Principles of Insurance

Abdul H. Rahman; Dick H. Harryvan

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ABDUL H. RAHMAN AND
DICK H. HARRYVAN

PRINCIPLES OF INSURANCE

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He has published in several prestigious refereed academic journals including the Journal of Banking and Finance, Journal of Asset Management, Journal of International Financial Markets, Institutions and Money, Journal of Financial and Quantitative Analysis, Journal of Futures Markets and Review of Economics and Statistics.

Dr. Rahman has consulted for global financial institutions in banking, asset management and insurance and currently facilitate high-level seminars for senior management professionals in retail and corporate banking, life and non-life insurance and institutional asset management.

I dedicate this eBook to my wife Ruth and daughters Sara and Lisa for their unfailing support of my professional pursuits.

Montreal, Canada, 2017.

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Drs. Dick Harryvan

Drs. Dick Harryvan is Vice Chairman of the Supervisory Board of NN Group in the Netherlands an insurance and asset management group operating in 18 countries. He is a former Executive Board Member of ING Group and CEO of ING Direct globally. After obtaining a Masters degree in business economics from Erasmus University, an international career in insurance and retail banking followed. For Nationale Nederlanden Group, he spent nine years in various functions in the US and Canadian business units. The next four years at the international division included amongst others responsibility for setting up new life insurance greenfields in Hungary and Czech Republic after the Wall fell. He then moved to ING Bank to develop retail banking internationally for ING Group. In a three-year incubator period, various concepts were tested with responsibility for the ING Direct pilot in Canada. After the initial success in Canada this concept was rolled out to nine countries growing to 24 million customers in the space of 10 years. Since retirement from the ING Executive Board in 2010 several supervisory board positions for ING Group subsidiaries followed. As one of the three founders, he set up the Retail Banking Academy in London in 2012 with the aim of making retail banking a recognised profession. Other positions include membership of the Board of the Dutch automobile association ANWB and partner and investment committee member at OGC a private equity fund investing in fintech startups.

I dedicate this book my wife Corrie without whom my career would not have been possible.

Rotterdam, The Netherlands, 2017.

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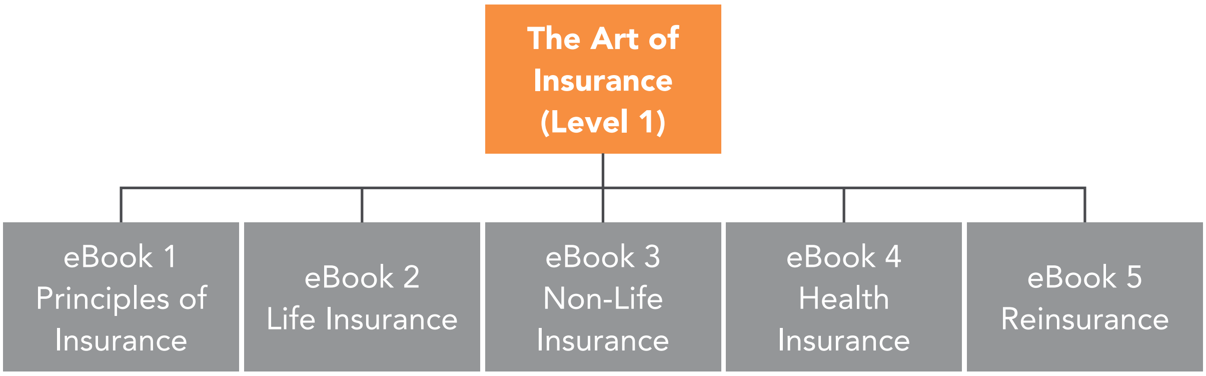
PREFACE

The ART OF INSURANCE is structured according to three levels – I, II and III, each with increasing knowledge and understanding of insurance as a business. The architecture of The Art of Insurance (Level I) is schematically summarised as follows:



Each eBook in Level I is developed independently and is self-contained. Hence, the reader is not required to complete the five eBooks sequentially. In addition, our approach focuses on the 'art' rather than the 'science' of insurance where we emphasise principles, concepts and intuition rather than mathematical proofs of complex theorems. While we recognise that the insurance business is founded on a high level of sophistication in probability and statistics, we avoid an overbearing level of jargon and a misplaced reliance on complex equations and formulae. In line with this objective, we present the most technical and mathematical analysis in the Endnotes Section or in an Appendix at the end of the respective chapter.

This eBook entitled the *Principles of Insurance* comprises five chapters. Chapter 1 considers the key principle of risk pooling – the very essence of insurance. Essentially, risk pooling is a form of diversification where the predictability of losses increases with the size of the pool but is constrained by the degree of correlation in losses. An Appendix to this chapter presents a review of fundamental concepts in probability and statistics that are required in the subsequent chapters. This review includes a discussion of the standard properties of the expectation operator, variance and coefficient of variation (also called 'risk index' in insurance) in the context of identically and independently distributed (IID) portfolio losses via the individual risk model.

Chapter 2 deals with the elements of the theory of interest and the calculation of present and future values of annuities. This is required to calculate insurance premiums based on expected present value of claim losses.

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Chapter 3 deals with premium calculation principles in insurance that include the equivalence principle with adjustments for safety, profit and expense loading. The principles of solidarity and mutuality are also discussed in the context of pricing principles.

Chapter 4 considers two key classes of risk sharing in insurance – reinsurance and deductibles. This chapter provides a brief introduction to reinsurance since a full treatment of this topic is considered in our eBook entitled *Reinsurance*. We provide a more detailed discussion of deductibles and policy loss limits as elements of risk-sharing between policyholders and the insurer. The relationship of deductibles with the principles of anti-selection (called adverse selection in economics) and moral hazard is also covered.

Chapter 6 is a final chapter comprising a list of references used in this eBook.

Finally, in order to inform the reader of the content in each of the subsequent eBooks in Level 1 of the Art of Insurance, we present summary a set of topics as follows:

Title of eBook	Summary of Covered Topics
Life Insurance	Features of Life Insurance Contracts; Modelling Mortality; Calculation of Premiums for Life Insurance; Life Annuities; Variable Annuities with Guarantees and Options.
Non-Life Insurance	Written Premium; Earned and Unearned Premium; Provision for Unexpired Risks; Claim Loss Reserving; Standard methods for IBNR Reserves (Chain Ladder Method, Loss Ratio Method; Bornhuetter-Ferguson Method); Combined Ratio (Claims Ratio, Expense Ratio).
Health Insurance	Features of Health Insurance Contracts; SLT and NSLT categories; Biometric State Models; Sickness Insurance; Accident Insurance; Disability Insurance; Long-Term Care Insurance; Critical Illness Insurance.
Reinsurance	Forms of Reinsurance (Treaty and Facultative; Proportional and Non-proportional); Reinsurance and Probability of Ruin; Annuity Reinsurance; Health Reinsurance; Pricing of Non-Life Reinsurance Contracts.

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1 PRINCIPLES OF RISK POOLING

1.1 INTRODUCTION

Consider the case of individuals who are faced with making savings decisions as part of their personal long-term financial planning. These decisions are facilitated by forecasts of future cash flows arising from employment and investment incomes over their expected remaining lifetimes. But in a world of uncertainty, there is a potential for unexpected interruptions in cash flows arising from random events that include an individual's premature death, temporary or permanent disability, critical illness, outliving retirement resources and damage to his/her property.

Evidently, the prediction of uncertain future events and their related financial and economic impact on the individual's savings decisions is of paramount importance. The actuary is a business professional who serves this objective through the use of mathematics and statistics to predict uncertain future events which are of concern to individuals and businesses in society.

Accordingly, we present in an Appendix to this chapter, the fundamental principles and concepts in probability and statistics which are required for the study of issues related to risk-pooling and risk-sharing in insurance.

This chapter deals with the principle of risk pooling which is considered by experts to be at the core of the insurance business. Olivieri and Pitacco (2010, page 54)[1] states that risk pooling "is of outstanding importance in risk theory and constitutes a kernel feature of the risk transfer process and the insurance process in particular".

Before we discuss this key principle, we introduce some relevant definitions and associated notation so as to clarify insurance concepts and principles presented in this eBook.

1.1.1 INSURANCE CONTRACTS

Chapter 4 of the International Financial Reporting Standards (IFRS)[2] defines an insurance contract as "contract under which one party (the insurer) accepts significant insurance risk from another party (the policyholder) by agreeing to compensate the policyholder if a specified uncertain event (the insured event) adversely affects the policyholder".

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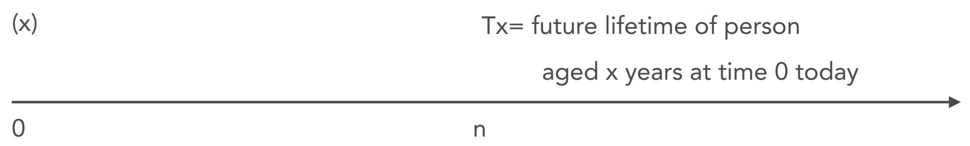
At the date of the policy issue, there is uncertainty regarding the benefit or claim payments (which are loss payments for the insurer). The sources of uncertainty are, in general, the frequency of claims within the same insurance contract, the size of the loss amount (i.e., severity) and the timing of claims. Premium payments made by the insured are also a source of uncertainty for the insurer since the insured may face cash flow problems arising from, for example, becoming unemployed. This is not the case for single premium policies since full payment is made at the time the policy is issued.

To illustrate the sources of uncertainty, we provide brief descriptions of life and non-life (also called, property and casuality) insurance contracts. This is because subsequent eBooks in Level 1 respectively entitled *Life Insurance, Non-Life Insurance and Health Insurance* cover these topics in full and in relevant detail.

1.1.2 LIFE AND NON-LIFE INSURANCE CONTRACTS

A **life insurance contract** is typically based on the mortality (i.e., death), survival or disability of the insured. For example, a **term life insurance contract** is defined as follows:



Notation: (x) represents a policyholder aged x years;
Policy is issued at time 0;
Term of life insurance policy is n years;

A term life insurance policy typically pays a fixed benefit called **sum insured (SI)** if the policyholder dies before the term of the contract ends (i.e., $Tx<n$). There is no payment if the insured survives the term of the contract. **The sources of uncertainty are the timing and the occurrence of the benefit.**

Figure 1.1: Illustration of a Term Life Insurance Contract

A term life insurance contract has a specific term that is set at the time the policy is issued. If we take the case of no fixed term, the payment of the benefit is certain providing the contract is in-force – that is, the insured will die with certainty. The only source of uncertainty is the timing of death. This is the case of a **whole life insurance contract**.

The two contracts considered so far are based on an insured event – the death of the insured.

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A **pure endowment insurance** contract is based on an insured event – survival. That is, payment is made if the insured survives the term of the contract. If the individual dies before the term ends, no payment is made. Since this contract has a fixed benefit amount that is promised to be paid at a fixed point in time, the only source of uncertainty is the survival of the insured.

We describe the main features of **non-life insurance contracts (also called property and casualty (P&C) insurance or general insurance)**. These contracts compensate the insured for insured events that cause damage to his/her property such as fire to residential property, loss from damage to automobiles, theft of personal assets or loss arising from third party liability. In non-life insurance, sources of uncertainty arise from frequency of claims during the term of the contract, the severity of each claim and the timing of each claim.

Comment

Health insurance contracts are classified according to whether they have the attributes of life insurance (and are called 'similar to life techniques (SLT)') or non-life insurance (and are called 'not-similar to life techniques (NSLT)').

Key Takeaway

The sources of uncertainty in life insurance contracts are the **timing** and **occurrence** of the insured event. Both sources of uncertainty underlie term life insurance contracts. The benefit is typically fixed.

Timing of the event is the only source of uncertainty for whole life insurance contracts that are in-force (i.e., still in effect).

For the case of non-life insurance contracts, the **amount (i.e., severity), timing and frequency (i.e., number of claims per policy term)** of the insured event are all sources of uncertainty.

These fundamental insurance contracts illustrate insurance as a risk-transfer mechanism. We describe this process in the next section.

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1.2 PURE RISK AND SPECULATIVE RISK

In general, risk is defined as the potential for the future outcome of a decision to be different from the outcome that is expected. Simply put, in the presence of risk, the actual outcome of a decision is unknown at the time the decision is made. This is because a risky decision involves several outcomes each associated with a probability of occurrence. Hence risk is the possibility of an outcome creating an adverse financial or economic impact.

Before we continue our discussion on the notion of risk, we defined a fundamental concept called a phenomenon. A **phenomenon** is an occurrence that can be observed. For example, a single toss of a coin is a phenomenon since its occurrence can be observed. The death of a person currently aged 40 years over the next five years is an occurrence that can be observed; in the same vein, damage to a specified building from fire over the next year is a phenomenon.

If a phenomenon has more than one outcome, then it is called **stochastic phenomenon**. An **event** is one of these possible outcomes.

In insurance, risk is founded on two characteristics – a phenomenon with random outcomes and the subject associated with the phenomenon. For example, 'the death of a person currently aged 40 years over the next five years' is an occurrence that can be observed and the subject is a person aged 40 years. Similarly, 'damage to a specified building from fire over the next year' can be observed and the subject is a specified building.

These two examples also show that risk has economic or financial consequences for the subject. Hence, we can define risk in insurance (i.e., **actuarial risk**) as follows: An actuarial risk of concern to a participant is an actuarial phenomenon together with one or more random variables, called economic consequences, that assign a positive (negative) number to outcomes that the participant finds desirable (undesirable) and zero to other outcomes" (Society of Actuaries (2008))[3].

We now consider the principle of risk pooling by first differentiating between pure risk and speculative risk.

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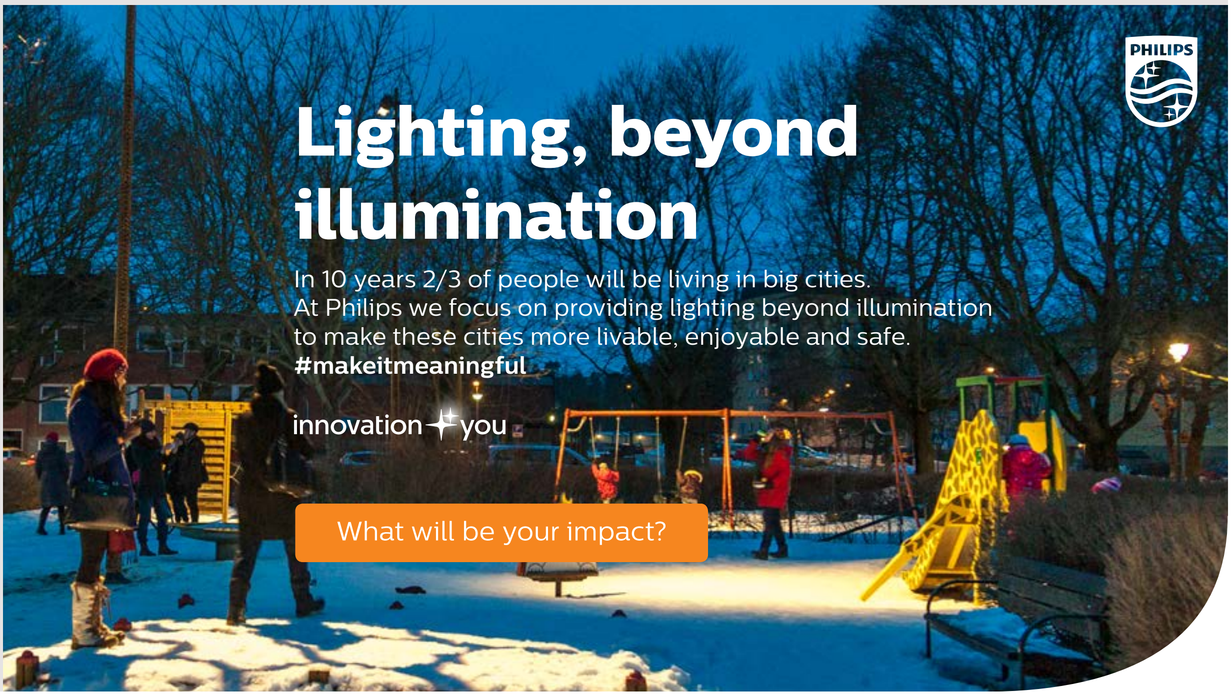
1.2.1 THE TRANSFER OF PURE RISKS

A pure risk has two possible outcomes which are determined by whether a stipulated event occurs or not. Each outcome has a probability measure assigned to it. For example, if the probability that an event occurs is 5%, then there is 95% chance it will not occur. Associated with these two outcomes are economic or financial consequences. There is a financial loss if the event occurs; there is no change in the individual's financial position, otherwise. Importantly, **there is no opportunity for a financial profit**.

To illustrate this definition, take the case of a healthy individual and the event that he/she may be disabled over the next year. A decision-maker assigns a probability to this event which has only two outcomes. If the individual is disabled during the next year, he/she may experience unexpected increases in medical expenses as well as unexpected loss of income both leading to a financial loss. If this individual is not disabled over the next year, there is no financial loss.



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Individuals are likely to mitigate the risk of financial loss through the purchase of insurance. By providing insurance coverage for pure risks from customers, an insurance company receives premiums and bears the risk of payouts. Since insured risks are random variables, not all policyholders will make claims for payments and hence there is an opportunity for the insurer to make a profit. Unlike pure risks, an insurance company faces **speculative risk** which share the two outcomes of pure risks but also permit the possibility of a financial profit. Simply put, an insurance company accepts and transforms a set of pure risks into speculative risks. This risk transfer process between a customer (or policyholder) and the insurance company is illustrated below:

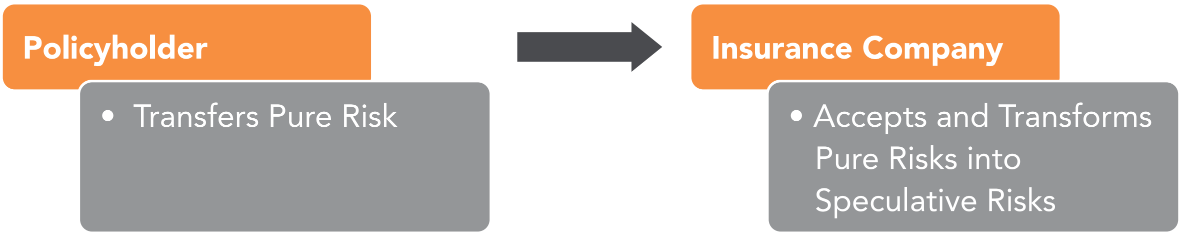

Figure 1.2: Risk Transfer and Transformation Process

A question that arises from this discussion is as follows: **are all pure risks insurable?**

1.2.2 INSURABLE RISKS

Insurable risks are pure risks with specific requirements. These are as follows:

1) The economic consequences of an insurance risk are determinable and measurable. This means that the insurance professional (e.g., actuary) is able to identify and estimate the probability and economic consequences of the phenomenon. For example, 'a person aged 40 years will die within the next five years' is a phenomenon which can be observed and for which the actuary is able to estimate the probability of occurrence from past data (e.g., life tables) or from specific mortality tables.
2) The insurance professional must be able to set premiums for each potential customer or class of customers. This includes the professional's ability to estimate potential claim losses which must be random and unintentional. For example, suicide is not covered in life insurance contracts since it is intentional.
3) Catastrophic events such as earthquakes and war are typically uninsurable in private insurance markets since these create general uncertainty. Catastrophic events create a high dependence among claim losses.

Comment

Unemployment is a pure risk which is not typically insurable in the private insurance market. This is because the employee can create his/her payoff by misbehaviour or by voluntarily quitting his/her job. In addition, there can be catastrophic risk associated with unemployment especially during market recessions and economic downturns.

Before we discuss the principle of risk pooling, it is necessary to introduce some fundamental concepts in probability and statistics. Our presentation is intuitive with technical details provided in the Appendix to this chapter.

1.3 CONCEPTS IN PROBABILITY AND STATISTICS

We begin with definitions of key concepts that are fully discussed in 'Principles Underlying Actuarial Science' by Mark Allaben *et al*, published in the July 2008 issue of the Actuarial Practice Forum, Society of Actuaries.

a) A **random variable** X is a function that assigns a numerical value to every possible outcome of a **phenomenon**. This numerical value is called a **probability** measure which is the likelihood of the occurrence of an event. It's value ranges from zero to unity with higher values indicating higher likelihood of occurrence. A value of unity means that the event is certain. Reflecting uncertainty, values of a probability measure range between 0 and 1 and sum to 1.
b) Probability as **relative frequency** means that we calculate the occurrence of the event as a ratio of the total number of outcomes. In most cases in actuarial science, the estimate of the probability measure is **subjective** in that it depends on the decision-maker's experience and expertise about the phenomenon.

Consider the following example that illustrates the concept of a phenomenon and the associated random variable.

Example 1

Toss a coin twice. This is a phenomenon since its occurrence can be observed. The set of all possible outcomes {HH, TT, HT, TH} where H = head and T = tail is called a **sample space**. With each outcome in the sample space, we assign a probability of occurrence. For this example, we calculate the probability as the relative frequency.

Sample Space	HH	TT	HT	TH
Probability of Outcome	¼	¼	¼	¼

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Here is a diagram illustrating a formal definition of a random variable (X).

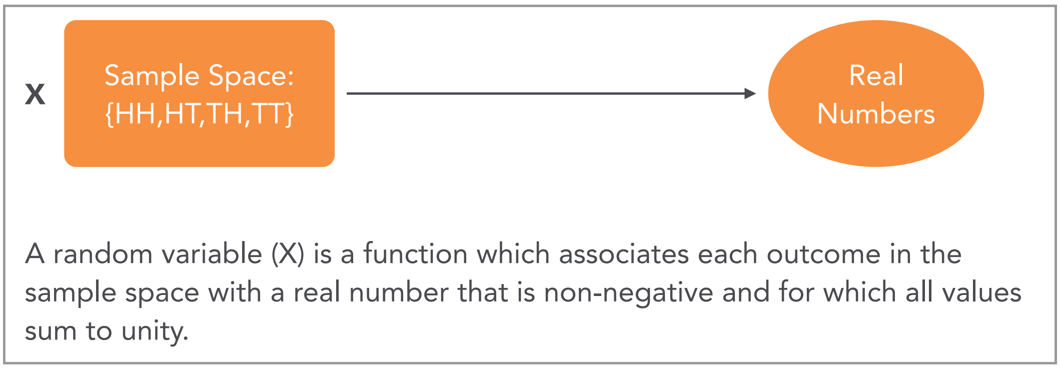

Figure 1.3: Random Variable

Suppose that we are interested in a particular event – that is, X which is the number of heads in each toss of a coin twice. Then we can tabulate the possible outcomes as follows:

Sample Space	HH	TT	HT	TH
X= Number of Heads	2	0	1	1

We observe that the probability that X = 0 is ¼ since there are 4 possible outcomes and there is only one outcome for which there is no head (TT). Similarly, the probability that X = 1 is 2⁄4 since there are two outcomes with exactly one head. Finally, the probability that X = 2 is ¼ since there is only one outcome with two heads.

This **probability distribution** is illustrated as follows:

X	0	1	2
Probability of X ≡ P(X)	¼	2/4	¼

Consider a general case as follows:

A random variable X has *n* possible outcomes represented as $\{X_1, X_2, \ldots, X_n\}$ with probability $\{p_1, p_2, \ldots, p_n\}$. The probability distribution is as follows:

X	X_1	X_2	…	…	…	…	…	…	X_n
P(X)	p_1	p_2	…	…	…	…	…	…	p_n

We now show the calculation of the **expected value** for a random variable, X.

1.3.1 EXPECTED VALUE OF A RANDOM VARIABLE

The **expected value E(X)** of a random variable is defined as follows:

$$E(X) = p_1X_1 + p_2X_2 + ... + p_nX_n \tag{1.1}$$

That is, each outcome is multiplied by its probability of occurrence and then all terms are added as shown in (1.1). Here is an example:

Example 2

A random variable X is defined as the number of claims made by a policyholder over the next year. Based on historical data, the probability distribution of X is shown below.

X	0	1	2	3
P(X)	0.50	0.25	0.20	0.05

The data shows that there is a 50% probability of no claim, 25% probability of a single claim, 20% probability of 2 claims and 5% probability of 3 claims. Note that X = number of claims is a discrete random variable.

Using (1.1), the expected number of claims, E(X) = (0.50 × 0) + (0.25 × 1) + (0.20 × 2) + (0.05 × 3) = 0.80.

We present properties of the expectation operator in the Appendix. We single out one important result that is stated as follows:

For random variables $\{X_1, X_2, ..., X_n\}$, let the sum of these variables be given as follows: $S = X_1 + X_2 + ... + X_n$. Then, $E(S) = E(X_1) + E(X_2) + ... + E(X_n)$ (1.2)

This a general result and holds true for any set of random variables.

Here is an example illustrating (1.2).

Example 3

For any two random variables X and Y where E(X) = 4 and E(Y) = 7, property (1.2) shows that:

$$E(X+Y) = E(X) + E(Y) = 4 + 7 = 11.$$

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Special Case

Assume that each random variable in (1.2) has the same expected value, i.e., $E(X_1) = E(X_2) = ... = E(X_n)$. Then $E(S) = E(X_1) + E(X_2) + ... + E(X_n)$ which implies that

$$E(S) = n \times E(X_1). \tag{1.3}$$

Here is an example:

Example 4

Consider an insurance portfolio comprises 100 policies each with a future claim loss over the next year represented by X_i, i=1,2,…,n. Let S = total portfolio claim loss so that $S = X_1 + X_2 + ... + X_n$. Assume that the expected value of each policy claim loss is identical and equal to \$200. By (1.3), the expected value of total portfolio claim loss is equal to $E(S) = n \text{ x } E(X_1) =$ 100 × \$200 = \$20,000.

Another important statistic that is useful in risk measurement in insurance is called the variance of a random variable.

1.3.2 VARIANCE OF A RANDOM VARIABLE

We know that a random variable can take on more than one value at some point in time in the future. Furthermore, each value of the random variable has a certain probability associated with it. This means that the values of the random variable have a degree of uncertainty with respect to the actual outcome.

As an example, consider a random variable X which represents the claim loss to the insurer from having issued a policy to the insured. The insurance company makes premium decisions based on the expected value of the claim loss. The risk to this decision is based in part, on the variability of possible values of the claim loss in relation to the expected value.

A measure of this dispersion or degree of variability of potential outcomes is called **variance.** A variance with a zero value indicates no dispersion around the expected value. A large value for the variance indicates that the expected dispersion is significant and hence decisions based on the expected value are relatively risky.

In simple words, the variance is a measure of the expected dispersion of the values of X about its expected value. A simplified formula is stated as follows:

$$\mathbf{Var(X) = E(X^2) - \left((E(X)\right)^2.} \tag{1.4}$$

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This formula defines the variance of a random variable X, in terms of the expected value of X and the expected value of X^2.

We illustrate the formula in (1.4) by an example.

Example 5

Refer to the information in Example 2.

X	0	1	2	3
X^2	0	1	4	9
P(X)	0.50	0.25	0.20	0.05

$E(X^2) = (0.50 \times 0) + (0.25 \times 1) + (0.20 \times 4) + (0.05 \times 9) = 1.5.$

E(X) = 0.80 from Example 2. Therefore, $\mathrm{Var}(X) = E(X^2) - \left((E(X)\right)^2 = 1.5 - (.80)^2 = 1.5 - .64 = 0.86$

Comment

One disadvantage of the variance is that its unit of measurement is not monetary. If the random variable X represents a claim loss amount stated in dollars, then its expected value is also expressed in dollars. But the unit of measurement of the variance is dollar squared, which is not monetary. For this reason, the standard deviation is considered.

The **standard deviation** is equal to the square root of the variance and is typically denoted as **σ (X)**. Clearly, by taking the square root, the unit of measurement of the standard deviation is monetary and it is the same as random variable X. Based on Example 5, $\sigma(X) = \sqrt{(0.86)} = 0.9274$.

With this presentation of expected value and variance of a random variable as well as additional information in the Appendix to this chapter, we now consider a key question:

How does the insurance company manage a portfolio including speculative risk?

The answer leads to a consideration of a fundamental principle of insurance – the principle of risk pooling.

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1.4 THE PRINCIPLE OF RISK POOLING

The concept of risk pooling in insurance is generally considered as the essence of insurance. Risk pooling is typically defined as a risk management process whereby similar actuarial risks are combined. We consider an aggregate loss analysis within the **individual risk model** in insurance.

1.4.1 AGGREGATE LOSS ANALYSIS

Consider a portfolio of an insurance company comprising *n* issued policies over a fixed period of time.

The aggregate loss random variable (S) is defined as follows:

$$S = X_1 + X_2 + ... + X_n \qquad \text{where } n \text{ is equal to the number of insureds.} \tag{1.5}$$

Hence each loss variable $X_i \geq 0$. This is because not all policyholders are likely to make a claim.

Assumptions for the Individual Risk Model

a) The loss variables, $X_1, X_2, ..., X_n$ are independent.
b) The claim losses are identically distributed. Assumptions a) and b) mean that we have identically distributed and independent (IID) losses. As shown in section A.2 of the Appendix to this chapter,
c) n is fixed; assumption b) is not required in the individual risk model

$$\begin{aligned} &E(S) = n \times E(X_1) = n \times x \times p; \\ &\sigma(X) = \sqrt{n} \times \sigma(X_1) = x \times \sqrt{np(1-p)}. \end{aligned} \tag{1.6}$$

Comment

a) From (1.6), the expected value of the total portfolio loss $E(S)$, increases with the size (*n*) of the portfolio.
b) Also from (1.6), the standard deviation of the total portfolio loss, $\sigma(X)$ increases with the square root of the size, $\sqrt{n}$. So the standard deviation increases at a slower rate than the expected value.

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Key Takeaway

Statements a) and b) above conclude that as the portfolio size increases, both the standard deviation and expected value of portfolio losses increase but the standard deviation increases at a slower rate than the expected value.

This result summarised in in the previous Key Takeaway, leads to a key statistic that is a measure of the portfolio's relative risk. It is called the **coefficient of variation (CV)** in statistics and a **risk index** in actuarial science.

1.4.2 RELATIVE PORTFOLIO RISK (CV)

The coefficient of variation (CV) of aggregate portfolio loss (S) is defined as the standard deviation of S divided by the expected value of S; that is, $CV(S) = \frac{\sigma(X)}{E(S)}$. (1.7)

The coefficient of variation or risk index for the aggregate portfolio loss (S) is:

$$CV(S) = \frac{\sigma(X)}{E(S)} == \frac{1}{\sqrt{n}} \times \sqrt{\frac{1-p}{p}} \tag{1.8}$$

The calculation details that lead to (1.8) are presented in section A.2 of the Appendix.

Equation (1.8) is illustrated in the table below for the case where we assume (without any loss of generality) that $p = 1\%$.

Value of *n*	1,000	10,000	100,000	1,000,000	10,000,000
CV(S)	0.3146	0.0995	0.0315	0.0099	0.0031

Observe from this table that as the number of policies increases steadily, relative risk declines towards a zero value. The number of policies matters in risk pooling.

The result implied from this table is key. It states that:

The insurer's relative risk declines to zero as portfolio size (*n*) increases to infinity. This is the principle of risk pooling and is of paramount importance in insurance.

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Comment

Risk pooling is dependent on portfolio losses being identically distributed and independent (IID). For example, if claim losses are dependent in the sense that the initiation of a claim automatically triggers other claims from insureds in the portfolio, then risk pooling will be less effective. In fact, contagion risks arising from catastrophic events (e.g., terrorism, epidemics, earthquakes, flooding) violate the independence assumption required for risk pooling.

Key Takeaway

For IID portfolio losses, the relative portfolio risk declines to zero as the portfolio size increases to infinity. This is a statement of the principle of risk pooling – the essence of insurance.

Comment

In chapter 3 we consider premium calculation principles and continue the discussion on risk pooling. In particular we show that, if the insurer's probability of insolvency (i.e., ruin probability in insurance language) is fixed, the safety loading factor converges to zero as the number of insureds approach infinity. This implies that each insured can obtain lower premiums as the size of the pool increases.

This concludes Chapter 1. In chapter 2, we consider the principles and concepts in the theory of interest with applications to annuities which serve to explain premium calculation principles in insurance covered in Chapter 3.

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APPENDIX

PRINCIPLES OF PROBABILITY AND STATISTICS: A REVIEW

This appendix summarises some of the technical results in probability and statistics required for a docussion of the implications of the principle of risk pooling.

We now introduce the Bernoulli random variable that is important in modeling insurance risk which typically has two outcomes for the insurance company – an outcome that results in a claim loss or an outcome that results in no claim loss.

A.1 Bernoulli Distribution

To motivate the definition of a Bernoulli variable, consider an event E defined as the death of a person currently aged 40 years over the next 10 years. This event has two possible outcomes – the person will die within the next 10 years or the person survives the next 10 years.

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Let p is the probability that the event E occurs. Associated with event E is a payoff or benefit variable for the policyholder which is a claim loss variable from the perspective of the insurer.

From the perspective of the insurer, let X be the claim loss variable with a value of $\boldsymbol{x}$ if E occurs and a value of zero otherwise. The random claim loss variable X is stated as follows:

$$X = \begin{cases} \text{x with probability p} \\ \text{0 with probability } 1 - \text{p} \end{cases}$$

Equivalently, $P(X = x) = p$ and $P(X = 0) = 1\text{-}p$

Comment

The traditional definition of an **indicator** variable is that it is a random variable that takes on a value 1 if success and 0 if not success (i.e., failure). Notice that in our definition of a Bernoulli variable in insurance, we specify a financial consequence to the outcomes.

We now derive the expected value and variance of the insured loss variable X.

A.1.1 Expected Value and Variance of the Bernoulli Loss Variable, X

We can write the distribution of a Bernoulli variable X as follows:

Claim Loss Variable, X	$X = x$	$X = 0$
Corresponding probability	p	1-p

The expected value of X is: $E(X) = (x \times p) + (0 \times (1 - p)) = xp$.
The formula for the variance of X that is presented in this chapter is $Var(X) = E(X^2) - (E(X))^2$.
Applying this formula for the loss variable X, $E(X) = xp \text{ and } E(X^2) = x^2 \times p + 0^2 \times (1 - p) = x^2 p$.
So $Var(X) = x^2 p - (xp)^2 = x^2 p(1 - p)$.

The standard deviation is defined as the square root of the variance. In this case, the standard deviation $\sigma(X) = x \times \sqrt{p(1 - p)}$.

Summary

For a Bernoulli variable X and probability $\boldsymbol{p}$ of outcome x;

$E(X) = xp$; $Var(X) = x^2p(1\text{-}p)$; $\sigma(X) = x \times \sqrt{p(1 - p)}$.

We now consider the binomial distribution which is useful from a portfolio perspective.

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A.2 Binomial Distribution

A Bernoulli distribution is based on a single trial of an experiment. For example, toss a coin once with the possibility of a head or a tail. But if we repeat this experiment, for example, 10 times then we have a **binomial distribution**.

Similarly, a life insurance policy based on the event that a **single individual** currently aged 40 years will die over the next years. The person will die or survive over the next 10 years. This is a Bernoulli distribution. Now consider a portfolio of *n* policies with IID claim losses. Then we have a binomial distribution. We summarise the expected value, variance and standard deviation of a binomial variable as follows:

Summary

For a Binomial variable X:

$$E(X) = nxp; \quad Var(X) = nx^2p(1\text{-}p); \ \sigma(X) = x \times \sqrt{np(1-p)} \tag{1.1}$$

We conclude with a key result underlying risk pooling.

Key Result

a) Assume that each claim loss has the same expected value, i.e., $E(X_1) = E(X_2) = \ldots = E(X_n)$. For $S = X_1 + X_2 + \ldots + X_n$; then $E(S) = n \times E(X_1)$.

b) Assuming mutual independence of claim losses and each has the same variance i.e., $Var(X_1) = Var(X_2) = \ldots = Var(X_n)$. For $S = X_1 + X_2 + \ldots + X_n$; then $Var(S) = n \times Var(X_1)$.

Note: The assumption of identically distributed (implying identical expected value and variance) and independent losses is denoted as IID claim losses.

Conclusion using (1.1):

$$CV(S) = \frac{\sigma(X)}{E(S)} = \frac{\sqrt{nx^2p(1-p)}}{nxp} = \frac{x\sqrt{np(1-p)}}{nxp} = \frac{1}{\sqrt{n}} \times \sqrt{\frac{1-p}{p}}.$$

Hence for any value of *p*, $\lim_{n\to\infty} CV(S) = \frac{1}{\sqrt{n}} \times \sqrt{\frac{1-p}{p}} \to 0$.

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2 THE THEORY OF INTEREST AND ANNUITIES[4]

2.1 INTRODUCTION

When dealing with money paid and/or received at different points in time, the decision-maker faces a fundamental fact: *different amounts of money are comparable only at the same point of time.* However, in a typical insurance agreement, the insurer agrees to pay the policyholder (or the designated beneficiary) a specified benefit or claim amount in the future contingent on the occurrence of an event (e.g., fire, death, survival, disability). The insured will make either a single premium payment at the date of the policy issue or a stream of periodic payments over time. This non-synchronous timing of cash flows requires that the decision-maker understand and apply the principles and concepts of the theory of interest and associated financial mathematics.

This introduction presents a brief review of the theory of interest which leads logically to the definition and illustration of the key concept of 'actuarial present value' necessary for the development of principles of pricing in insurance presented in Chapter 3.[5] The theory of interest summarises the concepts of nominal and effective interest rates as well as effective discount rates that underline the monetary accumulation process over time. Our discussion of the theory of interest also considers the calculation and interpretation of the present and future values of annuities immediate and annuities due that are required to convert single premiums into regular premiums (e.g., monthly or annual) over the lifetime of an insurance policy.

The theory of interest describes the process by which money is accumulated over time. Simple interest and compound interest are two central concepts that describe the accumulation function linking the initial invested amount to the accumulated amount at some point of time in the future. We now present the accumulation function and process using common actuarial terminology.

2.2 THE ACCUMULATION FUNCTION

We first consider the case of simple interest.

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2.2.1 THE CASE OF SIMPLE INTEREST

When money is deposited in a savings account based on **simple interest**, the same amount of interest income is credited to the account each period. Here is an illustration of this concept where $1,000 is deposited in a savings account paying a simple annual rate of interest of 4% for 5 years:

Year	Amount at the Beginning of Each Year	Interest Income Earned Based on Initial Deposit and 4% Simple Interest	Amount Accumulated at the End of Year
1	$1,000	0.04 x $1,000 = $40	$1,040
2	$1,040	0.04 x $1,000 = $40	$1,080
3	$1,080	0.04 x $1,000 = $40	$1,120
4	$1,120	0.04 x $1,000 = $40	$1,160
5	$1,160	0.04 x $1,000 = $40	$1,200

Observe that a constant amount ($40) is earned during the first year and for each subsequent year. Using common actuarial terminology, the accumulation process is described as follows:

The Accumulation Function for Simple Interest

A(0) is the initial investment based on a stated rate of interest per period (*i%);*
a(t) is the accumulation function at the end of *t* periods and,
A(t) is the accumulated amount at the end of *t* periods.

Then $\mathbf{A(t) = A(0) \times a(t) \text{ where } a(t) = (1 + it)}$. (2.1)

Returning to our illustration above, where A(0) = $1,000; i=4% and t=5. Then the accumulation function is a(5) = 1+ 0.04 x 5 = 1.20 and A(5) = A(0) x a(5) = $1,000 x 1.20 = $1,200.

We now introduce the case of compound interest.

2.2.2 THE CASE OF COMPOUND INTEREST

With **compound interest**, all interest income earned in previous periods are reinvested in subsequent periods. Simply put, **compounding** is a process wherein interest income earned at the end of each year is reinvested at the beginning of the next year to earn further interest income.

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We illustrate the compounding process using the previous illustration where $1,000 is invested at an annual rate of 4% for 5 years.

Year	Amount at the Beginning of Each Year	Interest Income Earned Based on Initial Deposit and 4% Compound Interest	Amount Accumulated at the End of Year
1	$1,000	0.04 x $1,000 = $40	$1,040
2	$1,040	0.04 x $1,040 = $41.60	$1,081.60
3	$1,081.60	0.04 x $1,081.60 = $43.264	$1,124.86
4	$1,124.86	0.04 x $1124.86 = $45.00	$1,169.86
5	$1,169.86	0.04 x $1,169.86 = $46.79	$1,216.65

Observe that the amount accumulated at the end of year 1 becomes the amount invested at the beginning of year 2. Interest income in year 2 is calculated on this accumulated amount. This accumulation process is called compounding and adds an additional $16.65 over 5 years compared to simple interest.

This accumulation process is described as follows:

The Accumulation Process for Compound Interest

$A(0)$ is the initial investment based on a stated rate of interest per period ($i\%$);
$a(t)$ is the accumulation function at the end of t periods and,
$A(t)$ is the accumulated amount at the end of t periods.

Then $\mathbf{A(t) = A(0) \times a(t) \text{ where } a(t) = (1+i)^t}$. (2.2)

Returning to our illustration above, $A(0) = \$1,000$; $i=4\%$ and $t=5$. Then the accumulated function for five years is $a(5) = (1.04)^5 = 1.21665$. Hence, the accumulated amount at the end of five years is $A(5) = A(0) \text{ x } a(5) = \$1,000 \text{ x } 1.21665 = \$1,216.65$.

A graphical depiction of the accumulation process described by equation (2.2) is presented below in Figure 2.1.

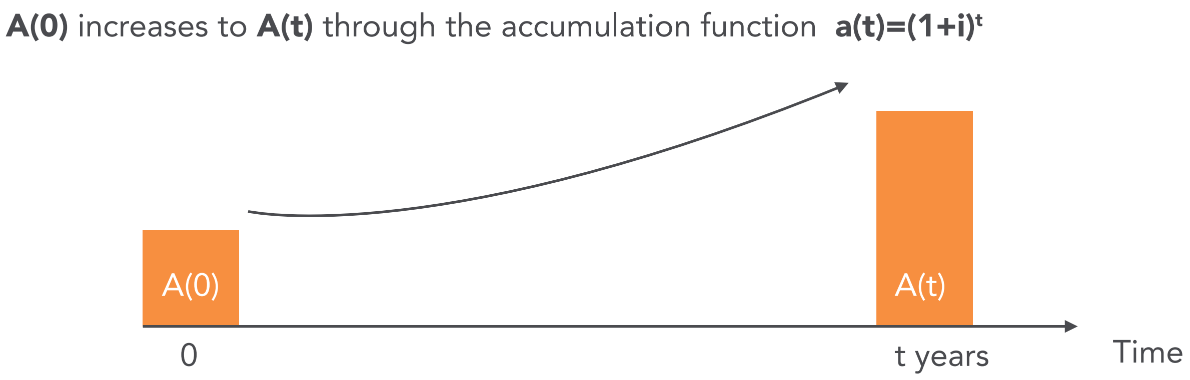

Figure 2.1: Illustration of Accumulation Function

Here is an example illustrating equation (2.2) for the calculation of the accumulated amount, A(t).

Example 1

An amount of €1,000 is deposited in a savings account paying an annual interest rate of 6%. What is the accumulated amount at the end of 12 years?

Based on equation (2.2), A(0) = €1,000 and the accumulated amount is A(12) = €1,000 x $(1+0.06)^{12}$ = €1,000 x 2.012196 = €2,012.196.

A special case of equation (2.2) is presented to answer the question: for a specified annual interest rate, how long does it take for money invested at this interest rate to double in amount?

The Rule of 72

Observe that the accumulated amount at the end of 12 years has approximately doubled. There is a simple rule that answers the question: *how long does it take for money to double for a specified and known interest rate?*

An approximate rule of thumb is $\mathbf{n = \frac{72}{i}}$ where *i* is stated in percentage terms (not a decimal). Hence for annual interest rate of 6%, the number of years is 72/6 = 12 years.[6] Similarly, for an 8% interest rate, the number of years is 72/9 = 8 years.

Equation (2.2) permits the calculation of a present value of a single future amount.

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2.2.3 THE PRESENT VALUE, A(0)

Note that from equation (2.2), we can obtain a formulation for the present value of a single future value A(t) as follows:

$$\mathbf{A(0)=\frac{A(t)}{(1+i)^t}} \tag{2.3}$$

Example 2

An insurance agent offers a plan that, under certain conditions will pay $100,000 at the end of 10 years. If the annual interest rate is 5%, what is the present value of the plan – that is, what is the fair value of this plan today?

We want calculate the present value A(0) based on the information that A(10) = $100,000,the time period is 10 years and the annual interest rate is 5%. Substituting into equation (2.3), we obtain:

A(0) = $\frac{\$100{,}000}{(1.05)^{10}}=\frac{\$100{,}000}{1.628895}=\$61{,}391.33$, indicating the price of the plan today is $61,391.33.

Comment on Common Actuarial Notation

In actuarial notation, $\mathbf{v\equiv\frac{1}{1+i}}$. So $\mathbf{A(0)=\frac{A(t)}{(1+i)^t}=A(0)\times\frac{1}{(1+i)^t}=A(0)\times v^t}$ (2.4)

We can interpret $\mathbf{v\equiv\frac{1}{1+i}}$ as the present value of 1 monetary unit (e.g. $, €) that is paid one year from today.

We now define **effective rate of interest** and differentiate this concept from the **nominal rate of interest** in the context of the frequency of compounding.

2.3 EFFECTIVE RATE OF INTEREST

We begin some common notation and terminology in the theory of interest. **Nominal interest rate** is the stated or quoted **annual** interest rate on, for example, loans and savings accounts. The **effective interest rate** takes into account the **frequency of compounding per year**.

To understand the effect of the frequency of compounding per year on the accumulated value, consider the case where the nominal annual interest rate is 6% compounded semi-annually for a period of 12 years. The accumulation factor = $\left(1+\frac{0.06}{2}\right)^{2\times12}=(1.03)^{24}=2.03274$ which is evidently higher than the accumulation factor of 2.01296 in example 1 when compounding was on an annual basis.

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If the frequency of compounding is monthly, the accumulation factor is given by the equation, $\left(1+\frac{0.06}{12}\right)^{12\times 12} = (1+0.005)^{144} = 2.050751$. We note that the accumulation factor increases with compounding frequency and so a higher accumulated value is obtained.

Comment

a) The annual nominal interest rate is the effective interest rate when interest payment is made annually or equivalently, when the compounding frequency is annual. For example, if the annual nominal rate of interest is 6% compounded annually, then the effective annual rate of interest is also 6%.

b) All else equal, the higher the frequency of compounding, the higher the accumulation factor.

This brings us to the formal definition of the effective interest rate corresponding to the annual nominal rate of interest.

In common actuarial notation, the annual nominal interest rate when the frequency of compounding during the year is *m* is given by the symbol, $\mathbf{i^{(m)}}$. The effective interest rate is given by the formula:

$$i_{eff} = \left(1+\frac{i^{(m)}}{m}\right)^{m} - 1 \tag{2.5}$$

Example 3

What is the effective annual interest rate for a loan that is quoted at 6% compounded daily (365 days)? In this case, $i^{(365)} = 0.06$ where $m = 365$. By equation (2.5), the effective annual interest rate is equal to: $i_{eff} = \left(1+\frac{0.06}{365}\right)^{365} - 1 = 1.061831 - 1 = 6.1831\%$.

2.3.1 SPECIAL CASE OF CONTINUOUS COMPOUNDING

Example 3 considered the case of daily compounding. The case of continuous compounding is when the frequency of compounding per year increases towards infinity. Let us discover the result from a simple illustration where the nominal interest rate is 6%. The data is shown in the table below:

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Nominal Interest Rate and Frequency of Compounding per Year	Formula for Effective Interest Rate	Effective Interest Rate
$i^{(365)} = 0.06$	$= \left(1 + \frac{i^{(365)}}{365}\right)^{365} \quad 1 = \left((1 + \frac{0.06}{365}\right)^{365} - 1$	6.1831%
$i^{(10,000)} = 0.06$	$= \left(1 + \frac{i^{(10,000)}}{10,000}\right)^{10,000} \quad 1 = \left((1 + \frac{0.06}{10,000}\right)^{10,000} - 1$	6.1836%
$i^{(100,000)} = 0.06$	$= \left(1 + \frac{i^{(100,000)}}{100,000}\right)^{100,000} \quad 1 = \left((1 + \frac{0.06}{100,000}\right)^{100,000} - 1$	6.1837%
$i^{(\infty)} = 0.06$	$\lim_{m \to \infty} \left(1 + \frac{i^{(m)}}{m}\right)^{m} = e^{i(\infty)} - 1 = e^{0.06} - 1$ where e = 2.718281828 approximately (Euler's constant)	6.1837%

This illustration shows that as the frequency of compounding per year becomes larger and approaches infinity, the effective interest rate approaches a bounded value, which is 6.1837%. As the last row indicates, this bounded value is obtained from the equation:

$$e^{i(\infty)} - 1 \text{ where e = 2.718281828 approximately known as Euler's constant.}^{7} \qquad (2.6)$$

Comment

The nominal annual interest rate that is compounded continuously, $i^{(\infty)}$ is called the **force of interest** with a symbol δ commonly used on actuarial science.
Here is an example of continuous compounding:

Example 4

What is the effective annual rate of 2% compounding continuously?

By equation (3.6), $e^{i(\infty)} - 1 = e^{0.02} - 1 = 1.0202 - 1 = 2.02\%$

We now reconsider the accumulation function by taking into consideration the frequency of compounding.

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2.4 ACCUMULATION FACTOR FOR COMPOUNDING FREQUENCY (M >1) PER YEAR

We note that equation (2.2) is the accumulation factor for the case when (*m=1*). For the case *m>1*, the accumulation factor is $\left(1+\frac{i^{(m)}}{m}\right)^{m\times t}$ where *n* is the number of time periods (e.g., years), m is the compounding frequency per year and $i^{(m)}$ is the annual nominal rate of interest. Substituting in equation (2.2) we obtain:

$$\mathbf{A(t)=A(0)\times\left(1+\frac{i^{(m)}}{m}\right)^{m\times t}} \tag{2.7}$$

Example 5

What is the accumulated amount at the end of 5 years if £1,000 is deposited today at 6 % per year compounded quarterly?

In this case, A(0)= £1,000; nominal interest rate = 6%; m=4 and t=5.

Hence, $A(5) = £1,000 \times\left(1+\frac{0.06}{4}\right)^{20}$ =£1,000 x 1.346855= £1,346.855.

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Comment

For the case of continuous compounding, the formula in (2.7) becomes:

$$\mathbf{A(t) = A(0) \times e^{\delta \times t}} \text{ where } \delta \text{ is the force of interest} \tag{2.8}$$

Example 6

Refer to the information in example 5 where £1,000 is deposited for 5 years at an annual rate of 6% but compounded continuously. Then $A(5) = 1{,}000 \times e^{0.06 \times 5}$=£1,000 x 1.349859 = £1,349.859

We now consider the effective rate of discount and relate it to the effective rate of interest.

2.5 EFFECTIVE RATE OF DISCOUNT

The process of accumulating money over time is the opposite of the process of discounting. This is observed from equations (2.2) and (2.3).

The effective rate of discount is defined as the interest income earned at the end of a period divided by the total amount at the **end** of the period. Here is an illustration:

A(0) Interest Rate of (i%) A(1) = A(0)(1+i)

0 1 Time

Interest Income Earned = A(0) x i and total amount at the end of the period is, A(0) =A(0) x (1 + i).

The effective rate of discount is: $\mathbf{d = \frac{A(0) \times i}{A(0) \times (1 + i)} = \frac{i}{1 + i}}$. (2.9)

Comment

Since $\mathbf{v = \frac{i}{1 + i}}$, then **d = iv**. It is also clear that **d = 1 – v** (2.10)

Figure 2.2: The Effective Discount Rate

This chapter concludes with a consideration of the present and future values of **annuities** that will be utilised to convert potentially prohibitive single premiums into monthly or annual premiums.

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2.6 PRESENT VALUE OF AN ANNUITY IMMEDIATE

Consider an individual who completed a financial transaction that promises to pay a fixed equal amount of money at equally-spaced time intervals (e.g., annually) for a specified number of time periods (e.g., years). Here is a diagram of a financial transaction involving a payment of C each year for *n* years where payment is made at the end of each year. This stream of payments is called an **annuity immediate**.

Figure 2.3 presents a graphical depiction of these periodic cash flows, C.

Figure 2.3: Annuity Immediate

In actuarial notation, the present value of an annuity immediate shown in equation (2.11) is represented by $\mathbf{a_{\overline{n}|i}}$.The present value of an annuity immediate for equal periodic payments (C) and interest rate *(i)* is given by the formula:

$$\mathbf{a_{\overline{n}|i} = C \times \frac{1 - v^n}{i}} \text{ where } \mathbf{v = \frac{i}{1+i}}. \tag{2.11}$$

The expression $\mathbf{\frac{1 - v^n}{i}}$ is called the present value factor and so the present value of an annuity immediate is the product of the equal periodic payment C and the present value factor. Here is an example which illustrates the application of this equation.

Example 7

An annuity immediate comprises an equal annual payment of \$500 for a period of 10 years. The appropriate interest rate is 6%. The present value factor is $\frac{1-\left(\frac{1}{1+0.06}\right)^{10}}{0.06} = 7.360087$.

The present value of the annuity immediate is the product of the annual payment and the present value factor $= \$500 \times 7.360087 = \$3,680.044$.

Comment

We will consider **deferred annuities** in the eBook entitled *Life Insurance.* Suffices to state that deferred annuities are those where the first payment is deferred to a point in the future. For an annuity immediate that pays \$500 at the end of each month for 20 years and deferred for 5 years, payments begin 5 years from today and end in the 25th year.

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2.6.1 ACCUMULATED VALUE (OR FUTURE VALUE) OF AN ANNUITY IMMEDIATE

The accumulated value (i.e., future value) of an annuity immediate at the end of the n^{th} period for a stream of equal periodic payments C and interest rate (i) is given by the formula:

$$\mathbf{s_{\overline{n}|i} = C \times \frac{(1+i)^n - 1}{i}} \tag{2.12}$$

Here is an example illustration the calculation of the future value of an annuity immediate.

Example 8

Refer to the information presented in Example 7. In this case, the future value $= \$500 \text{ x } \frac{(1.06)^{10} - 1}{0.06}$ =\$500 x 13.18079=\$6,590.397.

Technical Result

It can be shown by simple algebra that: $\mathbf{v^n} \times \mathbf{s_{\overline{n}|i} = a_{\overline{n}|i}}$.

This result is illustrated by the results from Examples 7 and 8. In Example 7, $\mathbf{a_{\overline{n}|i}}$ = \$3,680.044 and from Example 8, $\mathbf{s_{\overline{n}|i}}$ = \$6,590.397. Since $\mathbf{v^n} = \left(\frac{\mathbf{1}}{\mathbf{1+i}}\right)^{\mathbf{n}} = \left(\frac{1}{1.06}\right)^{10}$ = 0.558395, then $\mathbf{v^n} \times \mathbf{s_{\overline{n}|i}}$ = 0.558395 x \$6,590.397 = \$3680.044 which is exactly the value of $\mathbf{a_{\overline{n}|i}}$.

The present value of a **perpetuity immediate** can be obtained from the formula for an annuity immediate by taking the limit as the number of time periods (n) increases to infinity.

2.6.2 THE PRESENT VALUE OF A PERPETUITY IMMEDIATE

From the formula (2.11), we let $n \to \infty$ and hence $v^n \to 0$. We obtain that the present value of a perpetuity immediate is given by the formula:

$$\mathbf{a_{\overline{\infty}|i} = C \times \frac{1}{i}} \tag{2.13}$$

Example 9

Refer to the information in Example 7 except that the number of periods approaches infinity. The present value of an annuity immediate with an infinite stream of equal annual payments of \$500 and an annual nominal rate of interest of 6% is given as follows:

$$a_{\overline{\infty}|i} = C \times \frac{1}{i} = \$500 \text{ x } (1/0.06) = \$8333.33.$$

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We now consider the case of an annuity where the equal periodic payments are made at the **beginning** of each period. This is called an **annuity due**. This class of annuities is applicable to retirement plans where payments are typically made on retirement date. Annuities due also apply to the pricing of insurance contracts where payments are typically made by the insured at the date of policy issue. In addition, we will apply the concept annuity due in converting single premiums into a stream of regular premiums.

2.6.3 PRESENT VALUE OF AN ANNUITY DUE

Here is a graphical depiction of an annuity due for *n* periods and equal periodic payments C.

Figure 2.4: Annuity Due

Observe from Figure 2.4 that payments begin at time 0 and end at time *n-1.*

In actuarial notation, the present value of an annuity due shown in equation (2.14) is represented by $\mathbf{a_{\overline{n}|i}}$ and for equal periodic payments (C) and interest rate *(i),* the present value is given by the formula:

$$\mathbf{a_{\overline{n}|i}} = \mathbf{C} \times \frac{\mathbf{1 - v^n}}{\mathbf{d}} \text{ where } \mathbf{v} = \frac{\mathbf{i}}{\mathbf{1+i}} \text{ and } \mathbf{d = iv} \tag{2.14}$$

Here is an example which illustrates the application of this equation.

Example 10

An annuity due comprises an equal annual payment of \$500 for a period of 10 years. The appropriate interest rate is 6%. The present value factor is $\dfrac{1-\left(\dfrac{1}{1+0.06}\right)^{10}}{0.06 \times \dfrac{1}{1+.06}} = 7.801692$. The present value of the annuity immediate is product of the annual payment and the present value factor $= \$500 \text{ x } 7.801692 = \ \$3{,}900.846$.

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Technical Result

The present value of an annuity immediate is related to the present value of an annuity due as follows:

$\ddot{a}_{\overline{n}|i} = (1+i) \times a_{\overline{n}|i}$. This relationship is illustrated by examples 7 and 10 where the present value of an annuity immediate is $3,680.044 and the present value of the corresponding annuity due is $3,900.846. Clearly, $3,900.846 = 1.06 x $3,680.044 as predicted by the technical result.

Comment

The present value of an annuity due is equal to the present value of the corresponding annuity immediate multiplied by (1+ rate of interest).

2.6.4 ACCUMULATED (OR FUTURE VALUE) OF AN ANNUITY DUE

The accumulated value (i.e., future value) of an annuity due at the end of the n^{th} period for a stream of equal periodic payments C and interest rate (*i*) is given by the formula:

$$\ddot{s}_{\overline{n}|i} = C \times \frac{(1+i)^n - 1}{d} \tag{2.15}$$

Here is an example illustration the calculation of the future value of an annuity due.

Example 11

Refer to the information presented in Example 7. In this case, the future value = $500 × $\frac{(1.06)^{10}-1}{0.06 \times \frac{1}{1+0.06}}$ = $500 × 13.97164 = $6,985.821

Technical Result

It can be shown by simple algebra that: $v^n \times \ddot{s}_{\overline{n}|i} = \ddot{a}_{\overline{n}|i}$.

This result is validated by the results from Examples 10 and 11. In Example 10, $\ddot{a}_{\overline{n}|i}$ = $3,900.846 and from Example 11, $\ddot{s}_{\overline{n}|i}$ = $6,985.821. Since $v^n = \left(\frac{1}{1+i}\right)^n = \left(\frac{1}{1.06}\right)^{10}$ = 0.558395, then $v^n \times \ddot{s}_{\overline{n}|i}$ = 0.558395 x $6,985.821 = $3,900.846 which is exactly the value of $\ddot{a}_{\overline{n}|i}$.

The present value of a **perpetuity due** can be obtained from the formula for an annuity due by taking the limit as the number of time periods (*n*) increases to infinity.

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2.6.5 THE PRESENT VALUE OF A PERPETUITY DUE

From the formula (2.14), we let $n \to \infty$ and hence $v^n \to 0$. We obtain that the present value of a perpetuity immediate is given by the formula:

$$\mathbf{a}_{\overline{\infty}|\mathbf{i}} = \mathbf{C} \times \frac{\mathbf{1}}{\mathbf{d}} \tag{2.16}$$

Example 12

Refer to the information in Example 9 except that the number of periods approaches infinity. The present value of a perpetuity due with infinite stream of equal annual payments of \$500 and an annual nominal rate of interest of 6% is given as follows:

$$a_{\overline{\infty}|i} = C \times \frac{1}{d} = C \times \frac{1}{0.06 \times \frac{1}{1+0.06}} = \$500 \times 17.66667 = \$8833.33$$

Comment

Examples 9 and 12 show that the present value of a perpetuity due is equal to the present value of the corresponding perpetuity immediate multiplied by (1+ rate of interest). This is illustrated from the examples which show that \$8833.33 = (1.06) x \$8333.33.

We conclude this chapter by introducing a key concept in the development of premiums for insurance policies which we consider in Chapter 3. This is called **actuarial present value**.

2.7 ACTUARIAL PRESENT VALUE (APV)

Equation (2.3) shows the calculation of the present value for a fixed number of years (n), an assumed annual nominal interest rate (i) and a known and fixed future value (FV). These assumptions typically fit the context of banking and finance under the title of **time value of money**. However, the assumption of a known future value is not typical in insurance. In fact, payments by insurance companies are **contingent** on some insurance phenomenon as discussed in Chapter 1 and the ending Appendix. We illustrate this important concept of contingent payments that underpins the principles of pricing in insurance by way of an example.

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Example 13

Consider the case of an individual who is currently aged 45 years. This individual purchases a policy from an insurance company which will pay €100,000 in 15 years if he/she is alive at age 60 years. Observe that the insurance will make a payment only if the individual survives by age 60 years. Otherwise no payment is made. Such is payment is contingent on an event – survival for the next 15 years. Assume that an actuary for the insurance company estimates that for a selected cohort of individuals, the probability of life aged 45 years will survive the next 15 years is 0.95. Also assume that the appropriate rate of interest is 6%.

Actuarial Notation

In actuarial notation, ${}_{15}p_{45}$ is the probability of a life aged 45 years will survive 15 years. In other words, this notation expresses the probability that a life aged 45 years will have a future lifetime of at least 60 years. In this example, we assume that, ${}_{15}p_{45} = 0.95$.

There are two steps in the calculation of the actuarial present value.

First Step:

By equation (2.3), $PV = €100{,}000 \times \left(\frac{1}{1.06}\right)^{15} = €100{,}000 \times 0.417265 = €41{,}726.5$

Second Step:

The present value obtained in the first step is contingent on the survival of the individual. We calculate the expected present value as the present value obtained in the first step multiplied by the probability that the individual will survive the next 15 years.

Actuarial Present Value = $0.95 \times €41{,}726.5 = €39{,}640.18$

Key Equation

Actuarial Present Value = Probability of Payoff multiplied by the Present Value of the Payoff

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Key Comment

The Actuarial Present Value of the Contingent Payoff of an Insurance Contract is the Price (or Premium) evaluated at the Date of the Policy Issue.

i.e., Single Premium (at the date of policy issue) = Actuarial Present Value

With reference to example 13, the single premium at the date of policy issue is the actuarial present value of €39,640.18. But this value can be prohibitive for prospective customer. Hence the insurance company may offer monthly or annual level payments over the lifetime of the insurance contract. These are typically called **regular payments (premiums)**. The methodology for converting single premiums to a stream of regular premiums involves the application of life annuities due considered in the second eBook on life annuities and life insurance.

This concludes Chapter 2. Chapter 3 considers premium calculation principles in insurance.

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3 PREMIUM CALCULATION PRINCIPLES IN INSURANCE

3.1 INTRODUCTION

When a risk-averse individual purchases insurance, the insured may receive benefits (e.g., lumpsum in the event of disability or death) or periodic payments over a specified period (e.g., life annuities). In a similar vein, the insurance company may receive a single premium payment at the date of policy issue or a stream of periodic payments over a specified period of time. Both benefits and premium payments are random variables except in the case when the insurance contract is purchased by a single premium at the date of issue. In this context of random cash flows, we consider premium calculation principles (i.e., prices for insurance risk) that include the classical equivalence principle that is consistent with the law of large numbers. We also consider the effect on the safety loading factor for a fixed probability of ruin in the context of risk pooling[8].

We begin with the classical equivalence principle in the next section.

3.2 EQUIVALENCE PRINCIPLE

We develop the equivalence principle in several steps without considering expenses incurred by the insurer. This will be considered later in this chapter.

Assume that the present value of a future random variable is calculated at the time the policy is issued which is denoted as time 0.

Let PVB_0 be the present value of future benefits to be received by the insured should the insurable event occur; let PVP_0 be the present value of premiums to be paid by the insured for the future benefits.

From the perspective of the insurer, the net random future loss at time 0 ($\mathbf{L_0^n}$), is equal to the difference between the present value of benefits and the present value of premiums and is stated as follows:

$$\mathbf{L_0^n = PVB_0 - PVP_0} \tag{3.1}$$

The expected net random loss at time 0 for the insurer is:

$$\mathbf{E(L_0^n) = E(PVB_0) - E(PVP_0)} \tag{3.2}$$

The equivalence principle states that the premium is set such that, at the initial contract date, the expected value of the future loss is zero – that is, $\mathbf{E(L_0^n) = 0}$ (3.3)

Equation (3.3) implies that the premium should be set so that the expected present value of benefits equals the expected present value of premiums.

$$\mathbf{E(PVB_0) = E(PVP_0)} \tag{3.4}$$

Comment:

For a single premium insurance policy with net premium $\mathbf{P_0}$, equation (3.4) states that:

$$\mathbf{P_0 = E(PVB_0)} \tag{3.5}$$

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Key Point

Under the equivalence principle, the premium is set such that at the start of the contract the expected value of the future loss is zero – that is, $\mathbf{E(L_0^n) = 0}$. This implies that the expected present value of premiums paid to the insurer is equal to the expected present value of benefits paid by the insurer to the beneficiary.

Alternatively, at the date of policy issue, the actuarial present value of current and future premiums is equal to the actuarial present value of future benefits. The premium so obtained is called a **pure premium**. Pure premiums are premiums determined by the equivalence principle, ignoring expenses.

We illustrate the principle of equivalence for the case of a single premium and a single benefit.

Example 1

Consider the information presented in Example 13 of Chapter 2. In this example, the single premium is €39,640.18 and the present value of the future benefit is €41,726.50. We describe the net loss function for the insurance company. If the insured survives the next 15 years (with probability 95%), the insurance company incurs a loss which is equal to the present value of the benefit less the premium received at the policy date. If the insured does not survive the next 15 years (with probability 5%), the insurance company earns a profit which is equal to the premium received. Note that a profit is viewed as a negative loss. Accordingly, the net loss variable is as follows:

$$\mathbf{L_0^n = \begin{cases} PVB_0 - P_0 \text{ with probability, } 0.95 \\ -P_0 \text{ with probability, } 0.05 \end{cases}}$$

Hence the expected net loss is calculated as follows:
$E(L_0^n) = 0.95 \times (PVB_0 - P_0) + 0.05 \times (-P_0) = 0.95 \times PVB_0 - P_0 = 0.95 \times 41{,}726.5 - 0.05 \times 39{,}640.18$ = 39,640.18-39,640.18 = 0. In other words, the expected loss at the date of policy issue is zero, consistent with the principle of equivalence.

We now state the principle of equivalence which is the fundamental premium calculation principle as follows:

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Key Point

The equivalence principle states that the expected present value of premiums is equal to the expected present value of future losses. In the case of a single premium $\mathbf{P_0}$ and where **X** = present value of future payouts to the insured, the equivalence principle states that $\mathbf{P_0 = E(X)}$.

We now show that the equivalence principle is supported by the weak law of large numbers.

3.2.1 WEAK LAW OF LARGE NUMBERS (WLLN)

The equivalence principle finds validity from the weak law of large numbers and a formal statement is presented in the endnote[9]. In the context of insurance, the WLLN states that for identically distributed and independent losses, the probability that the average loss per contract is equal to its expected value approaches a value of unity (i.e., certainty) as the number of insureds in the portfolio increases to infinity. In other words, the WLLN provides high assurance that the average loss per contract ($\overline{S}$) is equal to its expected value of loss $E(X_1)$ as the number of insureds (i.e., risk units) becomes large without limit.

But the equivalence principle sets the premium equal to the expected value of the loss. Combining this result with the WLLN, we obtain an important result.

The premium (set according to the equivalence principle) will be sufficient to cover the average loss per contract. This will occur with probability close to 1 as the number of insureds approaches infinity.

The implication is, ignoring expenses, the insurer which sets premiums according to the equivalence principle will just breakeven.

We now present an example applying the equivalence principle that will highlight the role of a principle known as **mutuality**.

Example 2

An individual currently aged 60 years purchases an insurance contract that guarantees a payment of $100,000 if he/she survives the next 10 years. Payment will be made at the end of the term of 10 years. If the person dies before the term ends, no payment is made. An actuary calculates that the probability of this person surviving the next years is 90%. Based on an interest rate of 6%, the single premium based on the equivalence principle is calculated as the expected present value of the payout which is equal to $P_0 = 0.90 \times \frac{\$100,000}{(1.06)^{10}} = \$50,255.53$.

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The insurance company receives a single premium payment (with certainty) of \$50,255.53 and this amount is credited with the interest rate of 6% for 10 years. The accumulated amount of this investment is \$50,255.53 × $(1.06)^{10}$ =\$90,000.00. If the insured event occurs (i.e., the individual survives the next 10 years), then the payout is \$100,000 which exceeds the accumulated amount of \$90,000.00 from investing the premium.

This is not a surprising result since the probability of a payout is 90% and the accumulated amount by investing the single premium at 6% will be 90% of the payout amount of \$10,000 or \$90,000. This example illustrates the general conclusion in the technical result that follows:

Technical Result

The Equivalence Principle states that the premium ($\mathbf{\Pi}$) is given by the equation:

$\mathbf{\Pi = p \times \frac{X}{(1+i)^n}}$. Rewriting this equation, we obtain: $\mathbf{\Pi \times (1+i)^n = p \times X}$.

Since the probability $p < 1$, then crediting the premium with interest rate i for n years will accumulate to $\mathbf{p \times X < X}$.

How can this shortfall problem be addressed?

So far, the shortfall problem is considered from a single policyholder's perspective. By focusing on the portfolio of policies, we obtain a solution through the application of the principle of mutuality.

3.3 PRINCIPLE OF MUTUALITY

The principle of mutuality[10] simply states that policyholders with claims are subsidised by policyholders without claims. Specifically, policyholders who paid premiums exceeding their claim amounts subsidise those whose claim amounts exceed their premiums.

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Comment (Principle of Solidarity)

The principle of solidarity arises when homogenous risk classes are combined into a smaller number of rating classes – that is, for the purpose of setting premiums. For example, consider the simple case of two homogeneous risk classes – one class comprising similar high-risk individuals and the other class of similar low-risk individuals. The high-risk class is expected to pay a premium ($\mathbf{P_H}$) that is higher than for the low-risk class ($\mathbf{P_L}$). Suppose both risk classes are merged into a single rating class where all individuals pay a weighted average premium ($\mathbf{P_A}$), where $P_L < P_A < P_H$. Then, low-risk individuals will pay a relatively higher premium and high-risk individuals will be charged a relatively low premium. The net effect is that there is a cross-subsidy from low-risk individuals to high-risk individuals at the premium level.

Key Point

The principle of mutuality refers to the benefit or claim stage in the insurance process whereby policyholders who paid premiums exceeding their claim amounts subsidise those whose claim amounts exceed their premiums.

The principle of solidarity arises when the risk classes are combined into a smaller number of rating classes.

Comment

The principle of solidarity can create risk for the insurer that may be described as follows: If some low-risk individuals consider the average premium $\mathbf{P_A}$ to be too high relative to $\mathbf{P_L}$, they may leave the pool. Being a weighted average, $\mathbf{P_A}$ will increase causing more low-risk individuals to exit the pool. If this process continues, the pool will eventually comprise only high-risk individuals and the weighted average premium, $\mathbf{P_A}$ will increase towards $\mathbf{P_H}$. An insurer must be atuned to possible antiselection taking place. This is the case if relatively fewer low risk insureds and/or relatively more high risk insureds take the insurance coverage.

We now return to the equivalence principle for premium calculation with a consideration of safety loading. This is called the expected value premium principle.

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3.4 EXPECTED VALUE PREMIUM PRINCIPLE

Since future loss is a random variable, it is possible that the premium charged by the insurer according to the equivalence premium may not be sufficient to cover actual losses. In addition, there is potential for residual portfolio risk arising from probability estimates and interest rate forecasts applied in the calculation of actuarial present value. Hence the pure premium is increased by a **safety loading** so as to increase the probability that a loss will be avoided. Simply put, an insurer only has a chance to remain solvent if $\mathbf{P_0 > E(X)}$.

The expected value premium principle adds a safety loading to the pure premium for an individual policy holder (i.e., risk) as follows:

$\mathbf{\Pi_i = (1+\theta)E(X_i), \theta > 0}$ where $\boldsymbol{\theta}$ is the safety loading factor; *i* represents the i^{th} insured in the portfolio. (4.6)

We now demonstrate a key result linking the safety loading factor to the insurer's probability of ruin (i.e., insolvency).

3.4.1 SAFETY LOADING AND PROBABILITY OF RUIN

What do we mean by the probability of ruin (or insolvency)?

Typically, the insurer prefers to have a situation where the probability that aggregate losses exceed aggregate premiums is tolerably small – for example 0.5%. This value represents the risk tolerance level set by the senior management of the insurance company.

The question is: for a fixed level of probability of ruin, what is the effect on the safety loading factor as the portfolio size gets bigger and bigger?

We present an intuitive analysis of this problem as follows:

Suppose that the insurer sets its probability of ruin at a level of ε. The probability of ruin refers to the probability that the total amount of claim losses (S) for a period of time exceeds the total amount of premiums (Π) is equal to ε.

Formally, from a portfolio perspective the probability is ruin is stated as follows:

$$P(S > \Pi) = \varepsilon \tag{3.7}$$

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Here is a graphical illustration of the concept of 'probability of ruin' for a normal distribution.

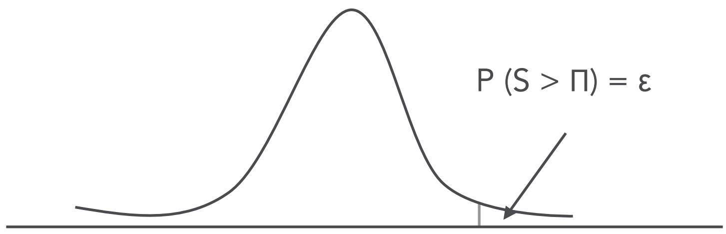

Figure 3.1: Probability of Ruin

We present the main result as a theorem with a proof presented in the endnote[11].

Theorem 3.1 (Equivalence Value Principle)

If the insurer sets it probability of ruin at a constant level (ε), then:

a) the safety loading factor goes to zero as the portfolio size approaches infinity;
b) The safety loading factor = $\Phi^{-1}(1-\varepsilon) \times \text{Risk Index}_{\text{Portfolio}}$.

Comment

Part a) of the theorem states that for a constant probability of ruin, the safety loading for the expected value principle converges to zero as the portfolio size approaches infinity. In this case, the premium is equal to the expected value of the claim loss for each policy holder. In other words, the individual policyholder pays a lower premium.

The portfolio size matters for the amount of the premium for the individual policyholder.

Part b) of the theorem provides a formula to calculate the safety loading factor which is related to the risk-index (i.e., coefficient of variation) of the portfolio of policyholders. The lower the risk index, the lower is the safety loading factor. The lower is the required probability of ruin, the higher is the safety loading factor, but this effect is diminished by a large portfolio size.

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Example 3

Case 1: Portfolio size is n =10,000

The board of directors of an insurance company has set the probability of ruin for the next year at a value of 0.5%. This is the value of ε in the formula in b) above. Assume that losses are IID and are normally distributed with expected value $E(X) = \$100$ and standard deviation $\sigma(X) = \$300$ for each insured.

From our calculations in Chapter 1, we have that for the portfolio, E(S) = n × E(X) = 10,000 × \$100 = \$1,000,000; and standard deviation σ(S) = sqrt(10,000) × \$300 = 100 × \$300 = \$30,000. Then the risk index for the portfolio is equal to the coefficient of variation CV(S) = \$30,000 / \$1,000,000 = 0.03.

For the standard normal distribution, $\Phi^{-1}(1-\varepsilon) = \Phi^{-1}(1-.005) = \Phi^{-1}(0.995) = 2.5758$. (Using Microsoft Excel, this value is obtained as NORMSINV(0.995).) Hence the safety loading factor $\theta = 2.5758 \text{ x } 0.03 = 0.0773 = 7.73\%$.

The premium for the individual is $\Pi_i = (1+\theta)\times E(X_{i)}) = (1.0773)\times\$100 = \$107.73$
We want to see the effect on the safety loading factor as we increase the portfolio size. The theorem predicts that the safety loading factor will decline.

Case 2: Portfolio size is increased to 20,000.

In this case, E(S) = n × E(X) = 20,000 × \$100 = \$2,000,000 and standard deviation σ(S) = sqrt(20,000) × \$300 = 141.42 × \$300 = \$42,426.41. Then the risk index for the portfolio is equal to the coefficient of variation CV(S) = \$42,426.71 / \$2,000,000 = 0.02
Hence the safety loading factor = 2.5758 × 0.02=0.0515= 5.15%

The premium for the individual policyholder is $\Pi_i = (1+\theta)\times E(X_{i)}) = (1.0515)\times\$100 = \$105.15$.

This example illustrates a key point:

Key Takeaway

For a fixed probability of ruin, the safety loading factor declines with an increase in the portfolio size. This provides a lower premium for the individual policyholder.

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Case 3: Probability of Ruin is set at a Lower Level

Suppose the probability of run is set at a lower level of 0.1%. In this case $\Phi^{-1}(1-\varepsilon)=\Phi^{-1}(0.999)$. Using the Microsoft Excel formula, we obtain NORMSINV(0.999) =3.09. Then the safety loading factor $= 3.09 \text{ x } 0.03 = 0.0927 = 9.27\%$.
The premium for the individual policyholder is $\Pi_i = (1+\theta)\times E(X_i) = (1.0927)\times \$100 = \$109.27$

> **Key Takeaway**
>
> Case 3 shows that if the insurance company wants a higher level for the probability of solvency (i.e., a lower probability of ruin), then the safety factor is also higher leading to a higher individual premium.

Case 4: A Higher Level of Standard Deviation

Suppose the standard deviation in Case 1 is increased from \$300 to \$500. All other information remain the same. Then risk index for the portfolio = \$50,000/\$1,000,000 = 0.05. Hence the safety loading factor is $\theta = 2.5758 \text{ x } 0.05 = 12.88\%$.

The premium for the individual policyholder is $\Pi_i = (1+\theta)\times E(X_i) = (1.1288)\times \$100 = \$112.88$.

> **Key Takeaway**
>
> Case 4 shows that, all else equal, a higher standard deviation (i.e., higher risk) leads to higher safety loading factor and a higher premium.

Important Comment

The higher safety factors in Cases 3 and 4 arising from lower probability of ruin or higher standard deviation can be countervailed by having a higher portfolio size. This is the key point of risk pooling considered to be the cornerstone of insurance.

It is also common to consider a constant fixed safety loading that is not related to the expected value as in (3.6). In this case, $\mathbf{\Pi_i = E(X_i) + m}$ (3.8)

Based on the analysis above with a fixed probability of ruin, it can be shown that:

Theorem 3.2

$$\mathbf{m = \frac{\Phi^{-1}(1-\varepsilon)\times\sigma(X_i)}{\sqrt{n}}} \tag{3.9}$$

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This theorem implies that as the portfolio size (n) increases, the safety loading (m) declines and the individual policyholder pays a premium that is closer to the expected value of claim loss as shown from (3.8). This result is similar to the conclusion of Theorem 3.1.

What can we conclude?

Whether the safety loading is proportional to expected value (i.e. $\Pi_i = (1+\theta)E(X_i), \theta > 0$ in (3.6) or a constant (i.e., $\Pi_i = E(X_i) + m$ in (3.8)), as the portfolio size increases, the safety loading declines leading to a lower premium for the individual policyholder.

Comment

There are several other specifications for premium calculation principles that incorporate a safety loading factor that directly relate to risk measures such as the variance and standard deviation or are utility-based. These are considered in the subsequent eBooks of Level 1.

Expense Loading

To this point in our discussion, we ignored the insurer's expenses arising from management, distribution, underwriting, claims settlement etc. This can be incorporated into (3.6) to arrive at the **gross premium** for the individual policyholder that is stated as follows:

$$\Pi_i = (1+\theta+\gamma)E(X_i), \text{ where } \theta = \text{safety loading, } \gamma = \text{expense loading, } \gamma > 0. \tag{3.7}$$

Example 4

Refer to example 3 for the case n =10,000. Suppose that the expense loading for the individual policy is 30%, the gross premium $= (1+ 0.0713+ 0.30) \text{ x } \$100 = \$137.73$

Comment

A profit loading (λ) may also be included in equation (3.7). In this case, we still use the term **gross premium**. Note that if the total amount of premiums is equal to total amount of claim losses for a specified time period, then the safety loading is profit for the insurer.

This concludes Chapter 3. Chapter 4 considers principles of risk sharing in insurance. In particular, issues related to reinsurance as well as deductibles and loss limit values are discussed and illustrated with examples.

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4 PRINCIPLES OF RISK SHARING

4.1 INTRODUCTION

As stated in chapter 1, an insurance contract is one under which the insurer accepts significant insurance risk (which is defined as 'pure risk') from a policyholder and agrees to compensate the policyholder if the insured event occurs and thereby adversely affects the policyholder. The principle of risk pooling also showed that for identically distributed and independent (IID) losses, the insurance company can mitigate its exposure by increasing its portfolio size. Theoretically, relative risk as measured by the coefficient of variation approaches zero as portfolio size increases without limit. But if the assumption of IID losses does not hold, then the severity of a single loss can be devastating for the insurance company. The insurance company may then seek to spread its risk through risk-sharing contracts in two common ways.

First, the insurance company can share its risks with another insurance company called a **reinsurer**. In this case the original insurance company is called the **primary insurer** or **ceding company**. Simply put the direct insurer purchases insurance from another insurance company. A graphical depiction (Figure 4.1) illustrates the risk-sharing principle of reinsurance.

Another principle of risk sharing is between the insurer and its policyholders whereby insurance contracts are structured by the insurer to include **deductibles and policy loss limits**. In this way, the policyholder is responsible for some of the risk transferred to the insurer.

This chapter considers both principles of risk sharing with greater coverage of deductibles and policy limits. This is because our eBook entitled *Reinsurance* provides an extensive coverage of this topic. In this chapter, we present a brief introduction of the fundamental issues in reinsurance and illustrate the basic concepts and principles for both proportional contracts (e.g., quota share and surplus share) and non-proportional contracts (e.g., excess of loss and stop loss) as classified in Figure 4.2 below.

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4.2 PRINCIPLE OF RISK SHARING – REINSURANCE

We illustrate the concept of reinsurance as a risk-sharing agreement between the primary insurer and reinsurer. Figure 4.1 illustrates this concept:

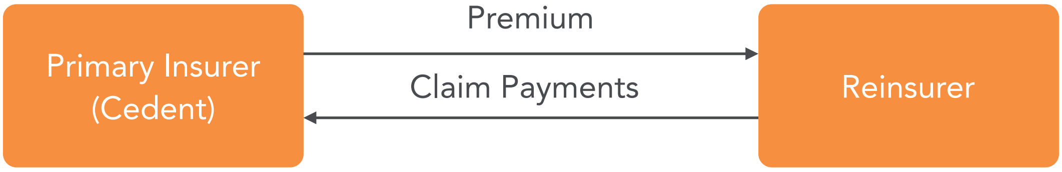
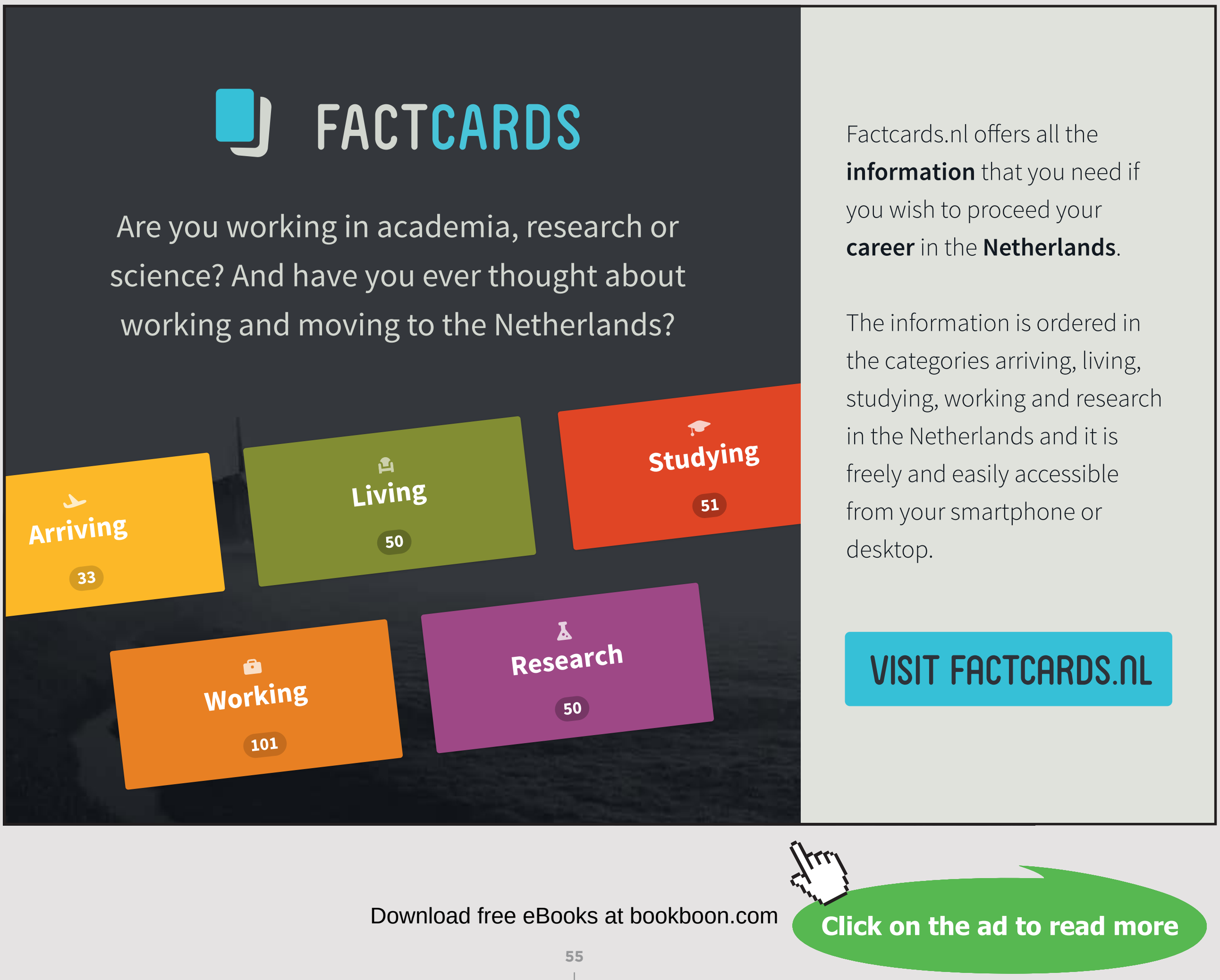


Figure 4.1: Reinsurance Risk Sharing



In a reinsurance agreement, a share of future losses incurred by the primary insurer that will be covered by the reinsurer is stipulated. In addition, the amount of premium that the primary insurer will pay (i.e., cede) to the reinsurer for this coverage is also specified. In the case of **treaty reinsurance**, a well-defined portfolio of risks (e.g., all automobile accidents in a specified geographical region over a specified period of time (e.g., a year) is defined in the treaty. It is typical (but not necessary) for the treaty to include obligatory cession by the primary insurer and automatic acceptance by the reinsurer. In the case of **facultative reinsurance**, the contract may also stipulate a specified risk for a specified location (e.g., losses from fire at a large commercial real estate over the next year). In this case, both the primary insurer and the reinsurer have the 'faculty' or option to accept or reject the agreement.

Key Takeaway

Facultative agreements cover individual risks especially the primary insurer's exposure to large single risks. Obligatory reinsurance treaties cover a portfolio of risks.

Reinsurance contracts may be classified according to the mechanism by which risk is transferred from the insurer to the reinsurer. The following diagram summarises this classification:

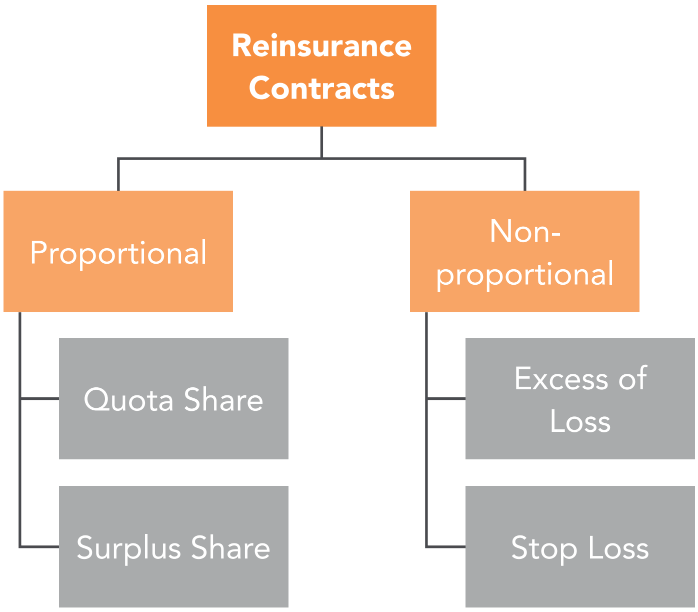


Figure 4.2: Classification of Reinsurance Contracts

Figure 4.2 shows that reinsurance contracts may be classified as proportional or non-proportional. In proportional reinsurance contracts both the insurer and reinsurer share losses and premiums according to a specified proportionality factor or share α where $0 < \alpha < 1$.

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We consider each of the four reinsurance arrangements listed in Figure 4.2. We begin with the two arrangements that are proportional.

4.2.1 QUOTA SHARE REINSURANCE

In this case, the primary insurer retains a **fixed** share **(α)** of both the claim loss payment and premiums paid by policyholders. For a portfolio of *n* policies (i.e., risks), let S = total claims; Π = total portfolio expense loaded premiums. Then the following equations describe a quota share reinsurance:

$$\Pi^{\text{retained}} = \alpha \times \Pi; \quad \Pi^{\text{ceded}} = (1-\alpha) \times \Pi \tag{4.1}$$

$$S^{\text{retained}} = \alpha \times S; \quad S^{\text{ceded}} = (1-\alpha) \times S \tag{4.2}$$

Comment

Since the premiums are expense-loaded, the reinsurer will typically reimburse the primary insurer for the expenses included in the ceded premiums. This reimbursement is called a **commission rate**. We illustrate these concepts by means of the next example.

Example 1

A primary insurer as ceding company has a quota share agreement in which 40% of all premiums and claim loss payments will be retained. The reinsurer will therefore pay 60% of loss payments and receive 60% of premiums less commission for expenses included in the premiums. Assume that the commission rate is 30% which matches an expense loading factor of 30%.

How are premiums and losses assigned to the direct insurer and the reinsurer if total premiums is $2,000 and the claim losses is $400,000?

In the case of losses, the reinsurer will pay 60% × $400,000 = $240,000. The insurer will pay 40% × $400,000 = $160,000.

Considering the premiums, the insurer will retain 40% of $2,000 = $800. The reinsurer will receive 60% of $2,000 = $1,200. But the reinsurer reimburses the primary insurer for expenses based on a commission rate of 30% of the premiums which is equal to $0.30 × 1,200 = $360. The amount of $360 is paid to the primary insurer to cover expenses.

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Comment

Reinsurance commission rate is justified because it compensates the primary insurer for expense loading included in ceded premiums.

4.2.2 SURPLUS REINSURANCE

For surplus reinsurance arrangements, a retention limit (also called a priority), **R** is established in the treaty whereby the primary insurer retains all risk up to the retention limit. The reinsurer takes all risk above the retention limit up to a maximum level, **M** which is also defined in the treaty. Risks in excess of M fall back to the primary insurer. However, this excess over M would typically be covered by a second surplus reinsurance treaty. Here are the details showing how the proportionality factor is established.

Let R be the retention limit which is the maximum amount the insurer wants to pay for a specific portfolio of risk. Let SI be the sum insured for this portfolio of risk. The primary insurer retention share (α) is defined as follows:

$$\alpha = \begin{cases} \mathbf{1 \text{ if } R \geq SI} \\ \mathbf{\dfrac{R}{SI} \text{ if } R < SI} \end{cases} \tag{4.3}$$

Equation (4.3) is summarised as $\boldsymbol{\alpha = \min\left(1, \frac{R}{SI}\right)}$.

Here are some examples illustrating the definition of the retention ratio in (4.3).

Example 2

Assume that the retention level for a surplus insurance is €100,000. For a sum insured of €65,000 that is clearly less than the retention limit, using (4.3), the value of α is 1 meaning that the primary insurer retains 100% of the loss.

Example 3

Assume that the retention level for a surplus insurance is €100,000. For a sum insured of €120,000 that is obviously higher than the retention limit, using (4.3), the value of $\alpha = \frac{R}{SI} = \frac{100{,}000}{120{,}000} = \frac{5}{6} = 83.33\%$ meaning that the primary insurer retains 83.33% of the loss.

Example 4

Using the information provided in Example 3, what is the reinsurer's share for a claim of €60,000?

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Since the value of α is equal to 5/6. This means that the primary insurer retains 5/6 of the value of the claim which is €60,000 × 5/6 = €50,000 and the reinsurer pays €60,000 × (1/6) = €10,000.

We now present illustrative examples of the two types non-proportional reinsurance arrangements.

4.2.3 EXCESS OF LOSS REINSURANCE

Here is an example that highlights the main idea behind this type of reinsurance arrangement. Assume that a claim loss X is equal to \$200,000 and the primary insurer pays an mount of the loss up to a value of \$80,000. The reinsurer pays the remaining amount up to a maximum value of \$100,000. The remaining amount is a spillover that is covered by the primary insurer.

This example is summarised using the typical terminology for this type of reinsurance.

Layer Number	Interval	Cover	Paid by:
0	(0, \$80,000]	Retention	Primary Insurer
1	(\$80,000, \$180,000]	\$100,000 XS \$80,000 XS ≡ Excess	Reinsurer
2	>\$180,000	Spillover	Primary Insurer

Note that for first layer (0), the value of \$80,000 is a deductible retained by the primary insurer. The phrase, \$100,000 XS \$80,000 means that the reinsurer covers \$100,000 in excess of the deductible of \$80,000. The first \$180,000 is shared between the primary insurer (\$80,000) and \$100,000 for the reinsurer. The remaining \$20,000 is spilled over to the primary insurer.

Generally, a 3-layer excess of loss reinsurance is described next:

Layer Number	Interval	Cover	Paid by:
0	(0, d]	Retention	Primary Insurer
1	(d, d + f]	d xs f	Reinsurer
2	> d + f	Spillover	Primary Insurer

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Example 5

An excess of loss reinsurance agreement is described as €300,000 XS €100,000. For a claim loss of €250,000, which one of the following correctly identifies the reinsurer share of the claim loss?

a) €250,000
b) €100,000
c) €200,000
d) €150,000

The correct answer is d) since the primary insurer pays the first €100,000 of the claim loss. The reinsurer pays any additional amount up to €300,000. Since the claim loss is €250,000, the reinsurer will pay the remaining €150,000.

Comment

A variation of excess of loss reinsurance is called **stop loss reinsurance**. While excess of loss reinsurance is typically based on single loss amounts, stop loss reinsurance is related to aggregate loss amounts net of other reinsurance contracts the primary insurer has already negotiated.

Key Takeaway

Excess of Loss reinsurance covers large losses beyond a retention limit (i.e., deductible) incurred by the primary insurer and for which risk pooling is not very effective. This has the effect of reducing the variability of potential future claim loss payments.

We now consider deductibles and policy limits which comprise another method of risk sharing between the insurer and policy holder.

4.3 DEDUCTIBLES AND POLICY LIMITS[12]

For non-life insurance, the insurer may be faced with claims with high severity (i.e., large claim loss amounts) or frequent small claims. In the case of unexpectedly large claim losses, the insurer is faced with the risk of insolvency. This case provides a rationale for policy limits that provide a maximum level of the insurer's claim loss amount. In the case of frequent small claim losses, average claim settlement costs are relatively high due to fixed administrative costs. The insurer is able to avoid small claim losses through the use of deductibles.

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Comment

Policies with no deductibles are insurance with **full coverage**; policies with positive deductibles provide **partial coverage**.

We present common types of deductibles and policy limits (i.e., stop loss contract features).

4.3.1 FRANCHISE DEDUCTIBLE

We provide an example to illustrate the main feature of a franchise deductible. Suppose that the insurer is faced with an actual claim loss equal to €2,000 and the insurance contract has a franchise deductible of €500. **When the loss amount exceeds the deductible, the insurer pays the loss amount in full**. In this case, the insurer pays the full loss amount of €2,000. **When the loss amount is less than or equal to the deductible, the insurer pays nothing.**

For a franchise deductible (**d**) and claim loss amount (**X**), the amount (**Z**) paid by the insurer is calculated as follows:

$$\mathbf{Z} = \begin{cases} \mathbf{0 \text{ if } X \leq d} \\ \mathbf{X \text{ if } > d} \end{cases} \tag{4.4}$$

Here are two examples that illustrate the formula in (4.4).

Example 6

The loss amount (X) is $200 and the franchise deductible is $300. What is amount of the compensation paid by the insurer to the policyholder?

a) $200
b) $300
c) $0
d) $100

The correct answer is c) since the loss amount (X) is less than the franchise deductible. The policyholder is responsible for loss amounts that are less than or equal to the deductible.

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Example 7

The loss amount (X) is \$1,200 and the franchise deductible is \$300. What is amount of the compensation paid by the insurer to the policyholder?

a) \$1,200
a) \$300
a) \$0
b) \$100

The correct answer is a) since the loss amount (X) is greater than the deductible. In this case, the insurer pays the full amount of the loss.

The most common deductible is called the **fixed amount deductible (also called an ordinary deductible)**.

4.3.2 ORDINARY DEDUCTIBLE

We provide an example to illustrate the main feature of a fixed deductible. Suppose that the insurer is faced with an actual claim loss equal to €2,000 and the insurance contract has a fixed amount deductible of €500. **When the loss amount exceeds the deductible, the insurer pays the loss amount less the deductible.** In this case, the insurer pays loss amount of €2,000 – €500 = €1,500. **When the loss amount is less than or equal to the deductible, the insurer pays nothing.** In summary, the deductible is always subtracted from the loss amount.

For a fixed deductible **(d)** and claim loss amount **(X)**, the amount **(Z)** paid by the insurer is calculated as follows:

$$\mathbf{Z} = \begin{cases} \mathbf{0 \text{ if } X \leq d} \\ \mathbf{X - d \text{ if } X > d} \end{cases} \tag{4.5}$$

Here are two examples that illustrate the formula in (4.5).

Example 8

The loss amount (X) is \$200 and the fixed deductible is \$300. What is amount of the compensation paid by the insurer to the policyholder?

a) \$200
b) \$300
c) \$0
d) \$100

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The correct answer is c) since the loss amount (X) is less than the fixed deductible. The policyholder is responsible for loss amounts that are less than or equal to the deductible.

Example 9

The loss amount (X) is $1,200 and the fixed amount is $300. What is amount of the compensation paid by the insurer to the policyholder?

a) $1,200
b) $300
c) $900
d) $100

The correct answer is c) since the loss amount (X) is greater than the deductible. In this case, the insurer pays the amount of the loss after subtracting the amount of the deductible. This gives a value of $1,200 – $300 = $900.

Another type of deductible is called **fixed percentage deductible.**

4.3.3 FIXED PERCENTAGE DEDUCTIBLE

We provide an example to illustrate the main feature of a fixed percentage deductible. This case is sometimes called proportional co-insurance. Suppose that the insurer is faced with an actual claim loss amount equal to €2,000 and the insurance contract has a fixed percentage deductible of $\boldsymbol{\alpha}$= 90%. In this case, the policyholder pays 10% of the loss amount of €2,000 and the insurer pays the remaining 90%.

For a franchise deductible **(d)** and claim loss amount **(X)**, the amount **(Z)** paid by the insurer is calculated as follows:

$$\mathbf{Z} = \alpha \mathbf{X} \tag{4.6}$$

Here is an example that illustrates the formula in (4.6).

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Example 10

The loss amount (X) is \$1,200 and the fixed percentage deductible is 95%. What is amount of the compensation paid by the insurer to the policyholder?

a) \$1,200
b) \$1,140
c) \$60
d) \$0

The correct answer is b) since the insurer pays 95% of \$1,200 = \$1,140.

Our last example considers the case of a **limit value (u)**.

4.3.4 POLICY LOSS LIMIT (U)

We provide an example to illustrate the main feature of a policy loss limit which is essentially a stop loss feature. Suppose that the insurer is faced with an actual claim loss amount equal to €2,000 and the insurance contract has an upper limit value of **u** = €1,000. The insurer pays the insured an amount that is equal to **u** = €1,000. If the loss amount is €800, then the insurer pays the full loss amount since it is below the maximum value, **u**.

For a loss limit value (**u**) and claim loss amount (**X**), the amount (**Z**) paid by the insurer is calculated as follows:

$$\mathbf{Z} = \begin{cases} \mathbf{X} \text{ if } \mathbf{X} < \mathbf{u} \\ \mathbf{u} \text{ if } \mathbf{X} \geq \mathbf{u} \end{cases} \tag{4.7}$$

Here are two examples that illustrates the formula in (4.7).

Example 11

The loss amount (X) is \$200 and the loss limit value is \$1,000. What is amount of the compensation paid by the insurer to the policyholder?

a) \$200
b) \$300
c) \$0
d) \$100

The correct answer is a) since the loss amount (X) is less than the loss limit value.

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Example 12

The loss amount (X) is $1,200 and the loss limit value is $1,000. What is amount of the compensation paid by the insurer to the policyholder?

a) $1,200
b) $300
c) $0
d) $1000

The correct answer is d) since the loss amount (X) is greater than the upper limit value, u which is the maximum value the insured pays in all cases.

The final section of this chapter considers two important principles – the principle of anti-selection and the principle of moral hazard- and their respective relationships with deductibles.

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4.4 ANTI-SELECTION, MORAL HAZARD AND DEDUCTIBLES

We discuss the concept of anti-selection leading to an important prediction that links deductibles to policyholder's risk.

4.4.1 THE PRINCIPLE OF ANTI-SELECTION

The principle of anti-selection is linked to the concept of asymmetric information. In the context of insurance, asymmetric information is present when the insurance applicant has information that is not available to the insurer and importantly, this information is relevant to the outcome of their contractual relationship.

If **before** the transaction is completed, the potential customer has information that is not revealed to the insurer, then anti-selection may exist. Thee **principle of anti-selection** (also called negative selection or adverse selection in economics) is observed when the decision-maker inadvertently charges high-risk individuals premiums as if they are classified as low-risk. This negative selection creates a potentially adverse financial impact for the insurer since these high-risk policyholders are likely to make claims with greater frequency and severity as compared to their charged premiums.

As an illustration, consider the case where a person applies for a term life insurance but hides the information that he/she has a rare life-threatening disease. The insurance company does not know this information which is relevant for this potential contractual relationship. The risk for the insurance company is that it provides a life insurance policy to this individual who is charged a premium reflecting a too low probability of death. This example reflects the risk of anti-selection.

Comment

Anti-selection is a potential outcome of asymmetric information and occurs **before the policy is issued**.

One of the most important results from anti-selection theory is as follows:

4.4.2 PREDICTION OF ANTI-SELECTION THEORY

Assume that in the presence of asymmetric information, policyholders know their true level of risk (in terms of frequency and severity of claims) but the insurer does not. The central prediction of anti-selection theory is that high-risk policyholders tend to choose insurance policies with lower or no deductibles – that is, high insurance coverage.

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This is a very important prediction since the insurer in the presence of underwriting information deficit has some difficulty in separating high risk applicants for insurance from low-risk applicants. This prediction suggests that the insurer should take note that high-risk insurance applicants have a greater incentive to opt for full insurance coverage.

Key Takeaway

All else equal, high risk insurance applicants are likely to choose higher coverage (i.e., lower deductibles)

A related concept arising from asymmetric information is the principle of moral hazard.[13]

4.4.3 THE PRINCIPLE OF MORAL HAZARD

To illustrate the principle of moral hazard, consider the case of an individual who is issued an automobile insurance. **After** the policy is issued the individual takes actions (which where previously hidden from the insurer) that increase the probability of an accident and/or the loss amount from the accident. This is an example of moral hazard.

Let us formalise this example:

At the time the policy is issued, the insurer estimates the expected value of a loss amount (L) from an insured event as follows:

$$\mathbf{E(L) = p \times L} \tag{4.8}$$

In (4.8), **p** is the probability of the insured event.
Moral hazard occurs when the policyholder takes actions that increase **p** or **L.**

We summarise the distinction between anti-selection and moral hazard in Figure 4.3 below:

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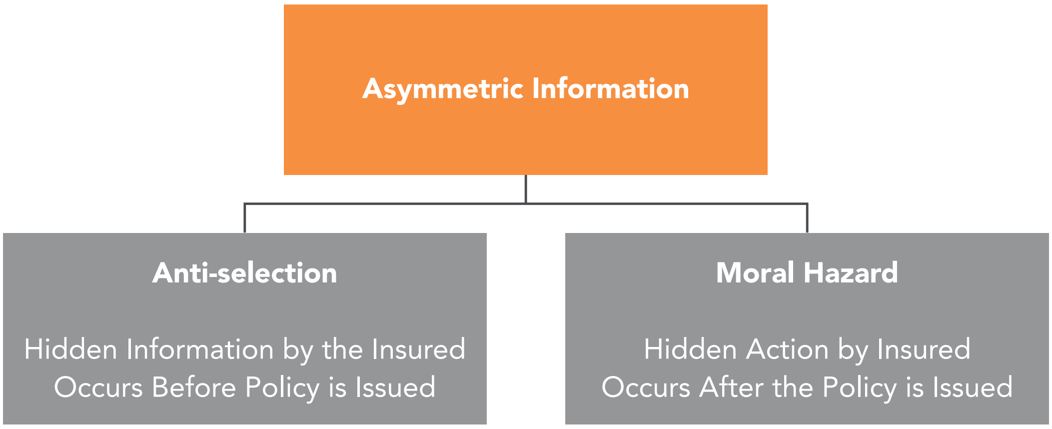

Figure 4.3: Anti-selection and Moral Hazard

In insurance, moral hazard is classified as **ex-ante moral hazard** and **ex-post moral hazard.** These two forms of moral hazard can be explained with reference to equation (4.8) where $\mathbf{E(L) = p \times L}$. We stated above that moral hazard occurs when the policyholder takes actions to increase the probability of a loss **(p)** and/or to increase the severity of the claim loss amount **(L).**

Ex-ante moral hazard refers to hidden actions taken by the policyholder that lead to an increase in the probability of a loss **(p)**. It is called 'ex-ante' which means that it takes place before the insured event occurs. This has the implication of affecting the probability of a loss.

Ex-post moral hazard refers to actions taken by the policyholder that lead to an increase in the probability of a loss (p) or in the severity of a claim loss amount (**L**). It is called 'ex-post' which means that it takes place after the insured event occurred. This has the implication of affecting the probability or severity (or size) of the loss amount since the loss will occur after the event itself.

We summarise these concepts in Figure 4.4 below:

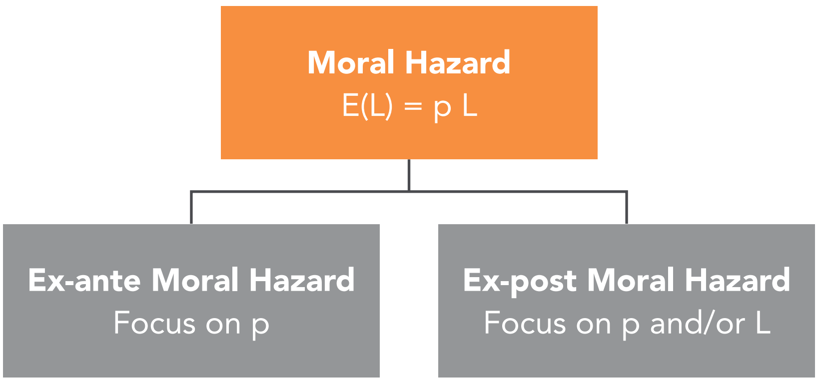

Figure 4.4: Theory of Moral Hazard

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4.4.4 PREDICTION OF MORAL HAZARD THEORY

Assume that in the presence of asymmetric information, policyholders know their true level of risk (in terms of frequency and severity of claims) but the insurer does not. The central prediction of moral hazard theory is that policyholders engage less in prevention when they choose high insurance coverage (i.e., tend to choose insurance policies with lower or no deductibles).

This prediction suggests that insurance contracts with relatively high deductibles can lower, for the insurer, the adverse impact of moral hazard. A deductible (i.e., partial insurance coverage) is a form of loss-sharing between the policyholder and the insurer and so gives the policyholder an incentive to reduce loss from an insured event.

Key Takeaway

All else equal, higher insurance coverage (low deductible) increases moral hazard risk.

This concludes this chapter and eBook on the Principles of Insurance which forms the foundation for the subsequent eBooks in Level 1.

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ENDNOTES

1. Refer to Olivieri and Pitacco, Introduction to Insurance Mathematics, Second Edition, Springer, 2010.
2. IFRS (2016) Chapter 4 deals with Insurance Accounting and Methods.
3. Refer to Principles Underlying Actuarial Science by Mark Allaben *et al,* published in the July 2008 issue of the Actuarial Practice Forum, Society of Actuaries.
4. An excellent reference is Lecture Notes in Actuarial Mathematics by Jerry Allan Veeh, 2006
5. The phrase 'pricing in insurance' is synonymous with 'calculation of premium in insurance'.
6. The exact formula is $n = \frac{\ln(2)}{\ln(1+r)}$
7. It is shown that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ =2.718281828 approximately.
8. An excellent reference is Straub's book on Non-Life Insurance Mathematics.
9. Using the notation of Chapter 1, $S = X_1 + X_2 + ... + X_n$, Let $\bar{S} = \frac{S}{n}$ and so $E(\bar{S}) = E(X_1)$. For identically distributed and independent losses, the (weak) law of large numbers (WLLN) states that for any $\varepsilon > 0, \Pr ob.\{|\bar{S} - E(\bar{S})| < \varepsilon\} = 1 \text{ as } n \to \infty.$

   Furthermore, the risk (as measured by the variance) that the average loss per contract will be higher than the premium approaches zero as the number of individual contracts approaches infinity.

   This result is shown from the following calculation.

   Recall from Chapter 1 that for the case of a portfolio of identically distributed and independent losses, $E(S) = n \times x \times p \quad and \quad Var(S) = n \times x^2 \times p(1-p)$**.**

   We now define the average loss per contract as $\bar{S} = \frac{S}{n}$.

   Then, $Var(\bar{S}) = \frac{1}{n^2}Var(S) = \frac{n \times x^2 \times p(1-p)}{n^2} = \frac{x^2 \times p(1-p)}{n} = \frac{Var(X_1)}{n}$.

   Observe that from equation (5.6), $\lim_{n\to\infty} Var(\bar{S}) = \lim_{n\to\infty} \frac{x^2 p(1-p)}{n^2} = 0$. This implies that for identically distributed and independent losses, the variance of the average loss per contract declines to a value of zero as the number of insurance policies in the portfolio increases to infinity.

10. The principle of mutuality is attributed to Karl Borch (1962).
11. Let us consider the expected value premium for a single policyholder (i.e., a single risk). Then,

    $\Pi_i = (1+\theta)E(X_i), \theta > 0$

    For simplicity, we assume that claim losses associated with each policy holders are normally distributed with expected value, $\mathbf{E(X_i)}$ and standard deviation, $\boldsymbol{\sigma}\mathbf{(X_i)}$**.** For *n* policyholders, the aggregate claim loss (as shown in Chapter 2) is as follows:

    $S = X_1 + X_2 + ... + X_n$, and for IID losses, the random variable S is normally distributed with expected value, $E(S) = n \times E(X_1)$ and $\sigma(S) = \sqrt{n} \times \sigma(X_1)$.

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Also, the total amount of premiums is Π which is equal to $n \times \Pi_i = n(1+\theta)E(X_1)$

What do we mean by the probability of ruin (or insolvency)?

Probability of ruin is:

$$\varepsilon = P(S > \Pi) = P(S > n(1+\theta)E(X_1) = P(S > nE(X_1) + n\theta E(X_1)) = P(S - E(S) > n\theta E(X_1)$$
$$= P\left(\frac{S - E(S)}{\sigma(S)} > \frac{n\theta E(X_1)}{\sigma(S)}\right)$$

Then $\frac{n\theta E(X_1)}{\sigma(S)} = \Phi^{-1}(1-\varepsilon) \Rightarrow \theta = \frac{\Phi^{-1}(1-\varepsilon) \times \sigma(S)}{n \times E(X_1)} = \frac{\Phi^{-1}(1-\varepsilon) \times \sigma(S)}{E(S)} = \Phi^{-1}(1-\varepsilon) \times CV(S)$

But we showed in chapter 2 that as the portfolio size (n) increases towards infinity, the risk index as measured by the coefficient of variation (CV(S)) approaches zero. Hence the safety loading factor (θ) goes to zero.

12. More details in Theory and Practice of Insurance by Francois Outreville.
13. The link between asymmetric information and moral hazard problem is found in Arrow (1963).

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