da

$$\frac{1}{y_1+1}+\frac{1}{y_2+1}-\frac{2}{\sqrt{y_1y_2}+1} = \frac{\left(\sqrt{y_1y_2}-1\right)\left(\sqrt{y_1}-\sqrt{y_2}\right)^2}{(y_1+1)(y_2+1)(\sqrt{y_1y_2}+1)} \ge 0$$

$n=2^k$ da tengsizlik o'rinli bo'lsin deb faraz qilsak, u holda $n=2^{k+1}$ da

$$\left(\frac{1}{1+y_1}+\frac{1}{1+y_2}+...+\frac{1}{1+y_{2^k}}\right)+\left(\frac{1}{1+y_{2^k+1}}+\frac{1}{1+y_{2^k+2}}+....+\frac{1}{1+y_{2^{k+1}}}\right)\geq$$

$$\geq\frac{2^k}{1+\sqrt[2^k]{y_1y_2...y_{2^k}}}+\frac{2^k}{1+\sqrt[2^k]{y_{2^k+1}y_{2^k+2}...y_{2^{k+1}}}}\geq\frac{2^{k+1}}{1+\sqrt[2^{k+1}]{y_1\cdot y_2....y_{2^{k+1}}}}$$

munosabat o'rinli va tengsizlik $n=2^k,\ k\in N,$ uchun isbotlandi. Endi tengsizlikni $n\in N$ uchun isbotlaymiz. Buning uchun $m=2^k>n, k\in N,$ uchun tengsizlik o'rinli deb faraz etsak va $y_{n+1}=y_{n+2}=....=y_m=\sqrt[n]{y_1y_2..y_n}$ deb olsak, u holda

$$\frac{1}{y_1+1}+\frac{1}{y_2+1}+...+\frac{1}{y_n+1}+\frac{m-n}{1+\sqrt[n]{y_1y_2...y_n}}\geq\frac{m}{\sqrt[n]{y_1y_2...y_n}+1}$$

bo'ladi. Bundan yuqoridagi tengsizlik isbotlanadi.

51. $x_i=\cos^2\alpha_i,\ \frac{\pi}{4}\leq\alpha_i<\frac{\pi}{2}$ belgilash kiritsak, u holda tengsizlik

$$\frac{\prod_{i=1}^{n}\cos^2\alpha_i}{\prod_{i=1}^{n}(1-\cos^2\alpha_i)}\leq\left(\frac{\sum_{i=1}^{n}\cos^2\alpha_i}{\sum_{i=1}^{n}(1-\cos^2\alpha_i)}\right)^n$$

ko'rinishga, yoki

$$\frac{1}{1+tg^2\alpha_1}+\frac{1}{1+tg^2\alpha_2}+...+\frac{1}{1+tg^2\alpha_n}\geq\frac{n}{1+\sqrt[n]{tg^2\alpha_1tg^2\alpha_2...tg^2\alpha_n}}$$

ko'rinishga keladi. Agar $tg^2\alpha_i=y_i$ deb belgilash kiritsak, $y_i\geq 1$ bo'ladi va

$\frac{1}{1+y_1}+\frac{1}{1+y_2}+...\frac{1}{1+y_n}\geq\frac{n}{1+\sqrt[n]{y_1y_2...y_n}}$ ni isbotlash kerak bo'ladi.Bu tengsizlik esa 50- masalada isbotlangan.

52. Tengsizlikni musbat $x_1,x_2,...,x_n$ sonlar uchun isbotlash yetarli. Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llab,

$$\frac{x_1}{1+x_1^2}+\frac{x_2}{1+x_1^2+x_2^2}+....+\frac{x_n}{1+x_1^2+x_2^2+...+x_n^2}<$$

$$<\sqrt{n\left(\left(\frac{x_1}{1+x_1^2}\right)^2+\left(\frac{x_2}{1+x_1^2+x_2^2}\right)^2+...+\left(\frac{x_n}{1+x_1^2+...+x_n^2}\right)^2\right)}$$

munosabatni hosil qilamiz. Bundan

$$\frac{x_1^2}{(1+x_1^2)^2}+\frac{x_2^2}{(1+x_1^2+x_2^2)^2}+..+\frac{x_n^2}{(1+x_1^2+x_2^2+...+x_n^2)^2}<1$$

ekanligini ko'rsatsak, yuqoridagi tengsizlik isbotlanadi.

$$\frac{{x_k}^2}{\left(1+x_1^2+x_2^2+...+x_k^2\right)}\le\frac{x_k^2}{(1+x_1^2+x_2^2+...+x_{k-1}^2)(1+x_1^2+...+x_k^2)}=$$

$$=\frac{1}{1+x_1^2+...+x_{k-1}^2}-\frac{1}{1+x_1^2+...+x_k^2}$$

bo'lgani uchun

$$\sum_{k=1}^{n}\left(\frac{x_k}{1+x_1^2+...+x_k^2}\right)^2<1-\frac{1}{1+x_1^2+...+x_n^2}<1.$$

53. $9(a^3+b^3+c^3)\ge(a+b+c)^3$ tengsizlikdan foydalanib,

$$\sqrt[3]{\frac{1}{a}+6b}+\sqrt[3]{\frac{1}{b}+6c}+\sqrt[3]{\frac{1}{c}+6a}\le\sqrt[3]{9\left(\frac{1}{a}+6b+\frac{1}{b}+6c+\frac{1}{c}+6a\right)}=$$

$$=\frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{1}{abc}+3\frac{1-ab}{c}+3\frac{1-bc}{a}+3\frac{1-ac}{b}}=\frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{4-3\left((ab)^2+(bc)^2+(ca)^2\right)}{abc}}\le$$

$$\le\frac{3}{\sqrt[3]{3}}\sqrt[3]{\frac{4-3\frac{(ab+bc+ca)^2}{3}}{abc}}=\frac{3}{\sqrt[3]{abc}}$$

munosabatni hosil qilamiz.Endi $\frac{3}{\sqrt[3]{abc}}\le\frac{1}{abc}$ yoki $a^2b^2c^2\le\frac{1}{27}$ ekanligi

ko'rsatamiz: $(abc)^2=(ab)(bc)(ca)\le\left(\frac{ab+bc+ca}{3}\right)^3=\frac{1}{27}$.

54. $1-bc=a(b+c)$ ekanligidan

$$3abcx=3a\sqrt[3]{b^2c\cdot c^2b\cdot x^3}\le a(b^2c+c^2b+x^3)=ax^3+bc(1-bc)=$$

$$=ax^3+bc\left(\frac{2}{3}-bc\right)+\frac{1}{3}bc\le ax^3+\frac{1}{3}bc+\left(\frac{bc+(\frac{2}{3}-bc)}{2}\right)^2=ax^3+\frac{1}{3}bc+\frac{1}{9}$$

munosabatni hosil qilamiz. Xuddi shunday

$$3abcy\le by^3+\frac{1}{3}ac+\frac{1}{9},\ 3abcz\le cz^3+\frac{1}{3}ab+\frac{1}{9}$$

tengsizliklar o'rinli.bu tengsizliklarni hadma-had qo'shib isboti talab etilgan tengsizlikni hosil qilamiz.

55. Tengsizlikni ikkala qismiga umumiy maxraj tanlab,

$$\sum_{i=1}^{n}\frac{1}{a_i}-\sum_{i=1}^{n}\frac{1}{1+a_i}\ge\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}\cdot\sum_{i=1}^{n}\frac{1}{1+a_i}\quad \text{yoki}\quad n\sum_{i=1}^{n}\frac{1}{a_i(a_i+1)}\ge\sum_{i=1}^{n}\frac{1}{1+a_i}\sum_{i=1}^{n}\frac{1}{a_i}$$

munosabatni hosil qilamiz. Umumiylikni chegaralamasdan $a_1\ge a_2\ge...\ge a_n$ deb olsak, oxirgi tengsizlik Chebishev tengsizligia ko'ra o'rinlidir.

56. $a_1+a_2+....+a_{n-1}=k$ deb olamiz. U holda isboti talab etilgan tengsizlikka teng kuchli bo'lgan

$$\frac{a_0+k+a_n}{n+1}\cdot\frac{k}{n-1}\ge\frac{a_0+k}{n}\cdot\frac{a_n+k}{n}\Leftrightarrow$$
$$n^2k(a_0+a_n+k)\ge(n^2-1)(a_0+k)(a_n+k)\Leftrightarrow k(k+a_0+a_n)\ge a_0a_n(n^2-1).$$

tengsizlikni hosil qilamiz. $a_0+a_n\ge 2\sqrt{a_0a_n}$ ekanligidan $k\ge(n-1)\sqrt{a_0a_n}$ tengsizlikni isbotlasak yuqoridagi tengsizlik isbotlandi.

$a_{i-1}\cdot a_{i+1}\le a_i^2\quad (i=1,2,....,n-1)$ tengsizlikka ko'ra $\frac{a_0}{a_1}\le\frac{a_1}{a_2}\le\frac{a_2}{a_3}\le...\le\frac{a_{n-1}}{a_n}$ yoki $a_0a_n\le a_1a_{n-1}\le a_2a_{n-2}\le......$ munosabatni hosil qilamiz. Bundan,

$$K = a_1 + a_2 + ... + a_{n-1} = (a_1 + a_{n-1}) + (a_2 + a_{n-2}) + ... \geq 2\sqrt{a_1 a_{n-1}} +$$
$$+2\sqrt{a_2 a_{n-2}} + ... \geq 2\sqrt{a_0 a_n} + 2\sqrt{a_0 a_n} + ... = 2 \cdot \frac{n-1}{2}\sqrt{a_0 a_n} = (n-1)\sqrt{a_0 a_n}$$

ekanligi kelib chiqadi.

57. Umumiylikni chegaralamasdan $a \leq b \leq c$ deb olamiz. U holda o'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llab,

$$8abc = 8ab(1-a-b) \leq 2(a+b)^2(1-a-b) = 2(a+b)\left[(a+b)(1-a-b)\right] \leq \frac{a+b}{2}$$

tengsizlikni topamiz. $(a-c)(x-\frac{1}{2}) + (b-c)(y-\frac{1}{2}) \geq 0$ munosabat o'rinli ekanligidan $ax + by + cz \geq \frac{a+b}{2} \geq 8abc$ tengsizlikni to'g'riligini topamiz.

58. $x_{n+1} = 1 - x_1 - x_2 - ... - x_n$ bo'lsin. U holda $x_{n+1} > 0$ va o'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llasak

$$1 - x_i = x_1 + x_2 + ... + x_{n+1} - x_i \geq n\sqrt[n]{\frac{x_1 x_2 ... x_{n+1}}{x_i}} \quad (i = 1, 2, ..., n+1)$$ munosabatni olamiz.

Bu tengsizliklarni hadma-had ko'paytirib,

$$\prod_{i=1}^{n+1}(1-x_i) \geq \prod_{i=1}^{n+1} n\sqrt[n]{\frac{x_1 x_2 ... x_{n+1}}{x_i}} = n^{n+1}x_1 x_2 ... x_n x_{n+1} = n^{n+1}x_1 x_2 ... x_n(1 - x_1 - x_2 - ... - x_n)$$

yoki $(1-x_1)(1-x_2)...(1-x_n)(x_1 + x_2 + ... + x_n) \geq n^{n+1}x_1 x_2 ... x_n(1 - x_1 - x_2 ... - x_n)$ tengsizlikni hosil qilamiz.

59. Umumiylikni chegaralamasdan $0 \leq a \leq b \leq c \leq 1$ deb olamiz. U holda $(1-a)(1-b) \geq 0$ yoki $a + b + c \leq a + b + 1 \leq 2 + ab$<2(1+ab) ekanligini topamiz. Bundan,

$$\frac{a}{bc+1} + \frac{b}{ac+1} + \frac{c}{ab+1} \leq \frac{a}{ab+1} + \frac{b}{ab+1} + \frac{c}{ab+1} = \frac{a+b+c}{ab+1} < \frac{2(ab+1)}{ab+1} = 2$$

munosabatni hosil qilamiz.

60. $b(a-1)+a(c-1)\geq 0 \Leftrightarrow ab+ac\geq b+a \Leftrightarrow \frac{a+b}{b+c}\leq a$ va

$d(c-1)+c(a-1)\geq 0 \Leftrightarrow dc+ca\geq d+c \Leftrightarrow \frac{d+c}{d+a}\leq c$ ekanligi rashan. Bundan

$$\frac{4(a+c)}{(b+d)}\geq \frac{4(a+c)}{4}=a+c\geq \frac{a+b}{b+c}+\frac{d+c}{d+a}$$

munosabat o'rinli ekanligi kelib chiqadi.

61. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llab $\sqrt{\frac{a}{b+c-ta}}=\frac{a\sqrt{1+t}}{\sqrt{(a+at)(b+c-ta)}}\geq \frac{2a\sqrt{t+1}}{a+b+c}$ tengsizlikni topamiz. Xuddi shunday $\sqrt{\frac{b}{a+c-tb}}\geq \frac{2b\sqrt{1+t}}{a+b+c}$, $\sqrt{\frac{c}{a+b-tc}}\geq \frac{2c\sqrt{1+t}}{a+b+c}$ tengsizliklar o'rinli. Bu tengsizliklarni hadma-had qo'shib, isboti talab etilgan tengsizlikni hosil qilamiz.

62. $(ab+1-c)^2+(bc+1-a)^2+(ac+1-b)^2\geq 0$ ekanligidan, yuqoridagi tengsizlikni o'rinli bo'lishi kelib chiqadi.

63. $\frac{1}{1+x_i^2}=y_i$ $(\mathrm{i}=1,2,\ldots,2002)$ deb belgilash kiritsak, u holda $y_1+y_2+\ldots+y_{-2002}=1$ bo'ladi. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llasak, $1-y_i=y_1+y_2+\ldots+y_{2002}-y_i\geq 2001\sqrt[2001]{\frac{y_1y_2\ldots y_{2002}}{y_i}}$

$(i=1,2,\ \ldots\ 2002)$

ekanligini topamiz va bu tengsizliklarni hadma-had ko'paytirib

$$\prod_{i=1}^{2002}(1-y_i)\geq \prod_{i=1}^{2002}2001\sqrt[2001]{\frac{y_1y_2\ldots y_{2002}}{y_i}}=2001^{2002}y_1y_2\ldots y_{2002},\quad \prod_{i=1}^{2002}\frac{1-y_i}{y_i}\geq 2001^{2002}$$

yoki $\prod_{i=1}^{2002} x_i \ge 2001^{1001}$ tengsizlikni hosil qilamiz.

64. Musbat x, y, z va v sonlar uchun $\frac{x^2}{y}+\frac{z^2}{v}\ge\frac{(x+z)^2}{y+v}$ tengsizlik o'rinli ekanligidan foydalansak

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac}\ge\frac{(1+1)^2}{1+ab+1+bc}+\frac{1}{1+ac}\ge\frac{(1+1+1)^2}{3+ab+bc+ca}\ge$$

$$\ge\frac{9}{3+a^2+b^2+c^2}=\frac{3}{2}.$$

65. Tengsizlikni chap tomonini *T* bilan belgilab, o'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini qo'llasak, u holda

$$T=\left(\frac{ax}{\sqrt{ac}}+\frac{by}{\sqrt{ac}}+\frac{cz}{\sqrt{ac}}\right)\left(\frac{x\sqrt{ac}}{a}+\frac{y\sqrt{ac}}{b}+\frac{z\sqrt{ac}}{c}\right)\le$$

$$\le\frac{1}{4}\left(\left(\frac{ax}{\sqrt{ac}}+\frac{x\sqrt{ac}}{a}\right)+\left(\frac{by}{\sqrt{ac}}+\frac{y\sqrt{ac}}{b}\right)+\left(\frac{cz}{\sqrt{ac}}+\frac{z\sqrt{ac}}{c}\right)\right)^2$$

tengsizlik hosil bo'ladi. Ushbu tengsizliklar o'rinli ekanligidan

$$x\left(\frac{a}{\sqrt{ac}}+\frac{\sqrt{ac}}{a}\right)\le x\frac{a+c}{\sqrt{ac}},\ y\left(\frac{b}{\sqrt{ac}}+\frac{\sqrt{ac}}{b}\right)\le y\frac{a+c}{\sqrt{ac}},\ z\left(\frac{c}{\sqrt{ac}}+\frac{\sqrt{ac}}{c}\right)\le z\frac{a+c}{\sqrt{ac}}.$$

bu tengsizliklarni hadma-had qo'shib

$$S\le\frac{1}{4}\left(\frac{a+c}{\sqrt{ac}}(x+y+z)\right)^2=\frac{(a+c)^2}{4ac}(x+y+z)^2$$

munosabatni hosil qilamiz.

66. x>0, y>0 $\frac{1}{x+y}\le\frac{1}{4}\left(\frac{1}{x}+\frac{1}{y}\right)$ tengsizlik o'rinli va bu tengsizlikni qo'llab,

$$\frac{ab}{a+b+2c}+\frac{bc}{b+c+2a}+\frac{ac}{a+c+2b}\le\frac{ab}{4}\left(\frac{1}{a+c}+\frac{1}{b+c}\right)+\frac{bc}{4}\left(\frac{1}{a+c}+\frac{1}{b+a}\right)+$$

$$+\frac{ac}{4}\left(\frac{1}{a+b}+\frac{1}{b+c}\right)=\frac{1}{4(a+c)}\left(ab+bc\right)+\frac{1}{4(b+c)}\left(ab+ac\right)+\frac{1}{4(a+b)}\left(ac+bc\right)=$$

$$=\frac{1}{4}(a+b+c)$$

munosabatni hosil qilamiz.

67. Musbat x, y sonlar uchun $\frac{x^4+y^4}{x^3+y^3}\ge\frac{x+y}{2}$ (*) tengsizlik o'rinli. Chunki

$$2(x^4+y^4)\ge(x+y)(x^3+y^3)\Leftrightarrow x^4+y^4\ge x^3y+y^3x\Leftrightarrow(x-y)^2(x^2+xy+y^2)\ge 0.$$

Endi (*)dan foydalansak,

$$\frac{a^4+b^4}{ab(a^3+b^3)}+\frac{b^4+c^4}{bc(b^3+c^3)}+\frac{c^4+a^4}{ac(a^3+c^3)}\ge\frac{a+b}{2ab}+\frac{b+c}{2bc}+\frac{a+c}{2ac}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$$

munosabat hosil bo'ladi.

68. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini

qo'llab, $a+b+c+d+\underbrace{\frac{1}{32abcd}+...+\frac{1}{32abcd}}_{32ma}\ge 36\sqrt[36]{abcd\left(\frac{1}{32abcd}\right)^{32}}$

munosabatni topamiz.

Endi $36\sqrt[36]{abcd\left(\frac{1}{32abcd}\right)^{32}}\ge 18\Leftrightarrow 2\sqrt[36]{abcd\left(\frac{1}{32abcd}\right)^{32}}\ge 1\Leftrightarrow$

$\Leftrightarrow 2^{36}abcd\left(\frac{1}{32abcd}\right)^{32}=\frac{2^{36}}{2^{160}\left(abcd\right)^{31}}\ge 1\Leftrightarrow\left(abcd\right)\le\frac{1}{2^4}$ tengsizlikni isbotlasak

masala yechiladi. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini berilgan tenglikka qo’llab,

$1=a^2+b^2+c^2+d^2\geq 4\sqrt[4]{a^2b^2c^2d^2}=4\sqrt{abcd}\Leftrightarrow\ abcd\leq\frac{1}{2^4}$ ekanligini topamiz.

69. Berilgan tengsizlikni chap qismidagi qavslarni ochib o’rta arifmetik va o’rta geometrik miqdorlar haqidagi Koshi tengsizligini qo’llaymiz. U holda

$$(x^2+2)(y^2+2)(z^2+2)=x^2y^2z^2+2(x^2y^2+y^2z^2+z^2x^2)+4(x^2+y^2+z^2)+8=$$
$$=2(x^2y^2+1)+2(y^2z^2+1)+2(z^2x^2+1)+3(x^2+y^2+z^2)+x^2y^2z^2+2+x^2+y^2+z^2\geq$$
$$\geq 4(xy+yz+zx)+3(xy+yz+zx)+(x^2+y^2+z^2)+2+x^2y^2z^2=$$
$$=x^2y^2z^2+x^2+y^2+z^2+2+7(xy+yz+zx)$$

ekanligini topamiz. Bundan $x^2y^2z^2+x^2+y^2+z^2+2\geq 2xy+2yz+2zx$ (*)

tengsizlikni isbotlasak masala yechiladi.

Lemma: Istalgan *a, b, c* musbat sonlar uchun quyidagi

$(a+b-c)(b+c-a)(a+c-b)\leq abc$

tengsizlik o’rinli.

Isboti: Aytaylik *a+b-c=m, b+c-a=n, a+c-b=k* bo’lsin, bundan $a=\frac{m+n}{2}$,

$b=\frac{m+n}{2}$, $c=\frac{k+n}{2}$ tengliklarni topamiz. U holda $8mnk\leq(m+n)(n+k)(m+k)$

ekanligini ko’rsatish yetarli. Bu tengsizlikni ushbu: $m+n\geq 2\sqrt{mn}$, $n+k\geq 2\sqrt{nk}$,

$m+k\geq 2\sqrt{mk}$ tengsizliklarni hadma-had ko’paytirish natijasida hosil qilamiz.

$$(a+b-c)(b+c-a)(a+c-b)\leq abc\Leftrightarrow 3abc+a^3+b^3+c^3\geq$$
$$\geq ab(a+b)+bc(b+c)+ca(c+a)\geq 2(ab)^{\frac{3}{2}}+2(bc)^{\frac{3}{2}}+2(ca)^{\frac{3}{2}}.$$

Oxirgi tengsizlik istalgan a, b, c musbat sonlar uchun o’rinli ekanligidan,

quyidagicha $a=x^{\frac{2}{3}}$, $b=y^{\frac{2}{3}}$, $c=z^{\frac{2}{3}}$ belgilash olamiz. Bundan

$3(xyz)^{\frac{2}{3}}+x^2+y^2+z^2\geq 2xy+2yz+2zx$ tengsizlikni hosil qilamiz. Bu yerdan

quyidagi $2xy+2yz+2zx\leq x^2+y^2+z^2+3(xyz)^{\frac{2}{3}}\leq x^2+y^2+z^2+(xyz)+2$

munosabatni, ya’ni (*) to’g’ri ekanligini topamiz.

70. Berilgan tengsizlikni chap qismini S bilan belgilab, quyidagi usulda Koshi-Bunyakovskiy-Shvarts tengsizligini qo'llaymiz:

$$S=\left(\sum_{i=1}^{n}\left(\frac{\sqrt{a_i}}{a_i^2+x^2}\right)\left(\frac{1}{\sqrt{a_i}}\right)\right)^2\le\left(\sum_{i=1}^{n}\frac{a_i}{\left(a_i^2+x^2\right)^2}\right)\left(\sum_{i=1}^{n}\frac{1}{a_i}\right)\le\sum_{i=1}^{n}\frac{a_i}{\left(a_i^2+x^2\right)^2}.$$

Bizga ma'lumki $\left(a_i^2+x^2\right)^2>\left(a_i^2+x^2\right)^2-a_i^2>0$ (i=1, 2, …, n). Bundan

$\sum_{i=1}^{n}\frac{a_i}{\left(a_i^2+x^2\right)^2}<\frac{1}{2}\sum_{i=1}^{n}\left(\left(\frac{1}{a_i^2+x^2-a_i}\right)-\left(\frac{1}{a_i^2+x^2+a_i}\right)\right)$. Berilgan shartdan

$a_{i+1}\ge a_i+1$ yekanligini topamiz. Bundan $a_{i+1}^2-a_{i+1}+x^2\ge a_i^2+a_i+x^2$ va

$$\sum_{i=1}^{n}\frac{1}{\left(a_i^2+x^2-a_i\right)}\le\frac{1}{\left(a_1^2+x^2-a_1\right)}+\sum_{i=1}^{n}\frac{1}{\left(a_i^2+x^2+a_i\right)}-\frac{1}{a_n^2+x^2+a_n}\le\frac{1}{a_1^2+x^2-a_1}+$$

$$+\sum_{i=1}^{n}\frac{1}{\left(a_i^2+x^2+a_i\right)}\Leftrightarrow\frac{1}{2}\sum_{i=1}^{n}\left(\left(\frac{1}{a_i^2+x^2-a_i}\right)-\left(\frac{1}{a_i^2+x^2+a_i}\right)\right)\le\frac{1}{2a_1^2+2x^2-2a_1}$$

yoki $S\le\frac{1}{2a_1^2-2a_1+2x^2}$ munosabatni hosil qilamiz.

71. Quyidagi tenglikni qaraymiz $\left(x+y+z\right)^2=9$ yoki

$xy+yz+zx=\frac{9-x^2-y^2-z^2}{2}$. Bundan o'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini qo'llab,

$$xy+yz+zx=\frac{3x-x^2}{2}+\frac{3y-y^2}{2}+\frac{3z-z^2}{2}\le\sqrt{x}+\sqrt{y}+\sqrt{z}$$

munosabatni hosil qilamiz.

72. *a* va *b* sonlar musbat va butun ekanligidan, $a\sqrt{2}\ne b$ yoki $2a^2\ne b^2$ bulardan

$\left|2a^2-b^2\right|\ge1\Leftrightarrow\left|\left(a\sqrt{2}-b\right)\left(a\sqrt{2}+b\right)\right|\ge1\Leftrightarrow\left|a\sqrt{2}-b\right|\cdot\left|a\sqrt{2}+b\right|\ge1$ munosabatni hosil qilamiz. $0 < a\sqrt{2} + b < 2a + 2b = 2(a+b)$ ekanligidan

$\left|a\sqrt{2} - b\right| \geq \dfrac{1}{a\sqrt{2}+b} > \dfrac{1}{2(a+b)}$ munosabat kelib chiqadi.

73. Avvaliga $\sqrt{(a+c)^2 + (b+d)^2} \leq \sqrt{a^2+b^2} + \sqrt{c^2+d^2}$ (*) tengsizlikni isbotlaymiz: tengsizlikning ikkala qismini kvadratga oshirib, $\sqrt{(a^2+b^2)(c^2+d^2)} \geq ac + bd$ yoki $(ad-bc)^2 \geq 0$ munosabatni hosil qilamiz. Endi

$$\sqrt{a^2+b^2} + \sqrt{c^2+d^2} \leq \sqrt{(a+c)^2+(b+d)^2} + \frac{2|ad-bc|}{\sqrt{(a+c)^2+(b+d)^2}} \quad (**)$$

tengsizlikni isbotlaymiz: (**)ning ikkala qismiga umumiy mahraj tanlab va (*)dan foydalanib, $\left(\sqrt{a^2+b^2} + \sqrt{c^2+d^2}\right)^2 \leq (a+c)^2 + (b+d)^2 + 2|ad-bc|$ (***) munosabatni hosil qilamiz. Bundan

$$\sqrt{(a^2+b^2)(c^2+d^2)} \leq ac + bd + |ad-bc| \Leftrightarrow 2(ac+bd)|ad-bc| \geq 0$$

munosabatni hosil qilamiz. Demak, (***) isbotlandi. Bulardan esa isboti talab etilgan tengsizliklar kelib chiqadi.

74. Tengsizlikni ikkala qismini $\sqrt{abc}$ ga bo'lib,

$\sqrt{\dfrac{1}{bc}+\dfrac{1}{a}} + \sqrt{\dfrac{1}{ac}+\dfrac{1}{b}} + \sqrt{\dfrac{1}{ab}+\dfrac{1}{c}} \geq 1 + \dfrac{1}{\sqrt{bc}} + \dfrac{1}{\sqrt{ac}} + \dfrac{1}{\sqrt{ab}}$ tengsizlikni hosil qilamiz.

Endi, $\sqrt{\dfrac{1}{bc}+\dfrac{1}{a}} \geq \dfrac{1}{a} + \dfrac{1}{\sqrt{bc}}$ (*) ekanligini ko'rsatamiz: (*)ni ikkala qismini kvadratga oshirib,

$$\frac{1}{bc} + \frac{1}{a} \geq \frac{1}{a^2} + \frac{2}{a\sqrt{bc}} + \frac{1}{bc} \Leftrightarrow 1 \geq \frac{1}{a} + \frac{2}{\sqrt{bc}} \Leftrightarrow \frac{1}{b} + \frac{1}{c} \geq \frac{2}{\sqrt{bc}} \Leftrightarrow \left(\sqrt{b} - \sqrt{c}\right)^{-2} \geq 0$$

munosabatni hosil qilamiz. Xuddi shunday:

$$\sqrt{\frac{1}{ac}+\frac{1}{b}} \geq \frac{1}{\sqrt{ac}}+\frac{1}{b}, \sqrt{\frac{1}{ab}+\frac{1}{c}} \geq \frac{1}{\sqrt{ab}}+\frac{1}{c}$$

munosabatlar o'rinli. Bu tengsizliklarni hadma-had qo'shib, isboti talab etilgan tengsizlikni hosil qilamiz.

75. $a+b-c=x,\ b+c-a=y,\ c+a-b=z$ deb belgilash kiritsak, u holda $a=\frac{x+z}{2}, b=\frac{x+y}{2}, c=\frac{y+z}{2}$ tengliklarni topamiz va bu tengliklarni yuqoridagi tengsizlikka qo'yib, $\sqrt{\frac{x+z}{2}}+\sqrt{\frac{x+y}{2}}+\sqrt{\frac{y+z}{2}} \geq \sqrt{x}+\sqrt{y}+\sqrt{z}$ munosabatni hosil qilamiz. $\forall a,b>0$ sonlar uchun $\frac{\sqrt{a}+\sqrt{b}}{2} \leq \sqrt{\frac{a+b}{2}}$ (*) tengsizlik o'rinli. (*) dan foydalansak

$$\sqrt{x}+\sqrt{y}+\sqrt{z}=\frac{\sqrt{x}+\sqrt{y}}{2}+\frac{\sqrt{y}+\sqrt{z}}{2}+\frac{\sqrt{x}+\sqrt{z}}{2} \leq \sqrt{\frac{x+y}{2}}+\sqrt{\frac{y+z}{2}}+\sqrt{\frac{z+x}{2}}\ .$$

76. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llaymiz:

$$2(a^3b+b^3c+c^3a)+3 \geq (a^3b+a^3b+b)+(b^3c+b^3c+c)+(c^3a+c^3a+a) \geq$$
$$\geq 3a^2b+3b^2c+3c^2a$$

Bundan yuqoridagi tengsizlik isbotlandi.

77. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini qo'llab,

$$\left(1+\frac{a}{b}\right)\left(1+\frac{b}{c}\right)\left(1+\frac{c}{a}\right)=\left(\frac{a}{b}+\frac{a}{c}+\frac{a}{a}\right)+\left(\frac{b}{a}+\frac{b}{c}+\frac{b}{b}\right)+\left(\frac{c}{a}+\frac{c}{b}+\frac{c}{c}\right)-1 \geq$$
$$\geq 3\frac{b}{\sqrt[3]{abc}}+3\frac{a}{\sqrt[3]{abc}}+3\frac{c}{\sqrt[3]{abc}}-1=3\frac{a+b+c}{\sqrt[3]{abc}}-1 \geq 2\left(1+\frac{a+b+c}{\sqrt[3]{abc}}\right)$$

tengsizlikni hosil qilamiz.

78. Tengsizlikni chap qismini S bilan belgilab va Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llaymiz:

$S\left(\left(a_1^2+a_2^2\right)+\left(a_2^2+a_3^2\right)+...+\left(a_n^2+a_1^2\right)\right)\geq\left(a_1^2+a_2^2+...+a_n^2\right)^2$ bundan

$S\geq\frac{a_1^2+a_2^2+...+a_n^2}{2}\geq\frac{\left(a_1+a_2+...+a_n\right)^2}{2n}=\frac{1}{2n}$ tengsizlik hosil bo'ladi.

79. $a=\frac{x}{y},\ b=\frac{y}{z},\ c=\frac{z}{x}$ deb belgilash kiritsak, u holda yuqoridagi tengsizlik quyidagi $(x-y+z)(y-z+x)(z-x+y)\leq xyz$ ko'rinishga keladi. Bu tengsizlik **69-**misolda **lemma** sifatida isbotlangan.

80. Avvaliga quyidagi lemmani isbotlaymiz:

Lemma. Musbat x,y sonlar uchun $\frac{x^7+y^7}{x^5+y^5}\geq\frac{x^3+y^3}{x+y}$ munosabat o'rinli.

Lemmaning isboti: Haqiqatdan ham

$$\left(x^7+y^7\right)(x+y)-\left(x^3+y^3\right)\left(x^5+y^5\right)=\left(x^7y-x^5y^3\right)+\left(y^7x-y^5x^3\right)=$$

$$=x^5y\left(x^2-y^2\right)-y^5x\left(x^2-y^2\right)=\left(x^2-y^2\right)\left(x^5y-y^5x\right)=xy\left(x^2-y^2\right)^2\left(x^2+y^2\right)\geq0$$

tenglik $x=y$ bo'lganda bajariladi.

Lemmadan foydalansak,

$$\frac{a^7+b^7}{a^5+b^5}+\frac{b^7+c^7}{b^5+c^5}+\frac{c^7+a^7}{c^5+a^5}\geq\frac{a^3+b^3}{a+b}+\frac{b^3+c^3}{b+c}+\frac{c^3+a^3}{c+a}=$$

$$=\left(a^2-ab+b^2\right)+\left(b^2-bc+c^2\right)+\left(c^2-ac+a^2\right)=2\left(a^2+b^2+c^2\right)-(ab+bc+ca)$$

munosabat hosil bo'ladi.

Endi $2\left(a^2+b^2+c^2\right)-(ab+bc+ca)\geq\frac{1}{3}$ ekanligini ko'rsatamiz. Haqiqatdan ham $2\left(a^2+b^2+c^2\right)-(ab+bc+ca)\geq a^2+b^2+c^2\geq\frac{(a+b+c)^2}{3}=\frac{1}{3}$.

81. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligini quyidagi usulda qo'llaymiz:

$$x+y+\frac{2}{x+y}+\frac{1}{2xy}=\frac{x+y}{4}+\frac{x+y}{4}+\frac{1}{x+y}+\frac{1}{x+y}+\frac{x}{2}+\frac{y}{2}+\frac{1}{2xy}\geq$$

$$\geq 7\sqrt[7]{\frac{x+y}{4}\cdot\frac{x+y}{4}\cdot\frac{1}{x+y}\cdot\frac{1}{x+y}\cdot\frac{x}{2}\cdot\frac{y}{2}\cdot\frac{1}{2xy}}=\frac{7}{2}$$

82. Avvaliga quyidagi $f(x)=\frac{x}{1+x^2}$, $x\in[0;1]$ funktsiyani hossalarini o'rganamiz. Ko'rinib turibdiki, ushbu funktsiya ko'rsatilgan oraliqda qavariq funktsiyadir. U holda qavariq funktsiyalar uchun ushbu $g(x)+g(y)+g(z)\leq 3g\left(\frac{x+y+z}{3}\right)$ Iensen tensizligidan foydalanib,

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}=f(a)+f(b)+f(c)\leq 3f\left(\frac{a+b+c}{3}\right)=3f\left(\frac{1}{3}\right)=\frac{9}{10}$$

munosabatni hosil qilamiz.

83. Umumiylikni chegaralamasdan $a\geq b\geq c\geq d$ va $a^2\geq b^2\geq c^2\geq d^2$ deymiz. U holda Chebishev tengsizligini qo'llab, quyidagi

$\left(a^2+b^2+c^2+d^2\right)=\left(a+b+c+d\right)\left(a^2+b^2+c^2+d^2\right)\leq 4\left(a^3+b^3+c^3+d^3\right)$ yoki

$6\left(a^3+b^3+c^3+d^3\right)\geq\frac{3}{2}\left(a^2+b^2+c^2+d^2\right)$ (*) munosabatni hosil qilamiz.

Endi Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llasak,

$\left(a^2+b^2+c^2+d^2\right)\left(1+1+1+1\right)\geq\left(a+b+c+d\right)^2$ yoki $\frac{1}{2}\left(a^2+b^2+c^2+d^2\right)\geq\frac{1}{8}$

(**) munosabat hosil bo'ladi. (*) va (**) larni hadma-had qo'shish natijasida yuqoridagi isboti talab etilgan tengsizlik kelib chiqadi.

84. Istalgan natural n uchun $\frac{1}{2n-1}>\frac{1}{2n}$ ekanligini etiborga olsak,

$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n-1}>\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}$ munosabat o'rinlidir. Endi

$\underbrace{\frac{1}{2}+\frac{1}{2}+...+\frac{1}{2}}_{n}>\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}$ yoki $\frac{1}{2}>\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)\frac{1}{n}$ ekanligidan

foydalansak, U holda

$$1+\frac{1}{3}+...+\frac{1}{2n-1}=\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2n-1}>\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)\frac{1}{n}+\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)=$$
$$=\frac{n+1}{n}\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)$$

bo'ladi.

85. Berilgan tengsizlikda quyidagicha shakl almashtirish bajaramiz.

$\frac{1+a^2-(1-b^2)}{\sqrt{1+a^2}-\sqrt{1-b^2}}\leq\frac{a^2+b^2}{ab}$ yoki $ab\leq\sqrt{1+a^2}-\sqrt{1-b^2}$. Bu yerda

$a=tg\alpha,\ b=\sin\beta \quad \alpha,\beta\in\left(0;\frac{\pi}{2}\right)$ belgilash olsak, u holda

$tg\alpha\sin\beta\leq\frac{1}{\cos\alpha}-\cos\beta$ yoki $\cos(\alpha-\beta)\leq 1$ bo'ladi.

86. Yuqoridagi berilgan shartlarga ko'ra quyidagi tengliklarni yozamiz:

$$2\cdot 2\cdot x_2=2\cdot 2\cdot x_1-3x_1$$
$$2\cdot 3\cdot x_3=2\cdot 3\cdot x_2-3x_2$$

....

$$2\cdot n\cdot x_n=2\cdot n\cdot x_{n-1}-3x_{n-1}$$

va bu tengsizliklarni hadma-had qo'shib,

$3\sum_{i-1}^{n-1}x_i=2\cdot 2\cdot x_1+2\sum_{i=2}^{n-1}x_i-2\cdot n\cdot x_n=1+2\sum_{i=1}^{n-1}x_i-2\cdot n\cdot x_n$ yoki $nx_n=1-\sum_{i=1}^{n}x_i>0$

munosabatni hosil qilamiz.

87. O'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligi va $x\in\left(0;\frac{\pi}{2}\right)$ uchun $\sin x(\sin 2x-1)\le a$ yoki $\sin x\le\sqrt{\frac{tgx}{2}}$ tengsizliklarni o'rinli

$$\sin x_1\cdot\sin x_2\cdot\ldots\cdot\sin x_n\le\left(\frac{\sin x_1+\sin x_2+\ldots+\sin x_n}{n}\right)^n\le$$

ekanligini etiborga olib, $\le\left(\frac{\sqrt{tgx_1}+\sqrt{tgx_2}+\ldots+\sqrt{tgx_n}}{n}\right)^n\cdot 2^{-\frac{n}{2}}\le$

$$\le\left(\sqrt{\frac{tgx_1+tgx_2+\ldots+tgx_n}{n}}\right)^n\cdot 2^{-\frac{n}{2}}\le 2^{-\frac{n}{2}}$$

munosabatni hosil qilamiz.

88. Umumlashgan Koshi-Bunyakovskiy-Shvarts tengsizligini quyidagi usulda qo'llab, $\left(\frac{a^6}{x}+\frac{b^6}{y}+\frac{c^6}{z}\right)(x+y+z)(1+1+1)\ge\left(a^2+b^2+c^2\right)^3$ munosabatni hosil qilamiz. Bundan yuqoridagi isboti talab etilgan tangsizlik kelib chiqadi.

89. Ushbu $x^3+y^3\ge xy(x+y)$ tengsizlikdan foydalanamiz:

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\le\frac{1}{ab(a+b)+abc}+\frac{1}{bc(b+c)+abc}+$$

$$+\frac{1}{ca(c+a)+abc}=\frac{1}{a+b+c}\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)=\frac{1}{abc}$$

90. I-usul: Ushbu $1+2+\ldots+(n-1)=\frac{n(n-1)}{2}$ tenglikdan va o'rta arifmetik va o'rta geometrik miqdorlar haqidagi Koshi tengsizligidan foydalanamiz:

$$\frac{n(n-1)}{2}+x_1+x_2^2+x_3^3+\ldots+x_n^n=x_1+(1+x_2^2)+(2+x_3^3)+\ldots+\left((n-1)+x_n^n\right)=$$

$$=(x_1)+(1+x_2^2)+(1+1+x_3^3)+\ldots+\underbrace{(1+1+\ldots+1+x_n^n)}_{n-1}\geq$$

$$\geq x_1+2x_2+3x_3+\ldots+nx_n$$

$$x_1+x_2^2+\ldots+x_n^n=x_1+\left((x_2-1)+1\right)^2+\ldots+\left((x_n-1)+1\right)^n\geq x_1+\left(1+2x_2-1\right)$$

II-usul: Bernulli tengsizligidan quyidagi usulda foydalansak:

$$x_1+x_2^2+x_3^3+\ldots+x_n^n+\frac{n(n-1)}{2}=x_1+\left((x_2-1)+1\right)^2+\left((x_3-1)+1\right)^3+\ldots+\left((x_n-1)+1\right)^n+$$

$$+\frac{n(n-1)}{2}\geq x_1+\left(1+2(x_2-1)\right)+\left(1+3(x_3-1)\right)+\ldots+\left(1+n(x_n-1)\right)+\frac{n(n-1)}{2}=$$

$$=x_1+2x_2+3x_3+\ldots+nx_n$$

munosabat hosil bo'ladi.

91. Bu tengsizlikni chap qismini S bilan belgilab, quyidagi usulda Koshi-Bunyakovskiy-Shvarts tengsizligini qo'llaymiz:

$$S\left(a^2\left(b^2+1\right)+b^2\left(c^2+1\right)+c^2\left(a^2+1\right)\right)\geq\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2$$

bundan,

$$S\geq\frac{\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2}{a^2b^2+b^2c^2+c^2a^2+1}$$

tengsizlikni va undan

$$S\frac{\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2}{a^2b^2+b^2c^2+c^2a^2+1}\geq\frac{\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2}{\frac{\left(a^2+b^2+c^2\right)^2}{3}+1}=\frac{3}{4}\left(a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\right)^2$$

munosabatni hosil qilamiz.

92. $y_k=\dfrac{1}{x_k}$ almashtirish olsak, u holda,

$$\frac{1}{y_k}=x_k=\frac{1}{1+\frac{a_k}{y_{k-1}}} \Leftrightarrow y_k=1+\frac{a_k}{y_{k-1}}.$$

$y_{k-1}\geq 1,\ a_k\geq 1$ ekanligidan $\left(\frac{1}{y_{k-1}}-1\right)\left(a_k-1\right)\leq 0 \Leftrightarrow 1+a_k\frac{1}{y_{k-1}}\leq a_k+\frac{1}{y_{k-1}}$ bundan

$y_k=1+\frac{a_k}{y_{k-1}}\leq a_k+\frac{1}{y_{k-1}}$ munosabatni hosil qilamiz.

$$\sum_{k=1}^{n} y_k \leq \sum_{k=1}^{n} a_k+\sum_{k=1}^{n}\frac{1}{y_{k-1}}=\sum_{k=1}^{n} a_k+\frac{1}{y_0}+\sum_{k=1}^{n-1}\frac{1}{y_k}=A+\sum_{k=1}^{n-1}\frac{1}{y_k}<A+\sum_{k=1}^{n}\frac{1}{y_k}$$

$t=\sum_{k=1}^{n}\frac{1}{y_k}$ deb belgilash kiritamiz, bundan $\sum_{k=1}^{n} y_k \geq \frac{n^2 t}{},\ t>0$

tengsizlikni hosil qilamiz.

$$\frac{n^2}{t}\leq \sum_{k=1}^{n} y_k<A+t \Leftrightarrow t^2+At-n^2\geq 0 \Leftrightarrow t>\frac{-A+\sqrt{A^2+4n^2}}{2}=\frac{2n^2}{A+\sqrt{A^2+4n^2}}=$$

$$=\frac{2n^2}{\sqrt{A\left(A+\frac{4n^2}{A}\right)}+A}\geq \frac{2n^2}{\frac{A+A+\frac{4n^2}{A}}{2}+A}+\frac{n^2 A}{A^2+n^2}$$

93. $\frac{ab}{3a+b}\leq \frac{12b+a}{49}$ chunki $2(a-2b)^2\geq 0,\ \frac{bc}{b+2c}\leq \frac{8b+9c}{49}$ chunki $2\left(2b-3c\right)^2\geq 0,\ \frac{ac}{c+2a}\leq \frac{18c+a}{49}$ chunki $2\left(3c-a\right)^2\geq 0$, bu tengsizliklarni hadma-had qo'shib, isbotlash kerak bo'lgan tengsizlikni hosil qilamiz.

tenglik $a=2b=3c$ bo'lganda bajariladi.

94. x^9-1, x^3-1 ifodalar ishorasi bir xil hamda $x^4>0$ bo'lgani uchun

$$x^8-x^5-\frac{1}{x}+\frac{1}{x^4}=x^5(x^3-1)-\frac{x^3-1}{x^4}=\frac{(x^9-1)(x^3-1)}{x^4}\geq 0.$$

95. Ravshanki, $\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}=1$.

Koshi-Bunyakovskiy-Shvarts tengsizligiga ko'ra

$$\sqrt{x+y+z}\sqrt{\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}}\geq\sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}\,.$$

96.

$y_i=\frac{1998}{x_i+1998}$ almashtirish kiritamiz.

Ravshanki, $y_i\geq 0,\quad i=1,2,...,n$ va $y_1+y_2+...+y_n=1$.

Demak, $1-y_i=\sum_{j\neq i}y_j$.

Koshi tengsizligiga ko'ra $1-y_i\geq(n-1)\sqrt[n-1]{\prod_{j\neq i}y_j}$.

Bu tengsizliklarni barchasini ko'paytirsak,

$\prod_{i=1}^{n}(1-y_i)\geq(n-1)^n\prod_{i=1}^{n}y_i$ yoki $\prod_{i=1}^{n}\frac{1-y_i}{y_i}\geq(n-1)^n$ tengsizlikni hosil qilamiz.

$\frac{1-y_i}{y_i}=\frac{x_i}{1998}$ bo'lgani uchun bundan $x_1x_2...x_n\geq 1998^n(n-1)^n$ tengsizlikni hosil qilamiz.

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