O'ZBEKISTON RESPUBLIKASI OLIY VA O'RTA MAXSUS TA`LIM VAZIRLIGI

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# OLIY MATEMATIKA

**O‘quv qo‘llanma**

**5310100 Energetika**
**5310200 Elektr energetikasi**
**5310700 Elektr texnikasi, elektr mexanikasi va elektr texnologiyasi**
**5310800 Elektronika va asbobsozlik**
**5310900 Metrologiya standartlashtirish va mahsulot sifati menejmenti**
**5311000 Texnologik jarayonlar va ishlab chiqarishni avtomatlashtirish va boshqarish**

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**Qo'llanma "Oliy matematika" fanining qatorlar, n-o'lchovli , egri chiziqli va sirt integrallari, kompleks o'zgaruvchili funksiyalar, maydonlar nazariyasi va operatsion hisob bo'limlariga bag'ishlangan. Matematik tasdiqlar mantiqiy jihatdan asoslangan hamda misol va masalalar yechib ko'rsatilgan. Qo'llanma barcha texnika yo'nalishlarida ta'lim olayotgan talabalar uchun mo'ljallangan.**

## Annatatsiya

Ushbu o'quv qo'llanma  texnik yo’nalishlarida o'qiyotgan talabalar uchun mo'ljallangan.

Qo'llanmada oliy ta`limning birinchi va ikkinchi kurslarida talabalar tomonidan o'rganiladigan “Oliy matematika” kursining ayrim bo'limlari uchun zarur bo'lgan ma'lumotlar mavjud (sonli va funksional qatorlar, maydon nazariyasi elementlari, ko'p o'zgaruvchi funksiyalarining integrallari, kompleks o'zgaruvchining funksiyalari nazariyalari, operatsion hisob asoslari).

## Аннатация

Настоящое учебное пособие предназначено для студентов, изучающих высшую математику в том или ином объеме в технических учебных заведениях.

Учебное пособие содержит необходимый материал некоторым разделам курса высшей математике(числовые и функциональные ряды, элементы теория поля, интегралы функции многих переменных, теории функций комплексного переменного, основы операционного  исчисления), которые изучаются студентами на первом и втором курсе вуза.

## Annotation

This study guide is intended for students studying higher mathematics in one way or another in technical schools.

The manual contains the necessary material for  all sections of the course in higher mathematics (numerical and functional series, elements of field theory, integrals of the functions of many variables, theories of functions of a complex variable, the basics of operational calculus), which are studied by students in the first and second year of high school.

# KIRISH

Texnika va information texnologiyalar tez rivojlanayotgan davrda ishlab chiqarishda uchraydigan muammolarni tez va oqilona hal qilish uchun injener kadrlarni matematik tayorgarligi yuqori darajada bo‘lishini taqozo etadi. Oliy matematika fani injenerlarning ta’lim sifatini belgilovchi asosiy fundamental fandir. Amaliy masalalarni yechishga information texnologiyalarni qo‘llash injenerni matematik tayorgarligisiz amalga oshirib bo‘lmaydi. Injenerlarni matematik tayyorgarligi o ‘ziga xos xususiyatlarga ega.

Injener o‘rganilayotgan texnik jarayonlarni matematik modelini qurish va matematik usullardan foydalanib qo‘yilgan muammoni ham mantiqiy, ham miqdoriy jihatdan mukammal yechimni ajrata olishi kerak.

Qo‘llanmaning asosiy maqsadi - amaliy masalalarni yechishda matematik usullardan foydalanish ko‘nikmasini hosil qilishdan iborat.

Qo‘llanmada “Oliy matematika” fanining sonli va funksional qatorlar, n-o‘lchovli integrallar, egri chiziqli va sirt integrallari, maydon nazariyasi elementlari, kompleks o ‘zgaruvchili funksiyalar va operatsion bo ‘limlari yoritilgan.

Qo‘llanmaning oxiridagi ilovada Maple programmasidan foydalanib yechishga doir misollar yechib ko ‘rsatilgan. Ushbu qo‘llanma haqidagi tanqidiy fikr va mulohazalarini bildirgan kitobxonlarga mualliflar avvaldan minnatdordorchilik bildiradilar.

*Mualliflar.*

# 1-bob. Sonli qatorlar

## 1.1. Asosiy tushunchalar

Faraz qilaylik, biror

$$u_1; u_2; \ldots; u_n; \ldots \qquad (1.1.\ 1)$$

sonli ketma-ketlik berilgan bo‘lsin.

**1.1. 1-ta’rif.** Berilgan (1.1. 1) sonli ketma-ketlik hadlari vositasida tuzilgan

$$u_1 + u_2 + \ldots + u_n + \ldots \qquad (1.1.\ 2)$$

ifoda *sonli qator* deb ataladi. Bunda $u_n$ $(n \in N)$ qatorning $n$-hadi deyiladi, ba’zan uni *qatorning umumiy hadi* deb ham yuritiladi. Qatorni belgilash uchun (1.1. 2) dan tashqari $\sum_{n=1}^{\infty} u_n$ kabi yozuv ham ishlatiladi.

**1.1. 2-ta’rif.** Tayinlangan $n \in N$ uchun (1.1. 2) qatorning dastlabki $n$ ta hadlarining yig‘indisini uning $n$*-qismiy yig‘indisi* deb ataymiz va uni $S_n$ kabi belgilaymiz.

Bu ta’rif asosida

$$S_n = \sum_{i=1}^{n} u_i \qquad (1.1.\ 3)$$

ni olamiz. Bundan $n \in N$ ekanligidan $\{S_n\}$ sonli ketma-ketlikka ega bo‘lamiz va uni (1.1. 2) *qator qismiy yig‘indilarining ketma-ketligi* deyiladi.

**1.1. 3-ta’rif.** Agar (1.1. 2) sonli qator qismiy yig‘indilarining $\{S_n\}$ ketma-ketligi

$$S = \lim S_n$$

chekli limitga ega bo‘lsa, uni *qatorning yig‘indisi* deb ataladi va bu holda *qator yaqinlashuvchi* deb yuritiladi.

Agar $\lim S_n$ chekli limit mavjud bo‘lmasa, *qatorni uzoqlashuvchi* deb atalib, bu holda uning yig‘indisi mavjud emasdir. Bu o‘rinda quyidagi misollarni keltirishni lozim topdik.

1) $a + aq + aq^2 + \ldots + aq^{n-1} + \ldots$ qatorni qarasak, uning hadlari birinchi hadi $a$ dan maxraji esa $q$ ga teng bo‘lgan cheksiz geometrik progressiyani tashkil qiladi. Bu holda qatorning $n$-qismiy yig‘indisi $q \neq 1$ bo‘lganda

$$S_n = \frac{a(1-q^n)}{1-q}$$

ekanligi bizga ma’lumdir. $a = 0$ bo‘lganda qatorning yaqinlashuvchi ekanligini ko‘rish oson. Demak, $a \neq 0$ bo‘lgan holni tekshirish lozimdir. $|q| < 1$ bo‘lgan holda

$$\lim S_n = \frac{a}{1-q}$$

chekli limit mavjudligidan qatorning yaqinlashuvchi bo‘lishi kelib chiqadi. $a \neq 0 \wedge |q| > 1$ bo‘lgan hol uchun $\{S_n\}$ ketma-ketlik cheksiz limitga ega ekanligidan qatorning uzoqlashuvchi bo‘lishini ko‘ramiz.

Agar $q = 1$ bo‘lsa, $S_n = na$ bo‘lib, $a \neq 0$ bo‘lganda, $\lim S_n = \infty$ ekanligidan qator uzoqlashuvchi bo‘lishini ko‘ramiz. $q = -1$ bo‘lganda esa

$$S_n = \begin{cases} a, & n = 2m-1, \\ 0, & n = 2m, \; m \in N. \end{cases}$$

bo‘lib, $a \neq 0$ bo‘lgan holda $\{S_n\}$ ketma-ketlik limitga ega emasligidan qatorning uzoqlashuvchi bo‘lishi kelib chiqadi.

**2)** $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} + \ldots$ qatorni olsak, uning qismiy yig‘indilari uchun

$$S_1 = \frac{1}{1\cdot 2} = 1 - \frac{1}{2},$$

$$S_2 = S_1 + \frac{1}{1\cdot 3} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3},$$

$$. \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; .$$

$$S_n = S_{n-1} + \frac{1}{n(n+1)} = \left(1 - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1},$$

ga egamiz. Bundan

$$\lim S_n = \lim\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1.$$

Demak, berilgan qator yaqinlashuvchi va uning yig‘indisi 1 ga tengdir.

## 1.2. Sonli qator yaqinlashishining zaruriy sharti

Qatorlarni tekshirishda muhim ahamiyatga ega bo‘lgan masalalardan biri uning yaqinlashuvchi yoki uzoqlashuvchi ekanligini aniqlashdan iboratdir. Bu borada qator yaqinlashishining zaruriy sharti hamda bir qancha yetarli shartlari mavjud bo‘lib, bu bandda qator yaqinlashishining zaruriy shartini keltiramiz.

**1.2. 1-teorema.** Agar (1.1. 2) sonli qator yaqinlashuvchi bo‘lsa, uning n-hadi (umumiy hadi) cheksiz kichik miqdor, ya’ni $\lim u_n = 0$ bo‘ladi.

**Isbot.** Faraz qilaylik, (1.1. 2) qator yaqinlashuvchi va $\lim S_n = S$ bo‘lsin, bu yerda S – qator yig‘indisi bo‘lib, qandaydir tayin son ekanligi ayondir. U holda $\{S_n\}$ qismiy yig‘indilar ketma-ketligi uchun $n \to \infty \Leftrightarrow (n-1) \to \infty$ ekanligidan

$$\lim S_{n-1} = S$$

bo‘lishi ravshandir.

(1.1. 2) qatorning umumiy hadi uchun

$$u_n = S_n - S_{n-1}$$

tenglik o‘rinli bo‘lishiga ishonch hosil qilish osondir. Bundan

$$\lim u_n = \lim(S_n - S_{n-1}) = \lim S_n - \lim S_{n-1} = S - S = 0,$$

ya’ni $u_n$ ning cheksiz kichik miqdor ekanligi kelib chiqadi.

**1.2. 1-natija.** Agar (1.1. 2) qatorning umumiy hadi cheksiz kichik miqdor bo‘lmasa, u uzoqlashuvchi bo‘ladi.

**Eslatma.** Yuqorida isbotlangan shart faqat zaruriy bo‘lib, u qator yaqinlashuvchi bo‘lishi uchun yetarli emasdir.

Bunga misol tariqasida *garmonik qator* deb ataluvchi ushbu

$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}+...=\sum_{n=1}^{\infty}\frac{1}{n}$$

qatorni keltirish mumkin. Bu qator uchun qator yaqinlashishining zaruriy sharti bajariladi, ya’ni $\lim\frac{1}{n}=0$, ammo u uzoqlashuvchidir (bunday ekanligini keyinroq ko‘rsatamiz).

## 1.3. Qatorning qoldig‘i

**1.3. 1-ta’rif.** Qatorning dastlabki n ta hadlarini tashlab yuborishdan hosil qilingan qator berilgan *qatorning n-qoldig‘i* deyiladi.

Masalan, (1.1. 2) qatorning n-qoldig‘i

$$u_{n+1}+u_{n+2}+...+u_{n+k}+... \qquad (1.3.\ 1)$$

ko‘rinishda bo‘lib, uni qisqacha $\sum_{k=1}^{\infty}u_{n+k}$ kabi yozish mumkin.

Qator qoldig‘i ta’rifidan ko‘rinadiki, ular har bir berilgan qator uchun cheksiz ko‘pdir, hamda quyidagi tasdiq o‘rinlidir.

**1.3. 1-teorema.** Agar berilgan qator yaqinlashuvchi bo‘lsa, uning barcha qoldiqlari ham yaqinlashuvchi bo‘ladi va aksincha, agar qatorning birorta qoldig‘i yaqinlashuvchi bo‘lsa, u yaqinlashuvchi bo‘ladi.

**Isbot.** Aytaylik, (1.1. 2) qator yaqinlashuvchi bo‘lsin. U holda 1.1. 3-ta’rifga ko‘ra $S=\lim S_n$ chekli limit mavjuddir, bu yerda $S_n$ - qatorning n-qismiy yig‘indisi. Berilgan (1.1. 2) qatorning n-qoldig‘i bo‘lgan (1.3. 1) qatorning m-qismiy yig‘indisini $\sigma_n^{(m)}$ bilan belgilasak, u holda

$$S_{n+m}=S_n+\sigma_n^{(m)} \qquad (1.3.\ 2)$$

tenglik o‘rinli bo‘lishini ko‘rish osondir. Bunda $m\to\infty$ dagi limitga o‘tsak,

$$\lim_{m\to\infty}\sigma_n^{(m)}=S-S_n$$

chekli limitning mavjudligi, ya’ni (1.3. 1) ning yaqinlashuvchi ekanligi kelib chiqadi. Agar (1.3. 1) ning yig‘indisini $\sigma_n$ bilan belgilasak, bu holda

$$\sigma_n=S-S_n \qquad (1.3.\ 3)$$

tenglikka ega bo‘lamiz. Bu teoremaning birinchi qismining isbotidir, ikkinchi qismi ham Shunga o‘xshash isbotlanadi (mustaqil Shug‘ullaning).

**1.3. 1-natija.** Qatorning yaqinlashuvchi bo‘lishi uchun uning qoldig‘i cheksiz kichik miqdor bo‘lishi zarur va yetarlidir.

Bu natijaning isboti (1.3. 3) tenglik yordamida bajariladi (mustaqil).

**1.3. 2-teorema.** Agar qator yaqinlashuvchi (uzoqlashuvchi) bo‘lsa, uning tayin sondagi hadlarini tashlab yuborib yoki tayin sondagi yangi hadlarni unga qo‘shib yozib yohud tayin sondagi hadlarining o‘rinlarini almashtirib hosil qilingan qator ham yaqinlashuvchi (uzoqlashuvchi) bo‘ladi.

Isboti berilgan qatorning teoremada aytilgan hadlar ta’sir qilmaydigan qoldig‘i yordamida bajariladi.

**1.3. 3-teorema.** Agar qator yaqinlashuvchi (uzoqlashuvchi) bo‘lsa, uning barcha hadlarini bitta *c* songa ko‘paytirib hosil qilingan qator ham yaqinlashuvchi (uzoqlashuvchi $c \neq 0$ bo‘lganda) bo‘lib, yaqinlashuvchi bo‘lgan holda yangi qator yig‘indisi berilgan qator yig‘indisini *c* ga ko‘paytirish natijasidan iborat bo‘ladi.

**1.3. 4-teorema.** Agar $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorlar yaqinlashuvchi bo‘lsa, $\sum_{n=1}^{\infty}(a_n \pm b_n)$ qatorlar ham yaqinlashuvchi bo‘lib, $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$ o‘rinli bo‘ladi. Agar berilgan qatorlardan biri yaqinlashuvchi, boshqasi uzoqlashuvchi bo‘lsa, $\sum_{n=1}^{\infty}(a_n \pm b_n)$ qatorlarning ikkalasi ham uzoqlashuvchi bo‘ladi.

Oxirgi teoremalarning isbotini mustaqil bajaring.

## 1.4. Musbat hadli qatorlar va ularning yaqinlashish belgilari

Agar (1.1. 2) sonli qatorning barcha hadlari musbat bo‘lsa, uni *musbat hadli qator* deyiladi.

Musbat hadli qator qismiy yig‘indilarining ketma-ketligi o‘suvchi bo‘lishi ravshandir.

**1.4. 1-teorema.** Musbat hadli qator yaqinlashuvchi bo‘lishi uchun uning qismiy yig‘indilarining ketma-ketligi yuqoridan chegaralangan bo‘lishi zarur va yetarlidir.

**Isbot.** Chegaralangan monoton ketma-ketlikning chekli limiti mavjudligidan bu teoremaning isboti kelib chiqadi.

**1.4. 2-teorema** (Qatorlarni taqqoslash)**.** Agar musbat hadli $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ qatorlar berilgan bo‘lib, Shunday $n_0$ nomer mavjud va $a_n \leq b_n$, $n_0 < n \in N$ o‘rinli bo‘lsa, u holda

1) $\sum_{n=1}^{\infty} b_n$ yaqinlashuvchi bo‘lsa, $\sum_{n=1}^{\infty} a_n$ ham yaqinlashuvchi;

2) $\sum_{n=1}^{\infty} a_n$ uzoqlashuvchi bo‘lsa, $\sum_{n=1}^{\infty} b_n$ ham uzoqlashuvchi bo‘ladi.

**Isbot.** Faraz qilaylik, teorema shartlari bajarilsin va $\sum_{n=1}^{\infty} b_n$ yaqinlashuvchi bo‘lsin. Qatorlarning $n_0$-qoldiqlari bo‘lgan

$$\sum_{k=1}^{\infty} a_{n_0+k}, \quad \sum_{k=1}^{\infty} b_{n_0+k} \qquad (1.4.\ 1)$$

qatorlarni olsak, ularning m-qismiy yig‘indilari uchun

$$A_{n_0}^{(m)} = \sum_{k=1}^{m} a_{n_0+k}, \quad B_{n_0}^{(m)} = \sum_{k=1}^{m} b_{n_0+k}$$

larni yoza olamiz.

Agar $B_{n_0} = \sum_{k=1}^{\infty} b_{n_0+k}$ desak, u holda $A_{n_0}^{(m)} \le B_{n_0}^{(m)} < B_{n_0}$ ni olamiz va bundan $A_{n_0} = \lim_{m\to\infty} A_{n_0}^{(m)}$ chekli limitning mavjudligi kelib chiqadi. Bu esa $\sum_{n=1}^{\infty} a_n$ qatorning yaqinlashuvchi bo‘lishini isbotlaydi.

Agar $\sum_{n=1}^{\infty} a_n$ uzoqlashuvchi bo‘lsa, (1.4. 1) qoldiqlarning m-qismiy yig‘indilari uchun $A_{n_0}^{(m)} \le B_{n_0}^{(m)}$ o‘rinli bo‘lib, $\lim_{m\to\infty} A_{n_0}^{(m)} = \infty$ ekanligidan $\lim_{m\to\infty} B_{n_0}^{(m)} = \infty$ bo‘lishi kelib chiqadi. Bu $\sum_{n=1}^{\infty} b_n$ ning uzoqlashuvchi qator ekanligini isbotlaydi.

**1.4. 3-teorema.** Agar $\sum_{n=1}^{\infty} a_n$ va $\sum_{n=1}^{\infty} b_n$ musbat hadli qatorlar berilgan bo‘lib, noldan farqli

$$\lim \frac{a_n}{b_n} = c > 0$$

chekli limit mavjud bo‘lsa, berilgan qatorlar yaqinlashuvchi yoki uzoqlashuvchi bo‘lishi jihatidan bir xil bo‘ladi.

**Isbot.** $0 < \frac{c}{2} \le \frac{a_n}{b_n} \le \frac{3c}{2}$, $n_0 < n \in N$ o‘rinli bo‘ladigan $n_0$ nomerning mavjudligidan va oldingi teoremadan bu teoremaning isboti kelib chiqadi (mustaqil bajaring).

1.4. 2 va 1.4. 3-teoremalar qatorlarni taqqoslash (solishtirish) belgilari deb yuritiladi.

**Eslatma.** 1.4. 2-teoremaning $a_n \le b_n$ shartini $\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}$ shart bilan almashtirilgan taqdirda ham uning xulosasi o‘rinli bo‘ladi (mustaqil isbotlang).

**1.4. 4-teorema** (Dalamber belgisi). Agar $\sum_{n=1}^{\infty} a_n$ musbat hadli qator berilgan bo‘lib, Shunday $n_0$ natural son va musbat birdan kichik q son mavjud bo‘lib, $n_0 < n \in N$ bo‘lganda

$$\frac{a_{n+1}}{a_n} \le q \qquad (1.4.\ 2)$$

tengsizlik o‘rinli bo‘lsa, qator yaqinlashuvchi bo‘ladi, agar bu qator uchun

$$\frac{a_{n+1}}{a_n} \ge 1, \quad n_0 < n \in N \qquad (1.4.\ 3)$$

tengsizlik bajarilsa, u uzoqlashuvchi bo‘ladi.

**Isbot.** Faraz qilaylik, (1.4. 2) bajarilsin. Qatorning $n_0$ -qoldig‘in yozaylik:

$$\sum_{k=1}^{\infty} a_{n_0+k} .$$

(1.4. 2) dan, $\frac{a_{n+1}}{a_n} \le q = \frac{q^{n+1}}{q^n}$ ekanligidan hamda hadlari geometrik progressiyani tashkil qiluvchi $\sum_{k=1}^{\infty} q^{n_0+k}$ qator $|q| < 1$ bo‘lganda yaqinlashuvchi bo‘lishidan yuqoridagi eslatmaga ko‘ra berilgan qatorning yaqinlashuvchi ekanligi kelib chiqadi.

Agar (1.4. 3) o‘rinli deb faraz qilsak, undan

$$a_{n+1} \ge a_n , \quad n_0 < n \in N$$

kelib chiqadi, ya’ni qator umumiy hadi kamaymovchi ekanligidan uning limiti nolga teng bo‘la olmaydi. Bu esa qator yaqinlashuvchi bo‘lishining zaruriy sharti bajarilmasligini ko‘rsatadi. Demak, bu holda qator uzoqlashuvchi bo‘ladi.

Musbat hadli qator yaqinlashishining Dalamber belgisi sifatida yuqorida isbotlangan teoremaning quyidagi natijasi qo‘llaniladi.

**1.4. 1-natija.** Agar $\sum_{n=1}^{\infty} a_n$ musbat hadli qator uchun $\lim \frac{a_{n+1}}{a_n} = \ell$ limit mavjud bo‘lib, $\ell < 1$ bo‘lsa, qator yaqinlashuvchi $\ell > 1$ bo‘lganda esa uzoqlashuvchi bo‘ladi. Agar $\ell = 1$ bo‘lsa, u holda qatorning yaqinlashishi yoki uzoqlashishi haqidagi savolga bu natija javob berolmaydi, ya’ni boshqa usullardan foydalanish lozim bo‘ladi.

**1-misol.** $\sum_{n=1}^{\infty} \frac{1}{n!}$ qatorni tekshiring.

**Yechish.** $a_n = \frac{1}{n!}$ - umumiy had.

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim \frac{n!}{(n+1)!} = \lim \frac{n!}{n!(n+1)} = \lim \frac{1}{n+1} = 0 < 1.$$

Demak, Dalamber belgisiga ko‘ra qator yaqinlashuvchi ekanligi kelib chiqadi.

**2-misol.** $\sum_{n=1}^{\infty} e^n$ qatorni tekshiring.

**Yechish.** Umumiy had $a_n = e^n$.

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{e^{n+1}}{e^n} = \lim e = e > 2 > 1.$$

Dalamber belgisiga asosan berilgan qatorning uzoqlashuvchi ekanligini ko‘ramiz.

**3-misol.** $\sum_{n=1}^{\infty} \frac{1}{n}$ - qatorni tekshiring.

**Yechish.** Umumiy had $a_n = \frac{1}{n}$.

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim \frac{n}{n+1} = \lim \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1.$$

Dalamber belgisi masalani hal qilolmaydi. Bu garmonik qator bo‘lib, uning uzoqlashuvchi ekanligini keyinroq ko‘rsatamiz.

**1.4. 5-teorema** (Koshi belgisi)**.** Agar $\sum_{n=1}^{\infty} a_n$ musbat hadli qator uchun Shunday natural $n_0$ va musbat birdan kichik q son mavjud bo‘lib,

$$\sqrt[n]{a_n} \le q \quad (n_0 < n \in N)$$

tengsizlik bajarilsa, qator yaqinlashuvchi bo‘ladi. Agar berilgan musbat hadli qator uchun Shunday $n_0$ natural son mavjud bo‘lib,

$$\sqrt[n]{a_n} \ge 1 \quad (n_0 < n \in N)$$

tengsizlik o‘rinli bo‘lsa, qator uzoqlashuvchidir.

**Isbot.** Teoremaning birinchi qismining sharti bajarilsin. U holda

$$a_n \le q^n, \quad n_0 < n \in N$$

bo‘lib, berilgan musbat hadli qatorning $n_0$-qoldig‘i bilan $\sum_{k=1}^{\infty} q^{n_0+k}$ - yaqinlashuvchi qatorni taqqoslab, qator yaqinlashuvchi ekanligini ko‘ramiz.

Teoremaning ikkinchi qismining sharti bajarilgan taqdirda qatorning umumiy hadi yaqinlashishning zaruriy shartini qanoatlantirmasligiga ishonch hosil qilish osondir. Teorema isbotlandi.

Koshi belgisi deyilganda ko‘pincha bu teoremaning quyidagi natijasi tuShunilishini aytamiz.

**1.4. 2-natija.** Agar $\sum_{n=1}^{\infty} a_n$ musbat hadli qator uchun $\lim \sqrt[n]{a_n} = \ell$ limit mavjud bo‘lib, $\ell < 1$ bo‘lsa, qator yaqinlashuvchi, $\ell > 1$ bo‘lganda esa uzoqlao‘uvchi bo‘ladi. Agar $\ell = 1$ bo‘lsa, u holda qatorning yaqinlashishi yoki uzoqlashishi haqidagi savolga bu natija javob bera olmaydi, ya’ni boshqa yaqinlashish belgilaridan foydalanish kerak bo‘ladi.

**5-misol.** $\sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^{n^2}$ qatorni tekshiring.

**Yechish.** Musbat hadli qator umumiy hadi $a_n = \left( \frac{n}{n+1} \right)^{n^2}$.

$$\lim \sqrt[n]{a_n} = \lim \sqrt[n]{\left( \frac{n}{n+1} \right)^{n^2}} = \lim \left( \frac{n}{n+1} \right)^n = \lim \frac{1}{\left( \frac{n+1}{n} \right)^n} = \frac{1}{\lim \left( 1 + \frac{1}{n} \right)^n} = \frac{1}{e} < 1,$$

qator yaqinlashuvchi.

**1.4. 6-teorema** (Koshining integral belgisi)**.** Agar $\sum_{n=1}^{\infty} a_n$ musbat hadli qator berilgan bo‘lib, Shunday $n_0$ natural son hamda $[n_0;+\infty)$ oraliqda uzluksiz va o‘smovchi $f(x)$ funksiya mavjud bo‘lib, $n_0 \le n \in N$ bo‘lganda $f(n) = a_n$ bo‘lsa, u holda $\int_{n_0}^{+\infty} f(x)dx$ - xosmas integral yaqinlashuvchi (uzoqlashuvchi) bo‘lsa, berilgan qator ham yaqinlashuvchi (uzoqlashuvchi) bo‘ladi.

**Isbot.** Teorema shartlari bajarilganda, $f(x)$ - o‘smovchi ekanligidan

$$a_n \ge \int_{n}^{n+1} f(x)dx \ge a_{n+1} \text{ ,}$$

bundan $n_0 \le n \in N$ bo‘lganda

$$\sum_{k=0}^{m} a_{n_0+k} \ge \int_{n_0}^{n_0+m+1} f(x)dx \ge \sum_{k=1}^{m+1} a_{n_0+k} \qquad (m = n - n_0)$$

munosabatni olamiz. Bu yerda qator yaqinlashishining zaruriy sharti bajariladi deb faraz qilamiz. Bundan berilgan qator $n_0$ - qoldig‘ining m-qismiy yig‘indisi $\sigma_{n_0}^{(m)}$ uchun quyidagi kelib chiqadi:

$$\sigma_{n_0}^{(m)} + a_{n_0+m+1} \le \int_{n_0}^{n_0+m+1} f(x)dx \le a_{n_0} + \sigma_{n_0}^{(m)} .$$

Endi, $\int_{n_0}^{+\infty} f(x)dx$ xosmas integral yaqinlashuvchi bo‘lgan holni qarab, uning qiymatini $A$ ga teng deb faraz qilsak,

$$\sigma_{n_0}^{(m)} \le \int_{n_0}^{n_0+m+1} f(x)dx - a_{n_0+m+1} .$$

Bu tengsizlikdan $m \to +\infty$ $\sigma_{n_0}^{(m)}$ va $\int_{n_0}^{n_0+m+1} f(x)dx$ lar kamaymovchi hamda $\lim a_n = 0$ ekanligidan

$$\sigma_{n_0}^{(m)} \le A$$

ni olamiz. Oxirgidan qatorning $n_0$-qoldig‘i yaqinlashuvchi ekanligi 1.4. 1-teoremadan kelib chiqadi.

Agar xosmas integral uzoqlashuvchi bo‘lsa,

$$\int_{n_0}^{+\infty} f(x)dx = +\infty$$

bo‘lishi ravshandir va bu holda

$$\int_{n_0}^{n_0+m+1} f(x)dx \le a_{n_0} + \sigma_{n_0}^{(m)}$$

dan $m \to +\infty$ limitga o‘tib, $\lim \sigma_{n_0}^{(m)} = +\infty$, ya’ni $n_0$-qoldiq uzoqlashuvchi ekanligini olamiz. Teorema to‘liq isbotlandi.

**6-misol.** $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ $(\alpha \in R)$ qatorni tekshiring.

**Yechish.** 1) $\alpha \le 0$ bo‘lganda qator yaqinlashishining zaruriy sharti bajarilmasligini ko‘rish osondir, demak, bu holda qator uzoqlashuvchidir.

2) $0<\alpha<1$ bo‘lsin. Bu holda $f(x)=\frac{1}{x^{\alpha}}$ funksiyani olsak, u $[1;+\infty)$ oraliqda uzluksiz va kamayuvchi bo‘lib,

$$f(n)=\frac{1}{n^{\alpha}}$$

o‘rinlidir. bu holda

$$\int_{1}^{+\infty} f(x)dx=\int_{1}^{+\infty}\frac{dx}{x^{\alpha}}=+\infty$$

bo‘lishini ko‘rsatish mumkin. Bundan qatorning uzoqlashuvchi ekanligi kelib chiqadi.

3) $\alpha=1$ bo‘lsin (garmonik qator), bu holda $f(x)=\frac{1}{x}$ deb olish kerak bo‘lib,

$$\int_{1}^{\infty}\frac{dx}{x}=+\infty$$

ekanligini bilamiz. Demak, bu holda ham qator uzoqlashuvchi ekan.

4) $\alpha>1$ bo‘lsa, $f(x)=\frac{1}{x^{\alpha}}$ funksiya uchun

$$\int_{1}^{\infty}\frac{dx}{x^{\alpha}}=\frac{1}{\alpha-1}$$

xosmas integral yaqinlashuvchi ,1.4. 6-teorema asosida qatorning yaqinlashuvchi bo‘lishi kelib chiqadi.

Demak, berilgan qator $\alpha>1$ bo‘lgan holda yaqinlashuvchi bo‘lib, $\alpha \le 1$ bo‘lganda, jumladan garmonik qator ham uzoqlashuvchidir.

**7-misol.** $\sum_{n=1}^{\infty}\frac{\sin\frac{1}{n}}{n^{2}}$ - qatorni tekshiring.

**Yechish.** $f(x)=\frac{\sin\frac{1}{x}}{x^{2}}$ funksiya $[1;+\infty)$ oraliqda uzluksiz, musbat va kamayuvchi bo‘lib,

$$f(n)=\frac{\sin\frac{1}{n}}{n^{2}}=a_n$$

qatorning umumiy hadidan iboratdir.

$$\int_{1}^{+\infty} f(x)dx=\lim_{b\to+\infty}\left(-\int_{1}^{b}\sin\frac{1}{x}d\left(\frac{1}{x}\right)\right)=\lim_{b\to+\infty}\cos\frac{1}{x}\Big|_{1}^{b}=\lim_{b\to+\infty}\left(\cos\frac{1}{b}-\cos 1\right)=1-\cos 1.$$

Xosmas integral yaqinlashuvchi ekan, demak, 1.4. 6-teoremaga ko‘ra berilgan qator ham yaqinlashuvchidir.

## 1.5 . Leybnis teoremasi.

Agar sonli qator

$$a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n+1} a_n + \ldots \qquad (1.5\ .1)$$

ko‘rinishda bo‘lib, $\{a_n\}$ ketma-ketlikning barcha hadlari musbat (yoki manfiy) bo‘lsa, uni *ishorasi navbatlashuvchi qator* deb ataladi. Bunday qatorlar uchun *Leybnits belgisi* deb ataluvchi quyidagi tasdiq o‘rinlidir.

**1.5 .1-teorema(Leybnits).** Agar (1.5 .1) ishorasi navbatlashuvchi qator uchun yaqinlashishning zaruriy sharti bajarilib, uning hadlari absolyut qiymat jihatdan kamayuvchi bo‘lsa, u vaqtda bu qator yaqinlashuvchi bo‘lib, yig‘indisining absolyut qiymati birinchi hadining absolyut qiymatidan kichik bo‘ladi.

**Isbot.** Teorema shartlari bajariladi hamda aniqlik uchun $a_n > 0 \ (n \in N)$ deb faraz qilaylik, u vaqtda bu qatorning 2m-qismiy yig‘indisini

$$S_{2m} = (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2m-1} - a_{2m})$$

ko‘rinishda yozsak, qavslar ichidagi ayirmalarning har biri musbat son ekanligidan uning o‘suvchi ekanligiga ishonch hosil qilamiz.

Endi, $2 \le m \in N$ deb faraz qilib, bu qismiy yig‘indini quyidagicha

$$S_{2m} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{2m-2} - a_{2m-1}) - a_{2m}$$

ko‘rinishda yozib,

$$0 < S_{2m} < b = a_1 - (a_2 - a_3) < a_1$$

bo‘lishiga ishonch hosil qilamiz.

Demak, $\{S_{2m}\}$ - o‘suvchi va yuqoridan chegaralangan ketma-ketlik sifatida chekli limitga egadir:

$$\lim S_{2m} = S$$

hamda $0 < S \le b < a_1$ bo‘lishi ravshandir.

Agar (1.5 .1) ning $(2m+1)$-qismiy yig‘indisini olsak, uning uchun quyidagi munosabat o‘rinlidir:

$$S_{2m+1} = S_{2m} + a_{2m+1}.$$

Bundan

$$\lim S_{2m+1} = \lim S_{2m} + \lim a_{2m+1} = S + 0 = S$$

bo‘lishini olamiz.

Shunday qilib, (1.5 .1) qator yaqinlashuvchi va uning yig‘indisi uchun

$$0 < S < a_1$$

o‘rinli ekanligini isbotladik.

$a_n < 0 \quad (n \in N)$ bo‘lgan hol ham Shunga o‘xshash ko‘rsatiladi.

**Misol.** $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n} + \ldots$ qatorni tekshiring.

**Yechish.** Qator ishorasi navbatlashuvchi hamda uning hadlari absolyut qiymat jihatdan kamayuvchi bo‘lib, yaqinlashishning zaruriy sharti bajariladi. Demak, yuqoridagi teoremaga asosan u yaqinlashuvchi va uning yig‘indisi S uchun $0 < S < 1$ munosabat o‘rinlidir.

## 1.6 . O‘zgaruvchi ishorali sonli qatorlar

Agar

$$a_1 + a_2 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n \qquad (1.6\ .1)$$

sonli qator hadlari orasida musbatlari ham manfiylari ham mavjud bo‘lsa, uni *o‘zgaruvchi ishorali sonli qator* deyiladi.

**1.6.1-ta’rif.** Agar berilgan (1.6 .1) sonli qator hadlarining absolyut qiymatlaridan tuzilgan

$$|a_1| + |a_2| + ... + |a_n| + ... = \sum_{n=1}^{\infty} |a_n| \qquad (1.6\ .2)$$

musbat hadli qator yaqinlashuvchi bo‘lsa, berilgan qatorni *absolyut yaqinlashuvchi* deyiladi.

Berilgan (1.6 .1) sonli qator uchun

$$a_n^+ = \frac{|a_n| + a_n}{2}, \quad a_n^- = \frac{|a_n| - a_n}{2}$$

ifodalarni kiritib,

$$\sum_{n=1}^{\infty} a_n^+ \ , \qquad (1.6\ .3)$$

$$\sum_{n=1}^{\infty} a_n^- \qquad (1.6\ .4)$$

sonli qatorlarni yozsak, (1.6 .3) ni berilgan (1.6 .1) qator musbat hadlarini o‘z o‘rnida qoldirib, manfiy hadlari o‘rniga nol yozishdan, (1.6 .4) ni esa, (1.6 .1) ning har bir manfiy hadi o‘rniga uning absolyut qiymatini yozib, musbat hadlarini nol bilan almashtirishdan hosil qilish mumkinligiga ishonish osondir. Demak, aytilganlar asosida

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left(a_n^+ + a_n^-\right)$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(a_n^+ - a_n^-\right)$$

munosabatlarni yozib olamiz va ularning yordamida berilgan (1.6 .1) qatorning absolyut yaqinlashuvchi bo‘lishi uchun (1.6 .3) va (1.6 .4) qatorlarning yaqinlashuvchi bo‘lishlari zarur va yetarli ekanligini hamda *absolyut yaqinlashuvchi qator yaqinlashuvchi* bo‘lishini isbotlash qiyinchilik tug‘dirmaydi.

**1.6 .2-ta’rif.** Agar berilgan (1.6 .1) qator yaqinlashuvchi bo‘lib, uning hadlarining absolyut qiymatlari vositasida tuzilgan (1.6 .2) musbat hadli qator uzoqlashuvchi bo‘lsa, berilgan qatorni *shartli yaqinlashuvchi* deyiladi.

Bu o‘rinda, agar (1.6 .1) shartli yaqinlashuvchi bo‘lsa, (1.6 .3) va (1.6 .4) qatorlarning ikkalasi ham uzoqlashuvchi bo‘lishiga ishonch hosil qilish osonligini aytamiz (mustaqil Shug‘ullaning).

Shuningdek, absolyut yaqinlashuvchi qatorning hadlari o‘rnini ixtiyoriycha almashtirish natijasida uning yig‘indisi o‘zgarmasligini, shartli yaqinlashuvchi qator uchun esa bunday qilish natijasida uning yig‘indisini oldindan berilgan ixtiyoriy songa tenglashtirish mumkinligini, xattoki, uzoqlashuvchi qatorga aylantirish ham mumkinligini aytamiz.

**1-misol.** $\sum_{n=1}^{\infty}\frac{\cos n}{n\sqrt{n}}$ - qatorni tekshiring.

**Yechish.** $\sum_{n=1}^{\infty}\frac{|\cos n|}{n\sqrt{n}}$ - qatorni olsak, u berilgan qator hadlarining absolyut qiymatlari vositasida tuzilgan bo‘lib, musbat hadlidir. Shuningdek,

$$\frac{|\cos n|}{n\sqrt{n}} \le \frac{1}{n\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}$$

tengsizlik o‘rinli hamda $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ sonli qator yaqinlashuvchi ekanligidan (15.4-banddagi 4-misolda $\alpha=\frac{3}{2}>1$ bo‘lgan hol) berilgan qatorning absolyut yaqinlashuvchi bo‘lishi kelib chiqadi.

**2-misol.** $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ - qator tekshirilsin.

**Yechish.** Bu qator yaqinlashuvchi ekanligini 1.5 -bandda keltirilgan misolda ko‘rdik. Agar bu qator hadlarining absolyut qiymatlari vositasida tuzilgan $\sum_{n=1}^{\infty}\frac{1}{n}$ ni yozsak, u garmonik qatordan iborat bo‘lib, uzoqlashuvchi ekanligi ma’lum. Demak, berilgan qator shartli yaqinlashuvchi ekan.

## 1.7 . Kompleks hadli qatorlar

Agar

$$\sum_{n=1}^{\infty} z_n \qquad (1.7\ .1)$$

qatorning har bir hadi kompleks sondan iborat bo‘lsa, uni *kompleks hadli qator* deb ataladi.

Kompleks hadli (1.7 .1) qatorning umumiy hadini

$$z_n = x_n + iy_n\,, \quad n \in N$$

deb belgilasak $(x_n, y_n \in R)$, uni tekshirish ikkita haqiqiy hadli

$$\sum_{n=1}^{\infty} x_n\,, \quad \sum_{n=1}^{\infty} y_n \qquad (1.7\ .2)$$

qatorlarni tekshirishga keltiriladi.

Kompleks hadli (1.7 .1) qatorning yaqinlashishi (1.7 .2) sonli qatorlarning ikkalasining ham yaqinlashuvchi bo‘lishi bilan ekvivalent ekanligiga ishonch hosil qilish osondir.

Shuningdek, kompleks hadli qatorning absolyut va shartli yaqinlashishi tushunchalari ham yuqoridagidek bo‘lib, bu holda absolyut qiymat o‘rnida modul tushunchasidan foydalanish kerakligini aytamiz. Bu bilan Shug‘ullanishni o‘quvchining o‘ziga tavsiya qilish bilan bu bandni yakunlaymiz.

## Sonli qatorlarga doir mashqlar

1. Berilgan umumiy hadiga binoan qatorning dastlabki uchta hadini yozing:

1) $u_n = \dfrac{1}{n(n+1)}$.   2) $u_n = \dfrac{n^2}{n+1}$.   3) $u_n = \dfrac{(n!)^2}{(2n)!}$.

4) $u_n = \dfrac{2^n}{n!}$.   5) $u_n = \dfrac{3n}{n\sqrt{n}}$.   6) $u_n = \dfrac{1}{n^2}$.

2. Berilgan qatorni tekshiring:

1) $\dfrac{1}{2} + \dfrac{2}{2^2} + \ldots + \dfrac{n}{2^n} + \ldots$ .   Javob: yaqinlashuvchi.

2) $\dfrac{1}{\sqrt{10}} + \dfrac{1}{\sqrt{20}} + \ldots + \dfrac{1}{\sqrt{10n}} + \ldots$ .   Javob: uzoqlashuvchi.

3) $2 + \dfrac{3}{2} + \dfrac{4}{3} + \ldots + \dfrac{n+1}{n} + \ldots$ .   Javob: uzoqlashuvchi.

4) $\dfrac{1}{\sqrt[3]{7}} + \dfrac{1}{\sqrt[3]{8}} + \ldots + \dfrac{1}{\sqrt[3]{n+6}} + \ldots$ .   Javob: uzoqlashuvchi.

5) $\dfrac{1}{2} + \left(\dfrac{2}{3}\right)^4 + \left(\dfrac{3}{4}\right)^9 \ldots + \left(\dfrac{n}{n+1}\right)^{n^2} + \ldots$ .   Javob: yaqinlashuvchi.

6) $\dfrac{1}{2} + \dfrac{1}{5} + \dfrac{3}{10} + \ldots + \dfrac{n}{n^2+1} + \ldots$ .   Javob: uzoqlashuvchi.

7) $\dfrac{1}{2} + \dfrac{1}{5} + \dfrac{1}{10} + \ldots + \dfrac{1}{n^2+1} + \ldots$ .   Javob: yaqinlashuvchi.

3. Berilgan qatorni Leybnis belgisi yordamida tekshiring.

1) $\dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} - \dfrac{1}{5} \ldots + (-1)^n \dfrac{1}{n} + \ldots$ .   Javob: yaqinlashuvchi.

2) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(n+1)^2}$.   Javob: yaqinlashuvchi.

3) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(n+1)!}$.   Javob: yaqinlashuvchi.

4. Berilgan qatorni tekshiring:

1) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(2n-1)^2}$.   Javob: absolyut yaqinlashuvchi.

2) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n \cdot 2^n}$.   Javob: absolyut yaqinlashuvchi.

3) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt[5]{n}}$.   Javob: shartli yaqinlashuvchi.

4) $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(n+1)\ln(n+1)}$.   Javob: shartli yaqinlashuvchi.

## 1-bob bo‘yicha bilimingizni sinab ko‘ring.

1. Sonli qatorni ta’riflang.
2. Sonli qatorni qaysi holda yaqinlashuvchi (uzoqlashuvchi) deymiz?
3. Qator yaqinlashishining zaruriy shartini ayting.
4. Qatorning qoldig‘i nima?
5. Musbat hadli qatorning yaqinlashish belgilarini ayting.
6. Ishorasi navbatlashuvchi qator va Leybnis teoremasini ayting.
7. Ixtiyoriy ishorali qatorlar. Absolyut va shartli yaqinlashuvchi qatorlar tushunchalarini ayting.
8. Kompleks sonli qator tushunchasini keltiring.

# 2-bob . Funksional ketma-ketliklar va qatorlar

Agar $u_1(x), u_2(x), ..., u_n(x), ...$ ketma-ketlikning har bir hadi biror D sohada aniqlangan funksiyadan iborat bo‘lsa, uni *funksional ketma-ketlik*, uning vositasida tuzilgan $\sum_{n=1}^{\infty} u_n(x)$ ifodani esa *funksional qator* deb ataladi.

## 2.1 . Funksional ketma-ketlikning yaqinlashish sohasi va uning tekis yaqinlashishi tushunchasi

Agar $\{u_n(x)\}$ funksional ketma-ketlikning har bir hadi $D$ sohada aniqlangan bo‘lib, $x_0 \in D$ uchun tuzilgan $\{u_n(x_0)\}$ sonli ketma-ketlik yaqinlashuvchi (uzoqlashuvchi) bo‘lsa, funksional ketma-ketlik $x_0$ *nuqtada yaqinlashuvchi (uzoqlashuvchi)* va bunday nuqtalarning to‘plami uning *yaqinlashish (uzoqlashish) sohasi* deb ataladi.

Faraz qilaylik $D$ $\{u_n(x)\}$ funksional ketma-ketlikning yaqinlashish sohasidan iborat bo‘lsin. U holda sonli ketma-ketlikning limiti ta’rifi asosida $\forall x \in D$ bo‘lganda ixtiyoriy olingan musbat $\varepsilon$ son uchun, Shunday $S(x)$ son va $n_0 \in N$ topiladiki, $n_0 < n \in N$ bo‘lganda

$$|u_n(x) - S(x)| < \varepsilon$$

o‘rinli bo‘ladi va bunda $S(x)$ ni $\{u_n(x)\}$*funksional ketma-ketlikning limiti* deb ataladi. Bu yerda $S(x)$ son $x$ ning funksiyasi ekanligini ko‘rish qiyin emasdir, ya’ni yuqorida topilishi haqida aytilgan $n_0$ olingan $\varepsilon$ va $x$ larga bog‘liqdir.

Bu o‘rinda olingan musbat $\varepsilon$ son $\forall x \in D$ bo‘lganda $x$ ning barcha qiymatlari uchun bitta $n_0$ ning topilishi mumkin bo‘lgan hol muhim ahamiyatga ega ekanligini va bu holda D sohada *funksional ketma-ketlik tekis yaqinlashadi* deb atalishini aytamiz.

**2.1.1-ta'rif.** Agar $\{u_n(x)\}$ funksional ketma-ketlik M to'plamda $S(x)$ funksiyaga yaqinlashuvchi bo'lib, $\forall\varepsilon>0$ uchun Shunday $n_0$ natural son topilsaki, $\forall x\in M$ bo'lgan $x$ ning barcha qiymatlarida $n_0<n\in N$ uchun

$$|u_n(x)-S(x)|<\varepsilon$$

munosabat o'rinli bo'lsa, $\{u_n(x)\}$*funksional ketma-ketlik M to'plamda* $S(x)$ *ga tekis yaqinlashadi* deyiladi.

## 2.2 . Funksional qatorning yaqinlashish sohasi va uning tekis yaqinlashishi tushunchasi

Aytaylik,

$$\sum_{n=1}^{\infty}u_n(x) \qquad (2.2\ .1)$$

funksional qator berilgan bo'lib, uning barcha hadlari D to'plamda aniqlangan funksiyalar bo'lsin. Aga $x_0\in D$ bo'lganda $\sum_{n=1}^{\infty}u_n(x_0)$ sonli qator yaqinlashuvchi (uzoqlashuvchi) bo'lsa, (2.2 .1) *funksional qator* $x_0$ *nuqtada yaqinlashuvchi (uzoqlashuvchi),* $x_0$ nuqta esa (2.2 .1) *funksional qatorning yaqinlashish (uzoqlashish) nuqtasi* deyiladi.

(2.2 .1) funksional qatorning barcha yaqinlashish (uzoqlashish) nuqtalarining to'plami uning *yaqinlashish (uzoqlashish) sohasi* deb ataladi.

Masalan, hadlari geometrik progressiyani tashkil qiluvchi $\sum_{n=1}^{\infty}x^{n-1}=1+x+x^2+...+x^{n-1}+...$ funksional qatorni olsak, $|x|<1$ $(x\in(-1;1))$ bo'lganda uning yaqinlashuvchi, $|x|\geq 1$ $(x\in[-\infty;-1)\cup[1;+\infty))$ bo'lganda esa uzoqlashuvchi bo'lishini ko'rish osondir. Demak, bu funksional qatorning yaqinlashish sohasi $(-1;1)$ intervaldan, uzoqlashish sohasi esa $(-\infty;-1]\cup[1;+\infty)$ to'plamdan iborat ekan.

Agar (2.2 .1) funksional qatorning qismiy yig'indilari ketma-ketligini qarasak,

$$S_n(x)=\sum_{k=1}^{n}u_k(x) \qquad (2.2\ .2)$$

bo'lib, ular $\{S_n(x)\}$ funksional ketma-ketlikni tashkil qiladi va u (2.2 .1) funksional qatorning yaqinlashish sohasi bo'lgan M to'plamda biror $S(x)$ funksiyaga yaqinlashadi, ya'ni $\lim S_n(x)=S(x)$ bo'ladi va bunda $S(x)$ (2.2 .1) *funksional qatorning yig'indisi* deb ataladi.

**2.2 .1-ta'rif.** Agar (2.2 .1) funksional qatorning (2.2 .2) qismiy yig'indilaridan iborat bo'lgan $\{S_n(x)\}$ funksional ketma-ketlik M to'plamda $S(x)$funksiyaga tekis yaqinlashsa, (2.2 .1) *funksional qator M to'plamda (sohada) tekis yaqinlashadi* deyiladi.

Bu o'rinda quyidagi tasdiqni keltirish joizdir [1].

**2.2 .1-teorema.** (2.2 .1) funksional qator M to'plamda $S(x)$ ga tekis yaqinlashishi uchun

$$\lim_{n\to\infty} \sup_{x\in M} |S_n(x) - S(x)| = 0 \qquad (2.2\ .3)$$

bo‘lishi zarur va yetarlidir. Bu yerda $S_n(x)$ (2.2 .2) bilan aniqlangandir.

**Isbot.Zarurligi.** M to‘plamda (2.2 .1) funksional qator $S(x)$ ga tekis yaqinlashsin. U holda bu qatorning $\{S_n(x)\}$ qismiy yig‘indilari ketma-ketligi $S(x)$ ga M to‘plamda tekis yaqinlashadi. 2.1 .1-ta’rifga ko‘ra, $\forall\varepsilon > 0$ son olinganda Shunday $n_0 \in N$ topiladiki, $n_0 < n \in N$ va $\forall x \in M$ bo‘lganda

$$|S_n(x) - S(x)| < \frac{\varepsilon}{2}$$

bo‘ladi. bundan esa, $n_0 < n \in N$ uchun

$$\sup_{x\in M} |S_n(x) - S(x)| \le \frac{\varepsilon}{2} < \varepsilon$$

bo‘lishi kelib chiqadi. Demak, (2.2 .3) o‘rinli bo‘ladi.

**Yetarliligi.** (2.2 .1) funksional qator qismiy yig‘indilari ketma-ketligi $\{S_n(x)\}$ M to‘plamda limit funksiya $S(x)$ ga ega bo‘lib, (2.2 .3) bajarilsin. Limit ta’rifiga ko‘ra ,$\forall\varepsilon > 0$ son olinganda Shunday $n_0 \in N$ topiladiki, $n_0 < n \in N$ uchun $\sup_{x\in M} |S_n(x) - S(x)| < \varepsilon$ bo‘ladi. Bundan esa,

$|S_n(x) - S(x)| < \varepsilon, \quad \forall x \in M$

kelib chiqadi. Bu esa (2.2 .1) funksional qatorning M to‘plamda $S(x)$ ga tekis yaqinlashishini ko‘rsatadi.

## 2.3 . Mojarantlanuvchi funksional qatorlar

Aytaylik, (2.2 .1) funksional qator berilgan bo‘lib, uning hadlari M to‘plamda aniqlangan bo‘lsin.

**2.3 .1-ta’rif.** Agar (2.2 .1) funksional qatorning hadlari $\forall x \in M$ va $n \in N$ bo‘lganda

$$|u_n(x)| \le a_n \qquad (2.3\ .1)$$

munosabat o‘rinli bo‘lib,

$$\sum_{n=1}^{\infty} a_n \qquad (2.3\ .2)$$

musbat hadli qator yaqinlashuvchi bo‘lsa, (2.3 .2) qator (2.2 .1) funksional qatorning *mojaranti* deyiladi va bu holda (2.2 .1) ni *mojarantlanuvchi funksional qator* deb ataladi.

**2.3 .1-teorema** (Veyershtrass)**.** M to‘plamda mojarantlanuvchi funksional qator bu to‘plamda absolyut va tekis yaqinlashuvchi bo‘ladi.

**Isbot.** Faraz qilaylik, (2.2 .1) funksional qator M to‘plamda mojarantlanuvchi bo‘lib, uning mojaranti (2.3 .2) musbat hadli qatordan iborat bo‘lsin.

Aytaylik, $x_0 \in M$ bo‘lsin, u vaqtda (2.3 .1) dan musbat hadli qatorlarni taqqoslashga asoslangan yaqinlashish belgisiga ko‘ra $\sum_{n=1}^{\infty}|u_n(x_0)|$ qatorning yaqinlashuvchi ekanligini olamiz. Bu esa $\sum_{n=1}^{\infty}u_n(x)$ funksional qator M to‘plamning har bir nuqtasida absolyut yaqinlashuvchi ekanligini ko‘rsatadi. Endi (2.2 .1) funksional qator yig‘indisini $S(x)$ desak, u M to‘plamda aniqlangan ekanligi ravshandir. Funksional qatorning n-qoldig‘i uchun uning mojaranti ta’rifiga asosan $\forall x \in M$ bo‘lganda

$$|S(x) - S_n(x)| \le \sigma_n \qquad (2.3 .3)$$

tengsizlik o‘rinli bo‘lishini ko‘rsatish osondir. Bu yerda $\sigma_n$ (2.3 .2) yaqinlashuvchi musbat hadli qator n-qoldig‘idan iborat bo‘lib, u cheksiz kichik miqdordan iborat ekanligi ravshan, ya’ni $\forall \varepsilon > 0$ olinganda Shunday $n \in N$ topiladiki, $n_0 < n \in N$ bo‘lganda $\sigma_n < \varepsilon$ o‘rinli bo‘ladi. Bundan va (2.3 .2) dan (2.2 .1) funksional qatorning M to‘plamda tekis yaqinlashuvchi bo‘lishi kelib chiqadi. Teorema isbotlandi.

**Misol.** $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ funksional qator $(-\infty;+\infty)$ sohada mojarantlanuvchidir. Haqiqatdan ham, $\forall x \in R$ , $n \in N$ bo‘lganda

$$\left|\frac{\sin x}{n^2}\right| \le \frac{1}{n^2}$$

bo‘lib, $\sum_{n=1}^{\infty}\frac{1}{n^2}$ - musbat hadli qator yaqinlashuvchidir. Demak, oxirgi musbat hadli qator berilgan funksional qatorning mojarantidan iborat bo‘lib, bu funksional qatorning $(-\infty;\infty)$ to‘plamda absolyut va tekis yaqinlashuvchi ekanligi yuqorida isbotlangan teoremadan kelib chiqadi.

## 2.4. Tekis yaqinlashuvchi funksional qatorlarning xossalari

Faraz qilaylik, (2.2 .1) funksional qator berilgan bo‘lib, u $[a;b]$ kesmada yaqinlashuvchi bo‘lsin. Uning n-qismiy yig‘indisi $S_n(x)$ (2.2 .2) kabi aniqlangan bo‘lib, yig‘indisini $S(x)$ bilan belgilaylik. Mazkur bandda tekis yaqinlashuvchi funksional qatorlarning quyidagi asosiy xossalarini keltiramiz.

**1⁰.** Agar (2.2 .1) funksional qatorning har bir hadi $u_n(x)$ $(n \in N)$ $[a;b]$ kesmada uzluksiz bo‘lib, bu qator Shu kesmada tekis yaqinlashuvchi bo‘lsa, uning yig‘indisi $S(x)$ ham $[a;b]$ kesmada uzluksiz funksiya bo‘ladi.

**2⁰.** Agar (2.2 .1) funksional qatorning har bir hadi $u_n(x)$ $(n \in N)$ $[a;b]$ kesmada uzluksiz bo‘lib, funksional qator Shu kesmada $S(x)$ ga tekis yaqinlashuvchi bo‘lsa,

$$\int_a^b\left(\sum_{n=1}^{\infty}u_n(x)\right)dx = \sum_{n=1}^{\infty}\int_a^b u_n(x)dx \qquad (2.4.1)$$

bo‘ladi. Buni tekis yaqinlashuvchi funksional qatorni hadlab integrallash mumkinligi xossasi deb yuritiladi.

**3⁰.** (2.2 .1) funksional qator $[a;b]$ kesmada yaqinlashuvchi bo‘lib, uning yig‘indisi $S(x)$ bo‘lsin. Agar bu qatorning har bir hadi $u_n(x)$ $(n\in N)$ $[a;b]$ kesmada uzluksiz $u_n'(x)$ $(n\in N)$ hosilaga ega bo‘lib,

$$\sum_{n=1}^{\infty} u_n'(x) = u_1'(x) + u_2'(x) + \ldots + u_n'(x) + \ldots$$

funksional qator $[a;b]$ kesmada tekis yaqinlashuvchi bo‘lsa,

$$\left(\sum_{n=1}^{\infty} u_n(x)\right)' = \sum_{n=1}^{\infty} u_n'(x)$$

bo‘ladi. Buni funksional qatorni hadlab differensiallash mumkinligi xossasi deb yuritiladi.

Bu yerda keltirilgan xossalardan $2^0$-sining isbotini keltirish bilan chegaralanamiz va qolganlarini boshqa adabiyotlardan (masalan, [1]) o‘rganish mumkinligini aytamiz.

**$2^0$-xossasining isboti.** Berilgan (2.2 .1) funksional qator $[a;b]$ kesmada $S(x)$ funksiyaga tekis yaqinlashsin: $S(x)=\sum_{n=1}^{\infty} u_n(x)$. Unda tekis yaqinlashish ta’rifiga binoan, $\forall\varepsilon>0$ son olinganda Shunday $n_0\in N$ topiladiki, $n_0<n\in N$ va $\forall x\in[a;b]$ uchun

$$|S_n(x)-S(x)|<\varepsilon \qquad (2.4.2)$$

tengsizlik bajariladi, bunda $S_n(x)$ (2.2 .2) bilan aniqlanadi. Shu bilan birga $S(x)$ $[a;b]$ da uzluksiz bo‘lib, $\int_a^b S(x)dx$ mavjuddir. Shartga ko‘ra, $\int_a^b u_n(x)dx$ $(n\in N)$ mavjud. Endi

$$\sum_{n=1}^{\infty}\int_a^b u_n(x)dx$$

qatorning n-qismiy yig‘indisi uchun

$$\sum_{k=1}^{n}\int_a^b u_k(x)dx = \int_a^b\left(\sum_{k=1}^{n} u_k(x)\right)dx = \int_a^b S_n(x)dx$$

ga ega bo‘lamiz va

$$\int_a^b S(x)dx - \int_a^b S_n(x)dx = \int_a^b [S(x)-S_n(x)]dx$$

(2.4.2) dan foydalanib,

$$\left|\int_a^b S(x)dx - \int_a^b S_n(x)dx\right| \le \int_a^b |S(x)-S_n(x)|dx < \varepsilon(b-a).$$

Bundan

$$\lim_{n\to\infty}\left[\int_a^b S(x)dx-\int_a^b S_n(x)dx\right]=\lim_{n\to\infty}\left[\int_a^b\left(\sum_{n=1}^{\infty}u_n(x)\right)dx-\int_a^b\left(\sum_{k=1}^{n}u_n(x)\right)dx\right]=$$

$$=\lim_{n\to\infty}\left[\int_a^b\left(\sum_{k=1}^{\infty}u_k(x)\right)dx-\int_a^b\left(\sum_{k=1}^{n}u_k(x)\right)dx\right]=0,$$

ya'ni (2.4.1) o'rinli ekani kelib chiqadi.

## 2.5. Darajali qatorlar

### 2.5.1. Asosiy tushunchalar

Ushbu,

$$\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+...+a_nx^n+... \tag{2.5.1}$$

yoki

$$\sum_{n=0}^{\infty}a_n(x-a)^n=a_0+a_1(x-a)+a_2(x-a)^2+...+a_n(x-a)^n+... \tag{2.5.2}$$

ko'rinishdagi qator *darajali qator* deyiladi, bunda $a_n(n=0;1;2;...)$ o'zgarmas haqiqiy sonlar bo'lib, ularni darajali qatorning *koeffisientlari* deb ataladi.

Darajali qatorlar (2.2 .1) funksional qatorlarning xususiy, ya'ni

$$u_n(x)=a_{n-1}x^{n-1}\ \left(ёки=a_{n-1}(x-a)^{n-1}\right)$$

bo'lgan holidir.

(2.5.2) darajali qatorda $x-a=t$ o'rniga qo'yish (almashtirish) qilinsa, u (2.5.1) ko'rinishni oladi. Shu sababli (2.5.1) darajali qatorni tekshirish bilan kifoyalanamiz.

### 2.5.2. Darajali qatorning yaqinlashish radiusi va yaqinlashish intervali

**2.5.1-teorema** (Abel). Agar (2.5.1) darajali qator $x=x_0(x_0\neq 0)$ nuqtada yaqinlashuvchi bo'lsa, u $x$ ning $|x|<|x_0|$ tengsizlikni qanoatlantiruvchi barcha qiymatlarda absolyut yaqinlashuvchi bo'ladi.

**Isbot.** Modomiki (2.5.1) darajali qator $x=x_0$ $(x_0\neq 0)$ nuqtada yaqinlashuvchi ekan, u holda $\sum_{n=1}^{a}a_nx_0^n$ sonli qator yaqinlashuvchi bo'lib,

$$\lim a_nx_0^n=0$$

bo'lishi kelib chiqadi. Demak, $\{a_nx_0^n\}$ ketma-ketlik chegaralangan bo'lib, Shunday $M>0$ o'zgarmas son mavjud bo'ladiki,

$$|a_nx_0^n|\le M\quad(n=0;1;2;...)$$

tengsizlik bajariladi. Bundan

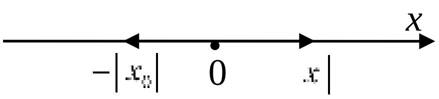
$$|a_n x^n| = \left| a_R x_0^n \cdot \frac{x^n}{x_0^n} \right| \le M \left| \frac{x}{x_0} \right|^n$$

bo‘lib, $|x| < |x_0| \Rightarrow \left| \frac{x}{x_0} \right| < 1$ ekanligidan

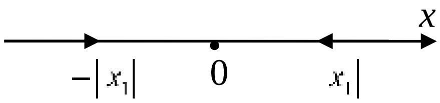
$$\sum_{n=0}^{\infty} M \left| \frac{x}{x_0} \right|^n$$

qator yaqinlashuvchi ekanligi kelib chiqadi. Bu esa (2.5.1) darajali qatorning absolyut yaqinlashuvchi bo‘lishini ko‘rsatadi. Teorema isbotlandi.

Bu teorema, agar $x = x_0$ $(x_0 \ne 0)$ nuqtada (2.5.1) darajali qator yaqinlashuvchi bo‘lsa, u $(-|x_0|; |x_0|)$ intervalda absolyut yaqinlashuvchi bo‘lishini ko‘rsatadi (2.5.1-rasm).




2.5.1-расм.




2.5.2-расм.

**2.5.2-teorema.** Agar (2.5.1) darajali qator $x = x_1$ nuqtada uzoqlashuvchi bo‘lsa, $x$ ning

$$|x| > |x_1| \tag{2.5.3}$$

tengsizlikni qanoatlantiruvchi barcha qiymatlarida uzoqlashuvchi bo‘ladi.

**Isbot.** Teskarisini faraz qilaylik, ya’ni (2.5.1) $|x| > |x_1|$ ni qanatlantiruvchi biror $x^*$ nuqtada yaqinlashuvchi bo‘lsin. U holda $|x_1| < |x^*|$ ekanligidan 2.5.1-teorema asosida (2.5.1) $x_1$ nuqtada absolyut yaqinlashuvchi bo‘ladi. Bu shartga ziddir. Demak, (2.5.3) ni qanoatlantiruvchi birorta nuqtada ham (2.5.1) yaqinlashuvchi bo‘lolmaydi. Teorema isbotlandi.

Bu teorema, agar $x = x_1$ nuqtada (2.5.1) uzoqlashuvchi bo‘lsa, u $(-\infty; -|x_1|) \cup (|x_1|; +\infty)$ to‘plamda uzoqlashuvchi bo‘lishini ko‘rsatadi (2.5.2-rasm).

Faraz qilaylik, (2.5.1) darajali qator $x = x_0$ $(x_0 \ne 0)$ nuqtada yaqinlashuvchi $x = x_1$ nuqtada esa uzoqlashuvchi bo‘lsin. U holda $|x_0| < |x_1|$ bo‘lishi aniq bo‘lib, yuqoridagi isbotlangan teoremalar asosida darajali qator $(-|x_0|; |x_0|)$ intervalda absolyut yaqinlashuvchi, $(-\infty; -|x_1|) \cup (|x_1|; +\infty)$ to‘plamda esa uzoqlashuvchi bo‘ladi. Agar darajali qator $(-|x_0|; |x_0|)$ intervalga tegishli bo‘lmagan biror $x^*$ nuqtada yaqinlashuvchi bo‘lsa,

$$|x_0| \le |x^*| < |x_1|$$

munosabat bajarilib, darajali qator yaqinlashuvchi bo‘ladigan nuqtalar uchun $\{|x^*|\}$ to‘plam chegaralangan ekanligidan uning aniq yuqori chegarasining mavjudligi kelib chiqadi. Uni $r$ bilan belgilaylik:

$$r = Sup\{|x^*|\} \tag{2.5.4}$$

Agar $|x| < r$ ni qanoatlantiruvchi $x$ ni olsak, u vaqtda Shunday $x^*$ mavjud bo‘lib, $|x| < |x^*| < r$ o‘rinli bo‘ladi va $x^*$ nuqtada (2.5.1) darajali qator yaqinlashuvchi bo‘lib, 2.5.1-teorema asosida uning $x$ nuqtada ham yaqinlashishi kelib chiqadi. Bu (2.5.1) ning $(-r;r)$ intervalda yaqinlashuvchi bo‘lishini ko‘rsatadi. Xuddi Shunga o‘xshash darajali qator $|x| > r$ bo‘lgan barcha $x$ nuqtalarda uzoqlashuvchi bo‘lishini ko‘rsatish mumkin (mustaqil bajaring). Shu sababli $(-r;r)$ ni darajali qatorning *yaqinlashish intervali* deb ataladi.

**2.5.1-ta’rif.** (2.5.4) munosabat bilan aniqlangan $r$ soni (2.5.1) darajali qatorning *yaqinlashish radiusi*, $(-r;r)$ esa *yaqinlashish intervali* deb ataladi.

Agar (2.5.1) darajali qator $\forall x \in R$ nuqtalarda (sonlar o‘qining barcha nuqtalarida) yaqinlashuvchi bo‘lsa $r = +\infty$, faqat $x = 0$ nuqtada yaqinlashuvchi bo‘lganda esa $r = 0$ deb qabul qilamiz.

(2.5.1) darajali qator uchun yaqinlashish radiusi $0 < r < +\infty$ bo‘lgan holda $x = \pm r$ nuqtada u yaqinlashuvchi yoki uzoqlashuvchi bo‘lishi mumkin. Buni (2.5.1) darajali qatordan $x = \pm r$ deb olingan sonli qatorni tekshirish natijasida aniqlanadi. Bu o‘rinda quyidagi misollarni keltirishni lozim topdik.

**1-misol.** Ushbu

$$1 + x + x^2 + ... + x^n + ...$$

darajali qator (geometrik qator) ning yaqinlashish radiusi $r = 1$, yaqinlashish intervali esa $(-1;1)$ ekanligiga ishonch hosil qilish oson. Bu qator $x = \pm 1$ nuqtada uzoqlashuvchidir, chunki

$$x = 1 \quad \Rightarrow \quad 1 + 1 + 1 + ... + 1 + ...,$$

$$x = -1 \quad \Rightarrow \quad 1 - 1 + 1 - 1 ... + (-1)^n + ...$$

sonli qatorlarning ikkalasi ham yaqinlashishning zaruriy shartini qanoatlantirmaydi. Demak, ko‘rilayotgan darajali qatorning yaqinlashish sohasi uning yaqinlashish intervali $(-1;1)$ dan iboratdir.

**2-misol.** Ushbu

$$1 + \frac{x}{1^2} + \frac{x^2}{2^2} + ... + \frac{x^n}{n^2} + ...$$

darajali qatorning yaqinlashish radiusini topish uchun Dalamber belgisidan foydalanamiz. Buning uchun bu qator hadlarining absolyut qiymatidan tuzilgan

$$1 + \sum_{n=1}^{\infty} \frac{|x|^n}{n^2}$$

musbat hadli qatorni tekshiramiz. Uning umumiy hadi

$$a_n = \frac{|x|^n}{n^2}$$

bo‘lib, Dalamber belgisini qo‘llash maqsadida quyidagi limitni hisoblaymiz:

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\dfrac{|x|^{n+1}}{(n+1)^2}}{\dfrac{|x|^n}{n^2}} = \lim \frac{n^2}{(n+1)^2}|x| = |x| \lim \left( \frac{1}{1 + \dfrac{1}{n}} \right)^2 = |x|.$$

Bu yuqoridagi musbat hadli qator Dalamber belgisiga ko‘ra $|x|<1$ bo‘lganda yaqinlashuvchi, $|x|>1$ bo‘lganda esa uzoqlashuvchi bo‘lishini ko‘rsatadi. Demak, berilgan qatorning yaqinlashish radiusi $r=1$.

Qaralayotgan darajali qatorda $x=\pm 1$ deb,

$$x=1 \quad \Rightarrow \quad 1+1+\frac{1}{2^2}+\frac{1}{3^2}...+\frac{1}{n^2}+...,$$

$$x=-1 \quad \Rightarrow \quad 1-1+\frac{1}{2^2}-\frac{1}{3^2}...+\frac{(-1)^n}{n^2}+...$$

sonli qatorlarni olamiz. Ularning ikkalasi ham yaqinlashuvchi ekanligiga ishonch hosil qilish qiyin emas. Demak, berilgan darajali qatorning yaqinlashish sohasi $[-1;1]$ kesmadan iborat bo‘lib, u yaqinlashish intervali $(-1;1)$ dan farq qiladi.

**3-misol.** Ushbu

$$\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+...+\frac{x^n}{n}+...$$

darajali qator uchun ham yaqinlashish radiusi $r=1$ ekanligiga oldingi misoldagi kabi ishonch hosil qilish osondir. Endi uni $x=\pm 1$ nuqtada tekshiraylik:

$$x=1 \quad \Rightarrow \quad 1+\frac{1}{2}+...+\frac{1}{n}+...,$$

$$x=-1 \quad \Rightarrow \quad -1+\frac{1}{2}-\frac{1}{3}...+\frac{(-1)^n}{n}+...$$

Bu sonli qatorlardan birinchisi garmonik qator ekanligidan u uzoqlashuvchi, ikkinchisi esa ishorasi navbatlashuvchi qator bo‘lib, Leybnis belgisi asosida yaqinlashuvchidir.

Demak, berilgan darajali qatorning yaqinlashish intervali $(-1;1)$ bo‘lgan holda yaqinlashish sohasi $[-1;1)$ oraliqdan iborat.

Shunday qilib, bu ko‘rilgan misollar asosida (2.5.1) darajali qatorning yaqinlashish sohasini aniqlash uchun uning yaqinlashish radiusini topib, yaqinlashish intervalining uchlarida darajali qatorni yaqinlashishga tekshirish kerakligini ko‘ramiz.

Darajali qatorning yaqinlashish radiusini hisoblashda quyidagi tasdiq qo‘l keladi.

**2.5.3-teorema.** Agar (2.5.1) darajali qator uchun

$$\lim\left|\frac{a_{n+1}}{a_n}\right|=\ell \qquad \left(\lim\sqrt[n]{|a_n|}=\ell\right)$$

limit mavjud bo‘lib, $\ell\neq 0$ chekli son bo‘lsa, darajali qatorning yaqinlashish radiusi $r=\frac{1}{\ell}$ bo‘ladi. Shuningdek, $\ell=0$ bo‘lganda $r=+\infty$, $\ell=+\infty$ bo‘lganda esa $r=0$ bo‘lishi ravshandir.

**Isbot.** $\sum\limits_{n=0}^{\infty}\left|a_n x^n\right|=\sum\limits_{n=0}^{\infty}|a_n|\cdot|x|^n$ musbat hadli qatorga Dalamber (Koshi) belgisini qo‘llaymiz:

$$\lim \frac{|a_{n+1}|\cdot|x|^{n+1}}{|a_n|\cdot|x|^n} = |x|\lim\left|\frac{a_{n+1}}{a_n}\right| = |x|\cdot\ell$$

$$\left(\lim \sqrt[n]{|a_n|\cdot|x|^n} = |x|\lim\sqrt[n]{|a_n|} = |x|\cdot\ell\right).$$

Ma’lumki, $|x|\cdot\ell<1$ bo‘lganda darajali qator yaqinlashuvchi $|x|\cdot\ell>1$ bo‘lganda esa uzoqlashuvchidir. Bundan teoremaning isbotiga kelamiz.

**4-misol.** $\sum_{n=1}^{\infty}\frac{x^n}{\sqrt{n}}$ darajali qatorni tekshiring.

**Yechish.** $a_n = \frac{1}{\sqrt{n}}$.

$$\ell = \lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}} = \lim\sqrt{\frac{1}{1+\frac{1}{n}}} = 1.$$

Yaqinlashish radiusi: $r = \frac{1}{\ell} = 1$, yaqinlashish intervali $(-1;1)$.

Endi yaqinlashish intervalining uchlarida darajali qatorni tekshiramiz.

$x=1 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ - uzoqlashuvchi,

$x=-1 \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}} = -1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\ldots+\frac{(-1)^n}{\sqrt{n}}+\ldots,$

ishorasi navbatlashuvchi qator bo‘lib, hadlari absolyut qiymat jihatidan kamayuvchi va nolga intiluvchi bo‘lgani uchun Leybnis belgisiga ko‘ra bu qator yaqinlashuvchidir. Demak, darajali qatorning yaqinlashish sohasi $[-1;1)$ oraliqdan iborat.

**5-misol.** $\sum\frac{x^n}{3^n}$ darajali qatorni tekshiring.

**Yechish.** $a_n = \frac{1}{3^n}$, $\lim\sqrt[n]{|a_n|} = \lim\sqrt[n]{\frac{1}{3^n}} = \frac{1}{3}$, $\ell = \frac{1}{3} \Rightarrow r = \frac{1}{\ell} = 3$.

Yaqinlashish intervali: $(-3;3)$

$x=3 \Rightarrow 1+1+1+\ldots+1+\ldots$ - uzoqlashuvchi,

$x=-3 \Rightarrow -1+1-1+\ldots+(-1)^n+\ldots$ - uzoqlashuvchi.

Demak, yaqinlashish sohasi $(-3;3)$.

### 2.5.3. Darajali qatorning xossalari

Aytaylik, (2.5.1) darajali qator berilgan bo‘lsin.

**$1^0$.** Agar (2.5.1) darajali qatorning yaqinlashish radiusi $r$ $(r>0)$ bo‘lsa, u holda bu qator $[a;b]\subset(-r;r)$ bo‘lgan kesmada tekis yaqinlashuvchi bo‘ladi.

**Isbot.** $\rho = \max\{|a|;|b|\}$ desak, $\rho < r$ hamda $[a;b] \subset (-\rho;\rho) \subset (-r;r)$ bo‘lishi ravshandir. Bundan

$$\sum_{n=0}^{\infty} |a_n||\rho|^n$$

sonli qator yaqinlashuvchi bo‘lib, u $[a;b]$ kesmada (2.5.1) darajali qatorning mojarantidan iborat bo‘lishini ko‘ramiz. Demak, 2.3 .1-teorema (Veyershtrass belgisi) asosida (2.5.1) darajali qatorning $[a;b]$ kesmada tekis yaqinlashuvchi ekanligi kelib chiqadi. Xossa isbotlandi.

**2⁰.** Agar (2.5.1) darajali qatorning yaqinlashish radiusi $r$ $(r > 0)$ bo‘lsa, u holda bu darajali qatorning yig‘indisi $S(x)$ $(-r;r)$ da uzluksiz funksiya bo‘ladi.

**Isbot.** $0 < \rho < r$ bo‘lgan sonni olsak, $[-\rho;\rho] \subset (-r;r)$ bo‘lib, $1^0$-xossaga ko‘ra darajali qator $[-\rho;\rho]$ kesmada tekis yaqinlashuvchi bo‘ladi. Darajali qatorning har bir hadi darajali funksiya sifatida uzluksiz bo‘lganligi sababli tekis yaqinlashuvchi funksional qatorning 2.4-bandda qayd qilingan $1^0$-xossasiga ko‘ra uning yig‘indisi $[-\rho;\rho]$ kesmada uzluksiz bo‘lishi kelib chiqadi. Bu esa xossaning isbotidir.

**3⁰.** Agar (2.5.1) darajali qatorning yaqinlashish radiusi $r$ $(r > 0)$ bo‘lsa, bu qatorni $[a;b] \subset (-r;r)$ bo‘lgan oraliqda hadlab integrallash mumkin. (Isbotini mustaqil bajaring).

**4⁰.** Agar (2.5.1) darajali qatorning yaqinlashish radiusi $r$ $(r > 0)$ bo‘lsa, uni $(-r;r)$ yaqinlashish intervalida hadlab differensiallash mumkin.

**Isbot.** (2.5.1) ni $(-r;r)$ yaqinlashish intervalida hadlab differensiallab,

$$\sum_{n=1}^{\ell} n a_n x^{n-1} \qquad (2.5.5)$$

darajali qatorga kelamiz. Agar $0 < \rho < r$ sonni olsak, $\rho < \rho_0 < r$ bo‘lganda $x = \rho_0$ nuqtada (2.5.1) yaqinlashuvchi ekanligi sababli, yaqinlashishning zaruriy sharti asosida

$$\left| a_n \rho_0^n \right| \le M \qquad (n \in N)$$

bo‘ladi, bu yerda *M* biror musbat sondir.

Endi $|x| \le \rho$ shartda (2.5.5) ning umumiy hadining qiymatini baholaylik:

$$\left| n a_n x^{n-1} \right| \le \left| n a_n \rho^{n-1} \right| \le n \left| a_n \right| \rho_0^{n-1} \cdot \left| \frac{\rho}{\rho_0} \right|^{n-1} \le n \frac{M}{\rho_0} q^{n-1}.$$

Bu yerda $q = \dfrac{\rho}{\rho_0} < 1.$

Shunday qilib, (2.5.5) darajali qatorning har bir hadi $|x| \le \rho$ bo‘lganda absolyut qiymat jihatidan

$$\sum_{n=1}^{\infty} n \frac{M}{\rho_0} q^{n-1} \qquad (2.5.6)$$

musbat hadli qator mos hadidan katta emasligini ko‘ramiz. (2.5.6) musbat hadli qatorni Dalamber belgisi yordamida yaqinlashuvchi ekanligini ko‘rsatish oson. Haqiqatdan ham,

$$\lim \frac{(n+1)\dfrac{M}{\rho_0} q^n}{n\dfrac{M}{\rho_0} q^{n-1}} = q < 1 .$$

Demak, (2.5.5) darajali qator $[-\rho;\rho]$ kesmada mojarantlanuvchidir. Uning yig‘indisi esa berilgan (2.5.1) darajali qator yig‘indisining hosilasidan iboratdir. $(-r;r)$ yaqinlashish intervalining ixtiyoriy nuqtasini biror $[-\rho;\rho]\subset(-r;r)$ bo‘lgan kesmaga tegishli qilib olish mumkinligidan yuqorida olingan natija $(-r;r)$ da o‘rinli bo‘lishi kelib chiqadi.

Endi (2.5.5) ning yaqinlashish radiusi ham $r$ dan iborat ekanligini ko‘rsataylik. $|x|<r$ bo‘lganda (2.5.5) ning yaqinlashuvchi ekanligini ko‘rdik. $|x|>r$ bo‘lganda (2.5.5) ning uzoqlashuvchi ekanligini ko‘rsataylik. Teskarisini faraz qilaylik, ya’ni $|x_1|>r$ bo‘lgan biror $x_1$ nuqtada (2.5.5) yaqinlashuvchi bo‘lsin. U vaqtda $r<|x_2|<|x_1|$ bo‘lgan $x_2$ nuqtada u absolyut yaqinlashuvchi bo‘ladi (Abel teoremasi).

(2.5.1) ni $x=x_2$ nuqtada tekshiramiz:

$$\left|a_n x_2^n\right| = |x_2|\cdot\left|a_n x_2^{n-1}\right| = \frac{|x_2|}{n}\left|n a_n x_2^{(n-1)}\right| .$$

Agar $n_0 = [|x_2|]$ desak, $n_0 < n \in N$ bo‘lganda oxirgidan

$$\left|a_n x_2^n\right| < \left|n a_n x_2^{n-1}\right|$$

bo‘lishini, bundan esa (2.5.1) ning $x_2$ nuqtada absolyut yaqinlashuvchi bo‘lishini ko‘ramiz. Bu shartga ziddir. Demak, $|x|>r$ bo‘lganda (2.5.5) uzoqlashuvchi bo‘ladi.

Bu xossadan darajali qatorni yaqinlashish intervalida xohlagan marta hadlab differensiallash mumkin degan xulosa kelib chiqadi.

### 2.5.4. Teylor qatori

Ma’lumki, $f(x)$ funksiya $x=a$ nuqtaning $\delta$ atrofida $(\delta>0)$ $(n+1)$ - tartibligacha differensiallanuvchi bo‘lsa, u holda

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + r_n(x) \qquad (2.5.7)$$

Teylor formulasi o‘rinli bo‘lar edi (10.8-bandga qarang), bu yerda $r_n(x)$ bu formulaning qoldiq hadi .

Agar $f(x)$ funksiya $x=a$ nuqtaning $\delta$ atrofida istalgan tartibli hosilaga ega, ya’ni cheksiz marta differensiallanuvchi bo‘lib, $n\to+\infty$ da

$$\lim r_n(x) = 0 \qquad (2.5.8)$$

o‘rinli bo‘lsa, u holda (2.5.7) dan $n\to+\infty$ dagi limitga o‘tib,

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots \qquad (2.5.9)$$

ni olamiz. (2.5.9) ning o‘ng tomonidagi darajali qatorni *Teylor qatori*, uning o‘zini esa $f(x)$*funksiyani* $x=a$ *nuqta atrofida Teylor qatoriga yoyish formulasi* deb ataladi.

Bu o‘rinda (2.5.9) formula (2.5.8) bajarilgan holdagina to‘g‘ri ekanligini aytamiz. Agar Teylor qatorida $a=0$ bo‘lsa,

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots \qquad (2.5.10)$$

ni olamiz va (2.5.10) ning o‘ng tomonini *Makloren qatori*, o‘zini esa $f(x)$*funksiyani Makloren qatoriga yoyish formulasi* deb atalishini aytamiz.

Bu yerda ba’zi-bir elementar funksiyalarni Makloren qatoriga yoyish formulalarini keltirish bilan cheklanamiz.

1) $f(x)=e^x$ funksiya ixtiyoriy $(-\delta;\delta)$ $(\delta>0)$ oraliqda istalgan tartibli hosilaga ega bo‘lib, uning Makloren formulasi

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+r_n(x)$$

ko‘rinishda va qoldiq Lagranj shaklida

$$r_n(x)=\frac{x^{n+1}}{(n+1)!}e^{\theta x} \qquad (0<\theta<1)$$

bo‘lishi ma’lumdir (10.8-bandga qarang).

Bularda $n\to+\infty$ dagi limitga o‘tib,

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\ldots+\frac{x^n}{n!}+\ldots \qquad (2.5.11)$$

ni olamiz. Bu $f(x)=e^x$ ning Makloren qatoriga yoyilmasidir. Uning yaqinlashish radiusini hisoblaylik:

$$\ell=\lim\left|\frac{a_{n+1}}{a_n}\right|=\lim\left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right|=\lim\frac{1}{n+1}=0.$$

Demak, yaqinlashish radiusi $r=+\infty$ bo‘lib, yoyilma sonlar o‘qining barcha nuqtalarida o‘rinli bo‘lishini ko‘ramiz. Bundan $x$ ning ixtiyoriy qiymati uchun

$$\lim r_n(x)=\lim\frac{x^{n+1}}{(n+1)!}e^{\theta x}=0$$

bo‘lishi ham kelib chiqadi.

2) $f(x)=\sin x$ funksiya ixtiyoriy $(-\delta;\delta)$ $(\delta>0)$ oraliqda istalgan tartibli hosilaga ega bo‘lib, uning Makloren formulasi

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+r_{2n+2}(x)$$

bo‘ladi. Bu formuladagi qoldiq hadning Lagranj ko‘rinishi

$$r_{2n+2}(x)=(-1)^{n+1}\frac{x^{2n+2}}{(2n+2)}\sin\theta x \qquad (0<\theta<1)$$

bo‘ladi. Bularda $n\to+\infty$ deb,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots \qquad (2.5.12)$$

yoyilmani olamiz. Bu $f(x) = \sin x$ funksiyaning Makloren qatoriga yoyilmasidir. Uni tekshiraylik. Dalamber belgisidan foydalanamiz:

$$\lim \left| \frac{\frac{x^{2n+3}}{(2n+3)!}}{\frac{x^{2n+1}}{(2n+1)!}} \right| = x^2 \lim \frac{1}{(2n+1)(2n+3)} = 0 \cdot x^2 = 0 < 1, \quad x \in R.$$

Demak, $\sin x$ ning Makloren qatori $\forall x \in R$ bo‘lgan nuqtada yaqinlashuvchidir, ya’ni $r = +\infty$, yaqinlashish sohasi $(-\infty; +\infty)$. Bundan

$$\lim \frac{x^{2n+2}}{(2n+2)!} \sin \theta x = 0$$

$\forall x \in R$ uchun o‘rinli bo‘lishi ham kelib chiqadi.

3) $f(x) = \cos x$ funksiya uchun (2.5.12) dan darajali qatorni hadlab differensiallab,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \qquad (2.5.13)$$

ni olamiz. Bu $f(x) = \cos x$ funksiyaning Makloren qatoriga yoyilmasidir.

4) $f(x) = e^{-x}$ funksiyaning Makloren qatoriga yoyilmasini (2.5.11) da $x$ ni $-x$ ga almashtirib olamiz:

$$e^{-x} = 1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^n}{(n)!} + \ldots, \quad x \in R.$$

5) $|x| < 1$ bo‘lganda

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \ldots$$

yoyilma o‘rinli ekanligiga ishonch hosil qilish oson. Bu tenglikning har ikki tomonini $[0, x]$ oraliq bo‘yicha integrallab, darajali qatorning hadlab integrallash mumkinligi xossasidan foydalanib,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^n \frac{x^{n+1}}{n+1} + \ldots, \quad x \in (-1;1]$$

yoyilmani olamiz. Bu $f(x) = \ln(1+x)$ funksiyaning Makloren qatoriga yoyilmasi bo‘lib, u $x \in (-1;1]$ bo‘lganda o‘rinlidir. Bunda $x$ ni $(-x)$ ga almashtirib,

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots - \frac{x^{n+1}}{n+1} - \ldots, \quad x \in [-1;1)$$

yoyilmani olamiz.

Oxirgilardan ularni hadlab ayirib,

$$\ln \frac{1+x}{1-x} = 2\left( x + \frac{x^3}{3} + \ldots + \frac{x^{2n+1}}{2n+1} + \ldots \right), \quad x \in (-1;1)$$

ga ega bo‘lamiz.

6. $|x| < 1$ bo‘lganda

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\ldots+(-1)^{n+1}x^{2n}+\ldots$$
yoyilma o‘rinlidir. Uni $(0;x)$ $(|x|<1)$ oraliqda hadlab intervallab,
$$arctgx=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots+(-1)^{n+1}\frac{x^{2n+1}}{2n+1}+\ldots,\quad x\in(-1;1)$$
yoyilmani olamiz. Bu $x\in[-1;1]$ bo‘lganda ham o‘rinli ekanligiga ishonch hosil qilish qiyin emas.

### 2.5.5. Eyler formulasi

$z\in K$ bo‘lganda, ya’ni kompleks sohada $e^z$ ko‘rsatkichli funksiyani (2.5.11) formula vositasida
$$e^z=1+z+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!}+\ldots \tag{2.5.14}$$
tenglik bilan aniqlaymiz. Bu kompleks sohadagi $e^z$ ko‘rsatkichli funksiyaning ta’rifidir. Uning aniqlanish sohasi butun kompleks tekislikdan iboratdir ($[2]$ 11-bobga qarang).

Agar (2.5.14) da $z=i\varphi,\ \varphi\in R$ desak va $i$ mavhum birlikning natural ko‘rsatkichli darajalarini eslasak, quyidagini olamiz:
$$e^{i\varphi}=\left(1-\frac{\varphi^2}{2!}+\frac{\varphi^4}{4!}-\frac{\varphi^6}{6!}+\ldots+(-1)^{n+1}\frac{\varphi^{2n}}{(2n)!}+\ldots\right)+$$
$$+i\left(\varphi-\frac{\varphi^3}{3!}+\frac{\varphi^5}{5!}-\frac{\varphi^7}{7!}+\ldots+(-1)^{n+1}\frac{\varphi^{2n+1}}{(2n+1)!}+\ldots\right).$$
Bundan (2.5.12) va (2.5.13) larni e’tiborga olgan holda
$$e^{i\varphi}=\cos\varphi+i\sin\varphi \tag{2.5.15}$$
ni olamiz. Bu Eyler formulasidir. Unda $\varphi$ ni $-\varphi$ bilan almashtirib,
$$e^{-i\varphi}=\cos\varphi-i\sin\varphi \tag{2.5.16}$$
ga kelamiz. Buni ham Eyler formulasi deb yuritiladi. (2.5.15) va (2.5.16) lar, ya’ni Eyler formulalari $\varphi\in K$ bo‘lganda ham o‘rinlidir. Bu formulalarni ba’zan
$$\begin{cases}\cos\varphi=\dfrac{1}{2}\left(e^{i\varphi}+e^{-i\varphi}\right),\\ \sin\varphi=\dfrac{1}{2i}\left(e^{i\varphi}-e^{-i\varphi}\right)\end{cases}$$
ko‘rinishda ham yoziladi.

Agar $z=x+yi$ bo‘lsa, $e^{x+yi}=e^x\cdot e^{yi}$ xossa o‘rinli deb faraz qilib, Eyler formulasi asosida
$$e^z=e^x\cdot(\cos y+i\sin y)$$
formulani olamiz. Bu olingan formula vositasida kompleks sohada $e^z$ ko‘rsatkichli funksiyaning moduli uchun
$$|e^z|=e^x$$

ni olamiz va bundan ko‘rsatkichli funksiya kompleks sohada ham nolga aylana olmasligiga ishonch hosil qilamiz.

### 2.5.6. Darajali qator yordamida integralni taqribiy hisoblash

Agar $f(x)$ funksiya $a$ nuqtaning biror $(a-\delta, a+\delta)$ atrofida $(\delta>0)$ istalgan tartibni hosilaga ega bo‘lib, uning $a$ nuqta atrofidagi Teylor formulasining qoldiq hadi hosila tartibi $n\to\infty$ da cheksiz kichik miqdor bo‘lsa, 2.5.4-bandda ko‘rilgan $f(x)$ ning $x=a$ nuqta atrofidagi

$$f(x)=f(a)+\sum_{n=1}^{+\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n, \quad x\in(a-\delta,\, a+\delta) \qquad (2.5.17)$$

Teylor qatoriga yoyilmasi o‘rinli bo‘ladi.

Faraz qilaylik, $[a;b]\subset(a-\delta,\, a+\delta)$ bo‘lsin. U holda $f(x)$ funksiyaning $[a;b]$ oraliq bo‘yicha aniq integrali mavjud bo‘lib, (2.5.17) ning har ikki tomonini bu oraliq bo‘yicha integrallab va o‘ng tomonidagi darajali qatorga hadlab integrallash mumkinligi xossasini qo‘llab,

$$\int_a^b f(x)dx=f(a)\cdot(b-a)+\sum_{n=1}^{\infty}\frac{f^{(n)}(a)}{n!}\cdot\frac{(b-a)^{n+1}}{n+1}=$$

$$=(b-a)\left[f(a)+\sum_{n=1}^{\infty}\frac{f^{(n)}(a)}{(n+1)!}\cdot(b-a)^n\right]$$

ni, ya’ni

$$\int_a^b f(x)dx=(b-a)\cdot\left(f(a)+\sum_{n=1}^{\infty}\frac{f^{(n)}(a)}{(n+1)!}\cdot(b-a)^n\right)$$

formulani olamiz. Unda o‘ng tomondagi sonli qatorni berilgan aniqlikda taqribiy hisoblab, aniq integralning taqribiy qiymatini topish mumkin.

**1-misol.** $\int_0^1 e^{-x^2}dx$ - integralni 0,001 aniqlikda taqribiy hisoblang.

**Yechish.** (2.5.11) da $x$ ni $-x^2$ bilan almashtirib,

$$e^{-x^2}=1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\ldots+(-1)^n\frac{x^{2n}}{n!}+\ldots, \quad x\in(-\infty;+\infty)$$

ni, ya’ni $e^{-x^2}$ funksiyaning Makloren qatoriga yoyilmasini olamiz. Uni $[0;1]$ oraliq bo‘yicha integrallaylik:

$$\int_0^1 e^{-x^2}dx=1-\frac{1}{1!\cdot 3}+\frac{1}{2!\cdot 5}-\frac{1}{3!\cdot 7}+\ldots+(-1)^n\frac{1}{n!(2n+1)}+\ldots .$$

Agar Teylor formulasi qoldig‘ining Lagranj shaklidan foydalansak,

$$\int_0^1 e^{-x^2}dx\approx 1-\frac{1}{1!\cdot 3}+\frac{1}{2!\cdot 5}-\frac{1}{3!\cdot 7}+\ldots+(-1)^n\frac{1}{n!(2n+1)}$$

taqribiy formulasining xatoligi uchun quyidagi bahoni olish mumkin:

$$\int_0^1|r_n(x)|dx\approx\int_0^1\frac{1}{(n+1)!}e^{-\theta x^2}x^{2n+1}dx<\frac{1}{(n+1)!}\int_0^1 x^{2n+1}dx=\frac{1}{(n+1)!(2n+2)}.$$

Agar taqribiy qiymatni biror $\varepsilon > 0$ aniqlikda hisoblash talab qilingan bo‘lsa, $n$ ning qiymatini

$$\frac{1}{(n+1)!(2n+2)} \le \varepsilon$$

tengsizlik vositasida aniqlaymiz. $\varepsilon = 0{,}001$ aniqlik talab qilinganligi uchun $n = 4$ bo‘lganda yuqoridagi tengsizlik bajarilishini tekshirib ko‘rish oson. Demak, $\varepsilon = 0{,}001$ aniqlikda

$$\int_0^1 e^{-x^2} dx \approx 1 - \frac{1}{1!\cdot 3} + \frac{1}{2!\cdot 5} - \frac{1}{3!\cdot 7} + \frac{1}{4!\cdot 9} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} = 0{,}7675 \approx 0{,}768 .$$

**2-misol.** $\int_0^1 \frac{\sin x}{x} dx$ ni 0,001 aniqlikda taqribiy hisoblang.

**Yechish.** (2.5.12) formuladan foydalanib,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \ldots, \qquad x \ne 0$$

ni olamiz. Oxirgini $[0;1]$ oraliqda hadlab integrallab,

$$\int_0^1 \frac{\sin x}{x} dx = 1 - \frac{1}{3!\cdot 3} + \frac{1}{5!\cdot 5} - \frac{1}{7!\cdot 7} + \ldots + (-1)^n \frac{1}{(2n+1)!(2n+1)} + \ldots$$

ni olamiz.

Agar $\sin x$ ning Makloren formulasi Lagranj ko‘rinishdagi qoldiq hadini eslasak $\frac{\sin x}{x}$ uchun

$$r_n(x) = (-1)^{n+1} \frac{x^{2n+1}}{(2n+2)!} \sin \theta x \quad (0 < \theta < 1)$$

ni olamiz. Buni $[0;1]$ oraliq bo‘yicha integrallab, qoldiq had integrali uchun quyidagi bahoga ega bo‘lamiz:

$$\left| \int_0^1 r(x) dx \right| < \frac{1}{(2n+2)!\cdot(2n+2)} .$$

Oxirgidan $\varepsilon = 0{,}001$ aniqlikka erishish uchun $n = 2$ deb olish kifoya ekanligiga ishonch hosil qilish qiyin emas.

$$\int_0^1 \frac{\sin x}{x} dx \approx 1 - \frac{1}{3!\cdot 3} + \frac{1}{5!\cdot 5} \approx 0{,}946 .$$

### 2.5.7. Differensial tenglamani qator yordamida integrallash

Bu bandda differensial tenglamaning xususiy yechimini qator yordamida qurishning ba’zi-bir misollarini keltiramiz.

Agar

$$y' = f(x; y), \; y\big|_{x=x_0} = y_0 \qquad (2.5.18)$$

Koshi masalasini olsak, uning yechimini ixtiyoriy tartibli hosilaga ega degan faraz asosida Teylor qatoriga yoyib,

$$y(x)=y_0+\sum_{n=1}^{\infty}\frac{y^{(n)}(x_0)}{n!}(x-x_0)^n \qquad (2.5.19)$$

ga ega bo‘lamiz. (2.5.19) dan $x=x_0$ bo‘lganda $y(x_0)=y_0$ bo‘lishini ko‘ramiz. Endi (2.5.18)dan

$$y'(x_0)=f(x_0\,;y_0)$$

ni olamiz. (2.5.18) dan differensial tenglamaning har ikki tomonini differensiallab,

$$y''=f'_x(x;y)+f'_y(x;y)\cdot y'$$

ni va bunda $x=x_0$ deb,

$$y''(x_0)=f'_x(x_0\,;y_0)+f'_y(x_0\,;y_0)\cdot y'(x_0)$$

ni olamiz. Xuddi Shunga o‘xshash bu jarayonni cheksiz davom ettirish natijasida (2.5.19) yoyilmaning barcha koeffisientlarini aniqlash mumkin.

**1-misol.** $y'=xy+y^2$, $x=0$, $y=1$ Koshi masalasini yeching.

**Yechish.** $y'(0)=0\cdot1+1^2=1$,

$$y''=y+xy'+2y\cdot y'=y+(x+2y)y',$$

$$y''(0)=1+(0+2\cdot1)\cdot1=3;$$

$$y'''=y'+(1+2y')y'+(x+2y)y''=(2+2y')y'+(x+2y)y'',$$

$$y'''(0)=(2+2\cdot1)\cdot1+(0+2\cdot1)\cdot3=10\,.$$

Va hokazo. Demak, yuqoridagilar asosida yechimni ifodalovchi qatorning birinchi 4 ta hadini yoza olamiz:

$$y=1+x+\frac{3}{2}x^2+\frac{4}{3}x^3+\ldots.$$

Yuqori tartibi differensial tenglama uchun qo‘yilgan Koshi masalasini ham yuqorida keltirilgan usul bilan darajali qator yordamida yechish mumkin.

**2-misol**. $y''=-yx^2$ differensial tenglamaning

$$y|_{x=1}=1,\qquad y'|_{x=0}=0$$

boshlang‘ich shartlarni qanoatlantiruvchi xususiy yechimini toping.

**Yechish.** Bu masalaning yechishini

$$y(x)=y_0+\sum_{n=1}^{\infty}\frac{y^{(n)}(0)}{n!}x^n$$

Makloren qatori ko‘rinishida izlasak, boshlang‘ich shartlar asosida $y_0=1$, $y'(0)=0$ bo‘lishini ko‘rish oson. $2\le n\in N$ bo‘lganda $y^{(n)}(0)$ larni berilgan differensial tenglamadan topamiz:

$$y''(x_0)=-y(0)\cdot0^2=0\,;$$

$$y'''(x)=-y'x^2-y\cdot2x,\ y'''(0)=0\,;$$

$$y^{IV}=-y''x^2-4xy'-2y\ y^{IV}(0)=-2\,;$$

va hokazo, Leybnis formulasiga binoan

$$y^{(k+2)}=-y^{(k)}x^2-2ky^{(k-1)}x-k(k-1)y^{(k-2)}.$$

Bunda $x=0$ deb,

$$y^{(k+2)}(0)=-k(k-1)y^{(k-2)}_{(0)}$$

yoki $k+2=n$ deb,

$y_{(0)}^{(n)} = -(n-3)(n-2)y_{(0)}^{(n-4)}$

ni olamiz .

Oxirgidan

$y_0^{IV} = -1\cdot 2$, $y_0^8 = -5\cdot 6 y_{(0)}^{(IV)} = (-1)^2\cdot(1\cdot 2)\cdot(5\cdot 6)$,

$y_0^{12} = -9\cdot 10 y_{(0)}^{(8)} = (-1)^3(1\cdot 2)\cdot(5\cdot 6)\cdot(9\cdot 10)$, ... ,

$y_{(0)}^{(4k)} = (-1)^k\cdot(1\cdot 2)\cdot(5\cdot 6)\cdot(9\cdot 10)\cdot\ldots\cdot[(4k-3)\cdot(4k-2)]$

ni va

$$y_{(0)}^{(4k+1)}(0) = y_{(0)}^{(4k+2)}(0) = y_{(0)}^{(4k+3)} = 0$$

bo‘lishini olamiz.

Demak, Koshi masalasining yechimi uchun

$$y = 1 - \frac{x^4}{4!}\cdot 1\cdot 2 + \frac{x^8}{8!}(1\cdot 2)\cdot(5\cdot 6) - \frac{x^{12}}{12!}(1\cdot 2)\cdot(5\cdot 6)\cdot(9\cdot 10) + \ldots. +$$
$$+ (-1)^k\frac{x^{4k}}{(4k)!}(1\cdot 2)\cdot(5\cdot 6)\cdot\ldots.\cdot[(4k-3)\cdot(4k-2)] + \ldots$$

darajali qatorga egamiz. Bu Makloren qatorini $x\in(-\infty;\infty)$ bo‘lganda yaqinlashuvchi ekanligiga ishonch hosil qilish qiyin emas (mustaqil bajaring).

## 2-bob ga doir mashqlar

1. Berilgan funksional qatorning yaqinlashish sohasini toping.

1) $\sum_{n=0}^{\infty}\frac{x^n}{2^n}$ Javob: $(-2;2)$ .

2) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n^2}$ Javob: $[-1;1]$

3) $\sum_{n=0}^{\infty}\frac{(100x)^n}{1\cdot 2\cdot\ldots\cdot(2n+1)}$ Javob: $(-\infty;+\infty)$

4) $\sum_{n=0}^{\infty}2^n\sin\frac{x}{3^n}$ Javob: $(-\infty;+\infty)$

5) $\sum_{n=1}^{\infty}\frac{x^n}{n+\sqrt{n}}$ Javob: $[-1;1)$

6) $\sum_{n=1}^{\infty}\frac{x^n}{n!}$ Javob: $(-\infty;+\infty)$

7) $\sum_{n=1}^{\infty}\frac{n!}{n^n}x^n$ Javob: $(-e;e)$

2. Berilgan qatorni berilgan oraliqda mojarantlanuvchi ekanlikka tekshiring.

1) $\sum_{n=1}^{\infty}\frac{x^n}{n^2}$, $(0\le x\le 1)$ Javob: mojarantlanuvchi.

2) $\sum_{n=1}^{\infty}\frac{x^n}{n}$ $(0\le x\le 1)$ Javob: mojarantlanuvchi emas.

3) $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ $(0\le x\le \pi)$ Javob: mojarantlanuvchi.

3. Berilgan funksiyani darajali qatorga yoying.

1) $\ln x$ ni $(x-1)$ ning darajalari bo‘yicha.

Javob: $(x-1)-\frac{1}{2}(x-1)^2+\ldots+(-1)^{n+1}\frac{(x-1)^n}{n}+\ldots$ , $x\in(0;2]$

2) $e^x$ ni $(x+2)$ ning darajalari bo‘yicha.

Javob: $e^{-2}\left[1+\sum_{n=1}^{\infty}\frac{(x+2)^n}{n!}\right]$, $x\in(-\infty;+\infty)$

3) $\frac{1}{x^2}$ ni $(x+1)$ ning darajalari bo‘yicha.

Javob: $\sum_{n=0}^{a}(n+1)(x+1)^n$ , $x\in(-1;0)$

4) $\sin^2 x$ ni $x$ ning darajalari bo‘yicha.

Javob: $\sum_{n=0}^{\infty}(-1)^{n-1}\frac{2^{2n-1}\cdot x^{2n}}{(2n)!}$, $x\in(-\infty;+\infty)$

5) $\frac{1}{1+x^2}$ ni $x$ ning darajalari bo‘yicha.

Javob: $\sum_{n=0}^{\infty}(-1)^n x^{2n}$ , $x\in(-1;1)$

6) $\cos^2 x$ ni $x$ ning darajalari bo‘yicha.

Javob: $1+\frac{1}{2}\sum\frac{(-1)^n(2x)^{2n}}{(2n)!}$, $x\in(-\infty;+\infty)$

7) $(1+x)^m$ ni $x$ ning darajalari bo‘yicha.

Javob: $1+\sum_{n=1}^{\infty}\frac{m\cdot(m-1)\cdot\ldots\cdot(m-n+1)}{n!}x^n$ , $x\in(-1;1)$

8) $\frac{1}{4-x^4}$ ni $x$ ning darajalari bo‘yicha.

Javob: $\sum_{n=0}^{\infty}\frac{x^{4n}}{4^{n+1}}$, $x\in\left(-\sqrt{2};\sqrt{2}\right)$

4. Berilgan differensial tenglamaning berilgan boshlang‘ich shartlarni qanoatlantiruvchi xususiy yechimini toping.

1) $y''=xy$; $x=0$, $y=1$, $y'=0$.

Javob: $1+\sum_{n=1}^{\infty}\frac{x^{3n}}{2\cdot3\cdot\ldots\cdot(3n)}$ , $x\in(-\infty;+\infty)$

2) $y''+xy'+y=0$ $x=0$, $y=0$, $y'=1$.

Javob: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n-1}}{1\cdot3\cdot\ldots\cdot(2n-1)}$, $x\in(-\infty;+\infty)$

3) $xy''+y'+xy=0$, $x=0$, $y=1$, $y'=0$.

Javob: $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(n!)^2 \cdot 2^{2n}}$, $x \in (-\infty;+\infty)$

4) $(1+x^2)y'' - xy' = 0$, $x = 0$, $y = 0$, $y' = 1$.

Javob: $\sum_{n=0}^{\infty}(-1)^{n+1} \frac{x^n}{n}$, $x \in (-1;1]$

5) $(1-x)y' = 1 + x - y$, $x = 0$, $y = 0$.

Javob: $x + \sum_{n=2}^{\infty} \frac{x^n}{(n-1)n}$, $x \in [-1;1]$

5. Berilgan differensial tenglama uchun qo‘yilgan Koshi masalasi yechimining Teylor qatoridagi dastlabki birnechta hadlarini yozing.

1) $y'' = yy' - x^2$; $x = 0$, $y = 1$, $y' = 0$.

Javob: $1 + x + \frac{x^2}{2!} + \frac{2x^3}{5!} + \frac{8x^4}{4!} + \frac{14x^5}{5!} + \ldots$

2) $y' = x^2 + y^2$, $x = 0$, $y = \frac{1}{2}$.

Javob: $\frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \frac{1}{6}x^3 + \frac{9}{32}x^4 + \ldots$ .

3) $y' = x^2 - y^2$, $x = 0$, $y = 0$.

Javob: $x + \frac{x^2}{2} + \frac{2x^3}{3} + \frac{11x^4}{2 \cdot 3 \cdot 4} + \ldots$

## 2-bob bo‘yicha bilimingizni sinab ko‘ring

1. Funksional ketma-ketlik va funksional qator tushunchalarini ayting.
2. Funksional ketma-ketlik va qatorning tekis yaqinlashishi qanday ta’riflanadi?
3. Darajali qator va uning yaqinlashish intervali nimadan iborat?
4. Darajali qatorning xossalarini ayting.
5. Teylor qatori nimadan iborat?
6. Darajali qator yordamida integralni hisoblashni tuShuntiring.
7. Darajali qator yordamida differensial tenglama qanday integrallanadi?

# 3-bob. Fur`e qatori

Mazkur bobda Fur`ening trigonometrik qatorini keltirish bilan chegaralanamiz.

## 3.1. Trigonometrik qator tushunchasi va davriy funksiyani trigonometrik qatorga yoyish

Ushbu

$$\frac{a_0}{2}+a_1\cos x+b_1\sin x+a_2\cos 2x+b_2\sin 2x+...+a_n\cos nx+b_n\sin nx+...$$

yoki , qisqacha

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{3.1.1}$$

ko‘rinishidagi funksional qator trigonometrik qator deb ataladi. Bunda $a_0, a_n, b_n$ $(n\in N)$ sonlar trigonometrik qatorning koeffisientlari deyiladi.

Agar (3.1.1) funksional qator yaqinlashuvchi bo‘lsa, uning yig‘indisidan iborat $f(x)$ $2\pi$ davrli davriy funksiya bo‘lishi ravshandir.

Endi $2\pi$ davrli $f(x)$ davriy funksiya berilganda uni (3.1.1) trigonometrik qator yig‘indisi kabi ifodalash masalasini qo‘yamiz. Bu masalani yechish maqsadida aytilgan $f(x)$ funksiyani bitta davr oralig‘i $(-\pi;\pi)$ da qarab, u

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right) \tag{3.1.2}$$

ko‘rilishda trigonometrik qator yoyilmasi sifatida ifodalangan va uning o‘ng tomonidagi funksional qator $(-\pi;\pi)$ oraliqda tekis yaqinlashuvchi deb faraz qilamiz. U vaqtda (3.1.2) ning har ikki tomonini $(-\pi;\pi)$oraliq bo‘yicha integrallab va o‘ng tomondagi funksional qatorni hadlab integrallab,hamda

$$\int_{-\pi}^{\pi}\cos nxdx=\int_{-\pi}^{\pi}\sin nxdx=0\ (n\in N)$$

bo‘lishini hisobga olib,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx \tag{3.1.3}$$

ni olamiz.

Agar (3.1.2) ning har ikki tomonini $\cos mx$ ga $(m\in N)$ ko‘paytirib, so‘ngra yuqoridagidek integrallash jarayonini bajarsak,

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx \tag{3.1.4}$$

ni, xuddi Shunga o‘xshash (3.1.2) ning har ikki tomonini $\sin mx$ ga $(m\in N)$ ko‘paytirib, so‘ngra integrallasak,

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx \tag{3.1.5}$$

ni olamiz.

Yuqoridagi (3.1.3)-(3.1.5) formulalar bilan aniqlangan $a_0, a_n, b_n \ (n \in N)$ larni $f(x)$ funksiyaning *Fur`e koeffisientlari* deb, va bunday koeffisientlarga ega bo‘lgan (3.1.1) trigonometrik qator esa $f(x)$ funksiyaning *Fur`e qatori* deb ataladi. (3.1.2) ni $f(x)$ funksiyani *Fur`e qatoriga yoyish formulasi* deyiladi.

Quyidagi tasdiqni isbotsiz keltiramiz.

**3.1.1-teorema.** Agar $2\pi$ davrli $f(x)$ davriy funksiya $[-\pi;\pi]$ kesmada bo‘lakli monoton va chegaralangan bo‘lsa, u holda bu funksiyaning Fur`e qatori barcha nuqtalarda yaqinlashuvchi bo‘ladi va uning $S(x)$ yig‘indisi uchun funksiya uzluksiz $x \in (-\pi;\pi)$ bo‘lgan nuqtalarda $S(x) = f(x)$, $c \in (-\pi;\pi)$ birinchi jins uzilish nuqtalarda esa

$$S(c) = \frac{f(c-0) + f(c+0)}{2}$$

hamda $S(-\pi) = S(\pi) = \frac{1}{2}[f(-\pi+0) + f(\pi+0)]$ o‘rinli bo‘ladi.

**Eslatma.** Agar $[a;b]$ kesmada $f(x)$ aniqlangan bo‘lib, bu kesmani chekli sondagi Shunday $x_1 < x_2 < ... < x_{n-1}$ nuqtalar vositasida $(a;x_1),(x_1;x_2),...,(x_{n-1};b)$ intervallarda ajratish mumkin bo‘lib, ularning har birida u monoton (keng ma’noda bo‘lishi ham mumkin) bo‘lsa, uni *bo‘lakli monoton* deb ataladi. Bu ta’rifdan ko‘rinadaki, $f(x)$ funksiya $[a;b]$ kesmada bo‘lakli monoton bo‘lsa, u faqatgina birinchi jins uzilishga ega bo‘lishi mumkin.

Bu teoremani isbotsiz qabul qilamiz va uning isbotini ba’zi-bir adabiyotlardan (masalan [3] ga qarang) o‘rganish mumkinligini aytamiz.

## 3.2 Funksiyani Fur`e qatoriga yoyish misollari

Bu bandda funksiyani Fur`e qatoriga yoyishga doir ba’zi-bir misollarni keltiramiz.

**1-misol.** $2\pi$ davrli $f(x)$ funksiya $(-\pi;\pi]$ oraliqda

$$f(x) = x$$

kabi aniqlangan bo‘lsin. Bu funksiya bo‘lakli monoton va chegaralangan ekanligi aniqdir (3.1.1-rasm). Demak, bu funksiyani Fur`e qatoriga yoyish mumkin.

Fur`e koeffisientlarini hisoblaymiz:

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = 0$ - toq funksiyaning simmetrik oraliq bo‘yicha integrali nolga tengligidan.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx dx = 0 .$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nxdx = \frac{2}{\pi}\int_0^{\pi} x\sin nxdx = \frac{2}{\pi}\left( -\frac{x}{n}\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nxdx \right) =$$

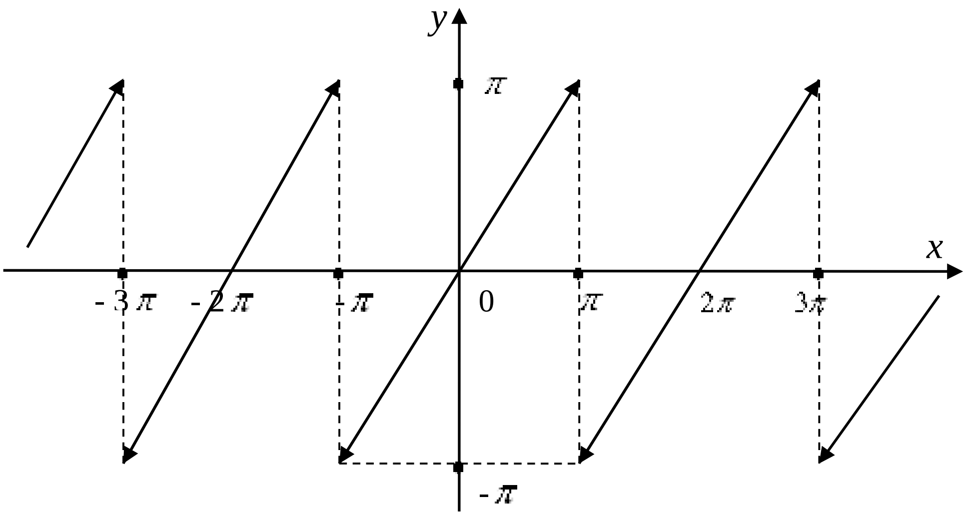
$$= \begin{vmatrix} u = dx, dv = \sin nxdx \\ du = dx, v = -\frac{1}{n}\cos nx \end{vmatrix} = -\frac{2}{\pi}\left( \frac{\pi}{n}\cos n\pi + \frac{1}{n^2}\sin nx\Big|_0^{\pi} \right) = \frac{2(-1)^{n+1}}{n}.$$

Shunday qilib,

$$S(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx,\ x \in (-\pi;\pi) \qquad (3.2.1)$$

ni olamiz.

Yuqoridagi teoremaga ko‘ra $x \in (-\pi;\pi)$ bo‘lganda $f(x) = S(x)$ o‘rinli bo‘lib, $S(-\pi) = S(\pi) = \frac{f(-\pi+0) + f(\pi+0)}{2} = \frac{-\pi+\pi}{2} = 0$,
3.1.1-rasmga qarang.




3.1.1-расм

**2-misol.** $f(x)$ davriy funksiya $2\pi$ davrli bo‘lib, $x \in [-\pi;\pi]$ bo‘lganda
$f(x) = |x|$
kabi aniqlangan. Bu funksiya bo‘lakli monoton va chegaralangan ekanligi aniq. Demak, uni Fur`e qatoriga yoyish mumkin (3.1.1-teorema asosida). Fur`e koeffisientlarini hisoblaymiz:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos nxdx = \frac{2}{\pi}\int_0^{\pi}xdx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \frac{2}{\pi}\frac{\pi^2}{2} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos nxdx = \frac{2}{\pi}\int_0^{\pi}x\cos nxdx = \frac{2}{\pi}\left( x\frac{1}{n}\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nxdx \right) =$$

$$= \frac{2}{\pi}\frac{1}{n^2}\cos nx\Big|_0^{\pi} = \frac{2}{\pi n^2}\left((-1)^n - 1\right);$$

$$\left| u = x, dv = \cos nxdx, du = dx, v = \frac{1}{n}\sin nx \right|$$

$a_n = -\frac{2}{\pi n^2}\left(1-(-1)^n\right),$

$n = 2m-1,\quad m \in N \;\; a_{2m-1} = -\frac{4}{\pi(2m-1)^2},$

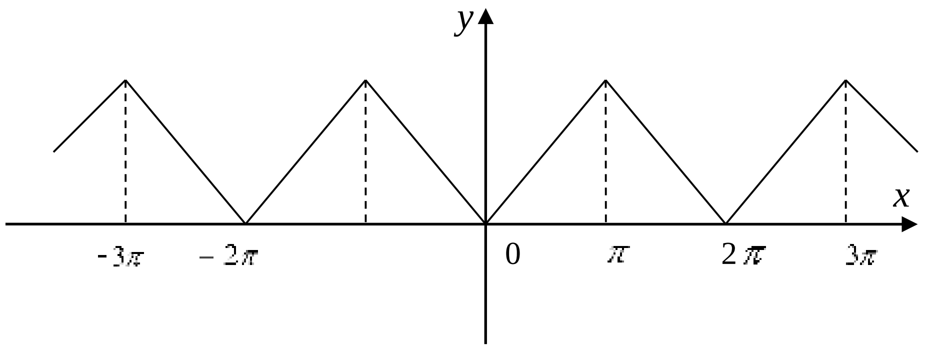
$n = 2m,\quad m \in N \;\; a_{2m} = 0.$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin nx dx = 0.$

Demak, $x \in (-\infty;+\infty)$ bo'lganda

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos(2m-1)x}{(2m-1)^2} \qquad (3.2.2)$$

3.1.2-rasmga qarang.




3.1.2-расм.

Olingan yoyilmada $x = 0$ desak,

$$\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} = \frac{\pi^2}{8}$$

ni olamiz.

**3-misol.** $f(x)$ $2\pi$ davrli funksiya bo'lib, u $x \in [-\pi;\pi]$ kesmada

$f(x) = x^2$

bilan aniqlangan bo'lsin. Bu funksiya ham bo'lakli monoton va chegaralangan ekanligi ravshan. Uning Fur`e koeffisientlari:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2 dx = \frac{2}{\pi}\frac{x^3}{3}\Big|_0^{\pi} = \frac{2}{\pi}\frac{\pi^3}{3} = \frac{2}{3}\pi^2;$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx dx = \frac{2}{\pi}\int_0^{\pi}x^2\cos nx dx = \frac{2}{\pi}\left(\frac{x^2}{n}\sin nx\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi}x\sin nx dx\right) =$$

$$= -\frac{4}{\pi n}\left(-\frac{x}{n}\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx dx\right) = -\frac{4}{\pi n}\left(\frac{(-1)^{n-1}\pi}{n} + \frac{1}{n^2}\sin nx\Big|_0^{\pi}\right) = (-1)^n\frac{4}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\sin nx dx = 0.$$

Fur`e qatoriga yoyilmasi:

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}, \qquad x \in (-\infty;+\infty)$$

Bunda $x = 0$ desak,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$$

tenglikni, agar $x=\pi$ desak,

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

ni olamiz.

## 3.3 Juft va toq funksiyalarning Fur`e qatori

Oldingi banddagi 1-misolda toq funksiyaning Fur`e qatoriga yoyilmasini ko‘rdik. Uning tarkibida faqat sinuslar , 2- va 3-misollarda esa juft funksiyani Fur`e qatoriga yoygan edik, ularning tarkibida ozod had va faqatgina kosinuslar qaytnashdi. Bu tasodifiy bo‘lmay umumiy tasnifga egadir.

Aytaylik, $f(x)$ davriy funksiya $2\pi$ davrli va $x\in(-\pi;\pi)$ oraliqda juft bo‘lsin. U holda uning Fur`e koeffisientlari uchun.

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{2}{\pi}\int_{0}^{\pi}f(x)\cos nxdx;$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{2}{\pi}\int_{0}^{\pi}f(x)\cos nxdx;$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=0,\quad n\in N$$

lar o‘rinli bo‘ladi. Demak, bu holda $f(x)$ ning Fur`e qatoriga yoyilmasi uchun

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx, \qquad (3.3.1)$$

$$a_0=\frac{2}{\pi}\int_{0}^{\pi}f(x)dx,\; a_n=\frac{2}{\pi}\int_{n=1}^{\pi}f(x)\cos nxdx,\; n\in N$$

ga ega bo‘lamiz.

Xuddi Shunga o‘xshash $f(x)$ $(-\pi;\pi)$ oraliqda toq bo‘lsin. U holda

$$f(x)=\sum_{n=1}^{\infty}b_n\sin nx, \qquad (3.3.2)$$

$$b_n=\frac{2}{\pi}\int_{0}^{\pi}f(x)\sin nxdx,\quad n\in N$$

ni olish qiyin emas.

Endi $f(x)$ funksiya $[0;\pi]$ oraliqda aniqlangan bo‘lsin. Agar uni $[-\pi;0]$ oraliqqa juft (toq) ravishda so‘ngra butun sonlar o‘qi bo‘ylab $2\pi$ davrli funksiya sifatida davom ettirsak, buning natijasida hosil bo‘lgan davriy funksiyaning Fur`e qatoriga yoyilmasi uchun (3.3.1.) ga ((3.3.2.) ga) ega bo‘lamiz.

Bu hol $[0;\pi]$ oraliqda *funksiyani kosinuslar (sinuslar) bo‘yicha Fur`e qatoriga yoyish* deb yuritiladi.

Masalan, $x \in [0;\pi]$ oraliqda $f(x)=x$ bo‘lsa, uning kosinuslar bo‘yicha yoyilmasi (3.2.2) dan sinuslar bo‘yicha yoyilmasi esa (3.2.1) dan iborat bo‘ladi.

## 3.4. $2l$ davrli funksiyaning Fur`e qatori

Aytaylik, $f(x)$ $2l$ davrli davriy funksiya bo‘lsin. Uni Fur`e qatoriga yoyish maqsadida

$$x=\frac{l}{\pi}t$$

almashtirish qilaylik. U holda $f\left(\frac{l}{\pi}t\right)$ funksiya t ning $2\pi$ davrli funksiyasi va uning Fur`e qatoriga yoyilmasi.

$$f\left(\frac{l}{\pi}t\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(\frac{l}{\pi}t\right)dt,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(\frac{l}{\pi}t\right)\cos ntdt,\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f\left(\frac{l}{\pi}t\right)\sin ntdt \qquad (3.4.1)$$

ko‘rinishda bo‘ladi.

Endi,

$$x=\frac{l}{\pi}t\Rightarrow t=\frac{\pi}{l}x\Rightarrow dt=\frac{\pi}{l}dx$$

ekanligini e’tiborga olib, (3.4.1) da dastlabki $x$ o‘zgaruvchiga qaytsak,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right),$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx,\quad a_n=\frac{1}{l}\int_{-l}^{l}f(x)\frac{\cos n\pi x}{l}dx, \qquad (3.4.2)$$

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}dx,\quad n\in N.$$

## 3.5. $[a,b]$ oraliqdagi Fur`e qatori

Aytaylik, $f(x)$ funksiya $[a,b]$ oraliqda aniqlangan, chegaralangan va bo‘lakli monoton bo‘lsin. Uni bu oraliqdan tashqariga $2l=b-a$ davr bilan davom ettirib, $2l$ davrli funksiyaga ega bo‘lamiz. Uni Fur`e qatoriga yoyish maqsadida, avvalo, bunday funksiya uchun.

$$\int_{a}^{a+2l}f(x)dx=\int_{-l}^{l}f(x)dx \qquad (3.5.1)$$

tenglik o‘rinli bo‘lishini ko‘rsataylik. Buning uchun $2l$ davrli funksiya bo‘lganda -$2l$ ham uning davriy bo‘lishini hisobga olamiz, ya’ni

$f(x-2\ell)=f(x)$

Endi $x=\xi-2l$ almashtirish yordamida

$$\int_{c}^{d} f(x)dx = \int_{c+2l}^{d+2l} f(\xi - 2l)d\xi = \int_{c+2l}^{d+2l} f(x)dx$$

ekanligini va bunda $c = -l$, $d = a$ deb

$$\int_{-l}^{a} f(x)dx = \int_{l}^{a+2l} f(x)dx$$

ni olamiz.

U holda

$$\int_{a}^{a+2l} f(x)dx = \int_{a}^{-l} f(x)dx + \int_{-l}^{l} f(x)dx + \int_{l}^{a+2l} f(x)dx = \int_{a}^{-l} f(x)dx + \int_{-l}^{l} f(x)dx + \int_{-l}^{a} f(x)dx = \int_{-l}^{l} f(x)dx.$$

Bu (3.5.1) ning to‘g‘ri ekanligining tasdig‘idir.

Endi $f(x)$ funksiyani $(-l;l)$ oraliqda (3.4.2) formula vositasida Fur`e qatoriga yoyib, bu yerda $\ell = \frac{1}{2}(b-a)$ ekanligini hamda (3.5.1) ni hiosbga olsak,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi nx}{b-a} + b_n \sin\frac{2\pi nx}{b-a}\right),$$

$$a_0 = \frac{2}{b-a}\int_{a}^{b} f(x)dx, \qquad a_n = \frac{2}{b-a}\int_{a}^{b} f(x)\cos\frac{2\pi nx}{b-a}dx,$$

$$b_n = \frac{2}{b-a}\int_{a}^{b} f(x)\sin\frac{2\pi nx}{b-a}dx, \quad n \in N$$

ni olamiz . Bu $f(x)$ funksiyani $[a,b]$ oraliq bo‘yicha Fur`e qatoriga yoyish formulalaridir.

## 3.6. Berilgan funsiyaga trigonometrik ko‘phad bilan o‘rtacha yaqinlashish

Mazkur bandda $[a;b]$ oraliqda aniqlangan $f(x)$ funksiyani boshqa biror $\varphi(x)$ funksiya bilan almashtirish natijasida yo‘l qo‘yilgan xatolikning baholaridan biri bo‘lgan o‘rtacha kvadratik chetlanish (og‘ish) hamda funksiyaga o‘rtacha yaqinlashish tushunchalari bilan tanishamiz.

Masalan, agar $[a;b]$ oraliqda aniqlangan $\{S_n(x)\}$ funksional ketma-ketlik berilgan bo‘lib, u biror $S(x)$ funksiyaga $[a;b]$ oraliqda tekis yaqinlashuvchi bo‘lishi uchun

$$\lim \sup_{x\in[a;b]} |S(x) - S_n(x)| = 0 \qquad (3.6.1)$$

bo‘lishi zarur va yetarli ekanligi ma’lum (2.2 .1-teorema)

Demak, agar $S(x)$ ni $S_n(x)$ bilan almashtirish natijasida yo‘l qo‘yilgan xatolikning bahosi sifatida

$$\varepsilon_n = \sup_{x\in[a;b]} |S(x) - S_n(x)|$$

ni qabul qilsak, $\varepsilon_n \to 0$ da, $S(x)$ funksiyaga $\{S_n(x)\}$ ketma-ketlikning tekis yaqinlashishi tushunchasiga kelishimiz tabiiydir. Buning uchun $S(x)$ funksiyadan kuchli bo‘lgan shartlar (masalan, $[a;b]$ kesmada bo‘lakli monotonlik va chegaralanganlik) talab qilinadi. Bu shartlarni yengillatish maqsadida xatolik bahosi uchun boshqacha mezon kiritish kerak bo‘ladi.

Agar

$$\delta_n^2 = \frac{1}{b-a}\int_a^b [S(x) - S_n(x)]^2 dx \qquad (3.6.2)$$

ko‘rinishdagi bahoni qabul qilsak, uni $[a;b]$ oraliqda $S_n(x)$ funksiyaning $S(x)$ dan o‘rtacha *kvadratik chetlanishi (og‘ishi)* deb ataladi. Bu holda $\delta_n^2 \to 0$ bo‘lishi uchun $S(x)$ va $\{S_n(x)\}$ funksional ketma-ketlik hadlari $[a;b]$ oraliqda integrallanuvchi bo‘lishi yetarli ekanligi isbotlangandir.

Bu yerda umumiy hadi

$$S_n(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n}(\alpha_k \cos kx + \beta_k \sin kx) \qquad (3.6.3)$$

n-tartibli trigonometrik ko‘phaddan iborat bo‘lgan funksional ketma-ketlikni qarab, $[-\pi;\pi]$ oraliqda aniqlangan $f(x)$ funksiyaga Shu ketma-ketlik yordamida yaqinlashish masalasini qaraymiz.

Agar $2\pi$ davrli $f(x)$ funksiya berilgan bo‘lsa, n-tartibli bo‘lgan (3.6.3) trigonometrik ko‘phad $\alpha_k$ va $\beta_k$ koeffisientlarini qanday tanlanganda (3.6.2) kabi aniqlangan o‘rtacha kvadratik chetlanish eng kichik bo‘lishi masalasini qo‘yaylik. Qaralayotgan hol uchun

$$\delta_n^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[f(x) - \left(\frac{\alpha_0}{2} + \sum_{k=1}^{n}(\alpha_k \cos kx + \beta_k \sin kx)\right)\right]^2 dx$$

ga egamiz. Bunda integral ostidagi ayirmaning kvadranti yoyilmasini ochib, so‘ngra hadlab integrallab va Fur`e koeffisientlaridan foydalanish natijasida

$$\delta_n^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx - \left[\frac{\alpha_0 a_0}{2} + \sum_{k=1}^{n}(\alpha_k a_k + \beta_k b_k)\right] + \left[\frac{\alpha_0^2}{4} + \frac{1}{2}\sum_{k=1}^{n}(\alpha_k^2 + \beta_k^2)\right]$$

ni olamiz.

Buning o‘ng tomoniga

$$\frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{n}(a_k^2 + b_k^2)$$

ifodani qo‘shib va ayirib, ba’zi-bir shakl o‘zgartirishlarni bajarib,

$$\delta_n^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx - \left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{n}(a_k^2 + b_k^2)\right] + \left[\frac{(\alpha_0 - a_0)^2}{4} + \frac{1}{2}\sum_{k=1}^{n}\left((\alpha_k - a_k)^2 + (\beta_k - b_k)\right)\right]$$

ni olamiz. Bunda $a_k$ va $b_k$ lar Fur`e koeffisientlari. Bu ifoda eng kichik qiymatiga $\alpha_0 = a_0$, $\alpha_k = a_k$, $\beta_k = b_k$ bo‘lganda erishishini ko‘rish oson.

Demak, (3.6.3.) n-tartibli trigonometrik ko’phad koeffisientlari Fur`e koeffisientlaridan iborat bo‘lganda uning $f(x)$ dan o‘rtacha kvadratik chetlanishi eng kichik, ya’ni

$$\delta_n^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx - \left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{n}(a_k^2 + b_k^2)\right] \qquad (3.6.4)$$

bo‘lar ekan. Bu *Fur`e koeffisientlarining minimallik xossasi* deb yuritiladi.

Endi $\delta_n^2 \geq 0$ ekanligini hisobga olsak (3.6.4) dan

$$\frac{a_0^2}{2} + \sum_{k=1}^{n}\left(a_k^2 + b_k^2\right) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

ga kelamiz. Buni *Bessel tengsizligi* deb yuritiladi.

Agar $f(x)$ funksiya 3.1.1-teorema shartlarini qanoatlantirsa, $\delta_n^2 \to 0$ bo‘lishini isbotsiz aytamiz. U holda (3.6.4) dan *Lyapunov tengligi* deb ataluvchi

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty}\left(a_k^2 + b_k^2\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

ga ega bo‘lamiz. Oxirgini $f(x)$ funksiyaning Fur`e qatoriga yoyilmasining har ikki tomonini $f(x)$ ga ko‘paytirib, so‘ngra uni integrallab ham olish mumkinligini aytamiz.

Shunday qilib, agar berilga $2\pi$ davrli $f(x)$ funksiya uchun Lyapunov tengligi bajarilsa, yuqoridagi $\delta_n^2$ o‘rtacha kvadratik chetlanish eng kichik ($n \to \infty$ da nolga teng) qiymatga erishadi (xususan, bo‘lakli monoton va chegaralangan funksiya uchun).

**Eslatma.** Agar $2\pi$ davrli $f(x)$ funksiya $[-\pi;\pi]$ oraliqda bo‘lakli uzluksiz, ya’ni u oraliqda yoki uzlukzis yoki chekli sondagi birinchi jins uzilish nuqtalarigagina ega bo‘lsa , Lyapunov tengligi bu funksiya uchun o‘rinli bo‘lib, unda

$$\frac{a_0^2}{2} + \sum\left(a_n^2 + b_n^2\right)$$

qatorning yaqinlashuvchi ekanligi va bundan esa

$$\lim a_n = \lim b_n = 0,$$

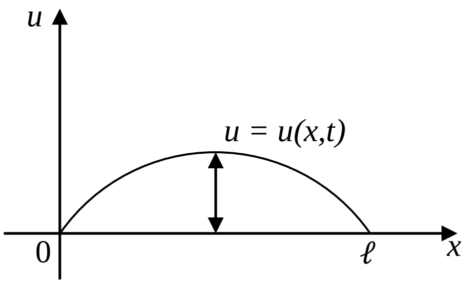
ya’ni Fur`e koeffisientlarining nolga intilishi kelib chiqadi.

## 3.7.Fur’e qatorini amaliy masalalarni yechishga tadbiqi.

Fur’e qatori yordamida amaliy masalalarni yechimini qurish va yechimini taxlil qilish yordamida qaror qabul qilish muhim ahamiyatga ega.

### 3.7.1 Fur’e qatorini tor tenglamasini yechishga tadbiqi.

Uzunligi $\ell$ ga teng bo‘lgan tor berilgan bo‘lib uchlari OX o‘qini $x = 0,\ x = \ell$ nuqtalariga mahkamlangan. Agar torning muvozanat hol ati buzilsa, tor tebrana boshlaydi. Biz torni erkin ko‘ndalang ($u$ o‘qi bo‘ylab)tebranishini o‘rganamiz.




Agar torning $t = t$ momentdagi hol atini $u\,(x, t)$ deb belgilasak, uni tebranish tenglamasi

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (3.7.1)$$

bu yerda $a$ - o‘zgarmas son. Demak, noma’lum $u\,(x, t)$ funktsiya torni $t = t$ momentdagi shaklini (formasini) aniqlaydi. (3.7.1) – yenglama ikkinchi tartibli chiziqli birjinsli xususiy hosilali differensial tenglamadan iborat. Torni uchlari
$x = 0 \;\; va \;\; x = \ell$ nuqtalarga mahkamlangani uchun,
$u(0, t) = 0 \quad va \quad u(\ell, t) = 0$(3.7.2) bo‘lishi ravshandir, bu shartlarga chegaraviy shartlar deyiladi.
Torni dastlabki hol ati va uni dastlabki tebranish tezligi
$u(x, 0) = \varphi(x), \qquad u'_t(x, 0) = \psi(x)$shartlar bilan xarakterlanadi (3.7.3)
$\varphi(x) \; va \;\; \psi(x)$ lar berilgan funktsiyalar.
Demak (3.7.1) tenglamani (3.7.2) chegaraviy va (3.7.3) boshlang‘ich shartlarni qanoatlantiruvchi $u\,(x, t)$ funktsiyani topishimiz kerak.

Yechish algoritmi: Dastlab (3.7.1) tenglamani (3.7.2) chegaraviy shartni qanoatlantiruvchi yechimi ni topamiz. Bevosita ko‘rsatish mumkinki $u\,(x, t) = 0$ funktsiya (3.7.2) shartni qanoatlantiruvchi yechimdir. Biz noldan farqli yechimni topishga harakat qilamiz; yechimni ikkita noma’lum funktsiyani ko‘paytmasi ko‘rinishda izlaymiz

$$u(x, \mathrm{t}) = \mathrm{X}\,(x)\mathrm{T}\,(\mathrm{t}) \qquad (3.7.4)$$

Bu yerda $\mathrm{X}\,(x)$ va $\mathrm{T}\,(\mathrm{t})$ funktsiyalar mos ravishda $x$ va t o‘zgaruvchilar bo‘yicha ikki marta differensiallanuvchi funktsiyalar, $0 \le x \le \ell, -\infty < t < +\infty$, $u(x, t)$ funktsiya noldan farqli funktsiya bo‘lgani uchun, $\mathrm{X}\,(x)$ va $\mathrm{T}\,(\mathrm{t})$ funktsiyalar noldan farqli, (3.7.4) ni (3.7.1) tenglamaga qo‘yib quyidagi tenglikni olamiz:

XT$''$ $= a^2$X$''$ T Bu yerdan $\frac{X''}{X} = \frac{T''}{a^2 T}$ tenglikni o‘ng tomoni T ga, chap tomon X ga bog‘liq emas.Bu nisbat- o‘zgarmas, unu-i$\lambda$ deb belgilaymiz

$$\frac{X}{X} = \frac{T'}{a^2 = T} = \lambda$$

bu yerdan,

$$X'' - \lambda X = 0 \qquad (3.7.5)$$

$$T' - \lambda a^2 T' = 0 \qquad (3.7.6)$$

Bu tenglamalar chiziqli o‘zgarmas koeffisientli birjinsli ikkinchi tartibli chiziqli tenglamalardir.

Agar biz (3.7.2) shartni e’tiborga olsak

$$u\,(0,t) = X(0)\ T(t) = 0, \qquad bu\ yerdan \quad X(0) = 0$$

$$u\,(\ell,t) = X(\ell)\ T(t) = 0, \qquad bu\ yerdan \quad X(\ell) = 0$$

(bu yerda $T(t)$ funktsiyani aynan nolga teng emasligini e’tiborga oldik),biz quyidagi xulosaga kelamiz: (3.7.5) va (3.7.6) tenglamalarni noldan farqli yechimini topishimiz kerak.Shu bilan birga,

$$X(0) = X(\ell) = 0 \qquad (3.7.7)$$

shart bajarilishi zarur.

Quyidagi mumkin bo‘lgan hol larni ko‘rib chiqamiz:

a) $\lambda = 0$ bo‘lsin. U hol da (3.7.5) tenglamani ko‘rinishi X$'' = 0$ Bu tenglamani umumiy yechimi $X(x) = C_1 x + C_2$.Bu yechim (3.7.7) shartni qanoatlantirishi kerak ekanligini e’tiborga olsak

$$X(0) = C_2 = 0\,, \quad X(\ell) = C_1 \ell = 0$$

demak $C_1 = C_2 = 0$.U hol da,$X(x) = 0$.Noldan farqli yechimni axtarganimiz uchun, bu yechimni e’tiborga olmaymiz.

b) $\lambda > 0$ bo‘lsin. U hol da (3.7.5) tenglamani ko‘rinishi X$'' - \lambda$X $= 0$ , xarakteristik tenglamasi $r^2 - \lambda = 0$ , $r_{1,2} = \pm\sqrt{\lambda}$. U hol da, umumiy yechim

$$\text{X}(x) = \text{C}_1 \ell^{\sqrt{\lambda}x} + C_2 \bar{\ell}^{\sqrt{\lambda}x}$$

Bu yechim (3.7.7) shartni qanoatlantirishi kerak $X(0) = C_1 + C_2 = 0$.Bu yerdan,$C_1 = -C_2$

$$\text{X}(\ell) = \text{C}_1 \ell^{\sqrt{\lambda}\ell} - C_2 \ell^{-\sqrt{\lambda}\ell} = \text{C}_1 \frac{\ell^{2\sqrt{\sqrt{\lambda}\ell}} - 1}{\ell^{\sqrt{\lambda}\ell}} = 0, \ell \neq 0$$ bo‘lgani uchun, $C_1 = 0$

bo‘lishi kelib chiqadi.Demak $C_2 = 0 \quad va \quad$ X$(x) \equiv 0$

c) $\lambda < 0$ bo‘lsin, $\lambda = -k$ deb belgilasak $(k > 0)$ , (3.7.5) va (3.7.2) tenglamalar quyidagi ko‘rinishga ega bo‘ladi:

$$X'' + kX = 0$$
$$T'' + k\,T = 0$$

Bu tenglamalar chiziqli bir jinsli o‘zgarmas koeffisientli tenglamalar bo‘lgani uchun, ularni xarakteristik tenglamalarining yechimlari

$$r_{1,2} = \pm i\sqrt{k} \quad ; \quad r_{1,2} = \pm ai\sqrt{k}$$

Demak,

$$X(x) = C_1 cos\sqrt{k}x + C_2 sin\sqrt{k}x$$
$$T(t) = D_1 cosa\sqrt{k}t + D_2 sina\sqrt{k}\mathrm{t}$$

Bu yerda,$C_1, C_2,$ $D_1,$ $va$ $D_2$- ixtiyoriy o‘zgarmas sonlar X $(x)$ funktsiya (3.7.7) shartni qanoatlantirishi kerak ,

$$X(o) = C_1 = 0, \; X(\ell) = C_2 sin\sqrt{K}\ell = 0$$

($C_2 \neq 0$ chunki $X(x)$ aynan nolga teng emas)

$$sin\sqrt{k}\,\ell = 0, \text{ ya'ni}$$

$$\sqrt{k}\,\ell = n\pi \quad n = 1, 2, \ldots$$

demak $k = \left(\frac{n\pi}{\ell}\right)^2, \; n = 1, 2, \ldots$
buni e’tiborga olsak

$$\left.\begin{aligned} X(x) &= c_2 sin\frac{n\pi}{\ell}x \\ T(t) &= D_1 cos\frac{an\pi}{\ell}t + D_2 sin\frac{an\pi}{\ell}t \end{aligned}\right\} \; n = 1,2, \ldots$$

U hol da,

$$u(x,t) = \left(A\cos\frac{\alpha n\pi}{\ell}t + B\sin\frac{\alpha n\pi}{\ell}t\right)sin\frac{n\pi}{\ell}x, n = 1,2, \ldots (3.7.8)$$

bu yerda,$A = C_2 D_1, \quad B = C_2 D_2$

Ko‘rsatish mumkinki ixtiyoriy musbat n va ixtiyoriy A va B lar to‘g‘ri keluvchi $u(x,\mathrm{t})$ funktsiya (3.7.1) tenglamani va (3.7.2) boshlang‘ich shartni qanoatlantiradi. Bu yerda biz $n$ - ni har-xil qiymatlariga mos keluvchi A va B larni qiymatlarini mos ravishda $A_n \ va \, B_n$ deb belgilasak, (3.7.8) yechimini quyidagicha yozishimiz mumkin.

$$u_n(x,t) = \left(A_n \, cos \frac{an\pi}{\ell} t + B_n \sin \frac{an\pi}{\ell} t\right) sin \frac{n\pi}{\ell} x$$

Bu yerda, $A_n \ va \, B_n$ lar ixtiyoriy sonlar$n = 1,2, \ldots$

$A_n \ va \, V_n$ o‘zgarmaslarni topish uchun

$$u(x,t) = \sum_{\ell=1}^{\infty} \left(A_n \, cos \frac{an\pi}{\ell} t + B_n \sin \frac{an\pi}{\ell} t\right) sin \frac{kn\pi}{\ell} \qquad (3.7.9)$$

funktsional qator (3.7.1) tenglamani,(3.7.2) – chegaraviy va (3.7.3) – boshlang‘ich shartni qanoatlantirishini talab qilamiz.

Agar (3.7.9) qatorni ikki marta hadma-had differensiallash mumkin bo‘lsa *x* va *t* o‘zgaruvchilar bo‘yicha, bu qatorni yig‘indisi *u(x, t)* funktsiya (3.7.1) tenglamani (3.7.2) shartni qanoatlartiradi. ((3.7.9) qatorni yaqinlashish masalasi bilan Shug‘ullanmaymiz.)

Demak, $A_n \ va \, \mu_n$ koeffisientlarni Shunday aniqlashimiz kerakki (3.7.9) qatorni yig‘indisi (3.7.1) tenglamani va (3.7.3) boshlang‘ich shartni qanoatlantirishi kerak. Buni e’tiborga olsak,

$$u(x,0) = \sum_{n=1}^{\infty} A_n \, sin \frac{n\pi x}{\ell} = \varphi(x) \qquad (3.7.10)$$

$$u'(x,0) = \sum_{n=1}^{\infty} \frac{\alpha n\pi}{\ell} B_n \, sin \frac{n\pi}{\ell} x = \psi(x) \qquad (3.7.11)$$

(3.7.10) va (3.7.11) lar mos ravishda $[0,\ell]$ kesmada $\varphi(x)$ va $\psi(x)$ funktsiyalarni sinuslar bo‘yicha Fur’e qatoriga yoyilmasidan ($\varphi(x)$ va $\psi(x)$ funktsiyalar Fur’e qatoriga yoyish shartini qanoatlantirishi kerak) iborat.

U hol da,

$$\left.\begin{aligned} A_n &= \frac{2}{\ell}\int_0^{\ell}\varphi(x)\ sin\frac{n\pi}{\ell}xdx \\ \frac{an\pi}{\ell}B_n &= \frac{2}{\ell}\int_0^{\ell}\psi(x)\ sin\frac{n\pi}{\ell}xdx \end{aligned}\right\}\quad n = 1,2,\dots$$

Demak, tor tenglamasini (3.7.2) va (3.7.3) shartlarni qanoatlantiruvchi yechimi:

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\,cos\frac{an\pi}{\ell}t + B_n\sin\frac{an\pi}{\ell}t\right)sin\frac{n\pi}{\ell}x$$

Bu yerda,

$$\left.\begin{aligned} A_n &= \frac{2}{\ell}\int_0^{\ell}\varphi(x)\ sin\frac{n\pi}{\ell}xdx \\ B_n &= \frac{2}{an\pi}\int_0^{\ell}\psi(x)\ sin\frac{n\pi}{\ell}xdx \end{aligned}\right\}\quad n = 1,2,\dots$$

## 3.8.Fur'e qatorining kompleks ko'rinishi.

Ayrim fizik masalalarni yechishda Fur'e qatorini kompleks ko'rinishidan foydalaniladi.

Davri $T = 2\pi$ ga teng bo'lgan $f(x)$ funktsiya Fur'e qatoriga yoyilgan bo'lsin:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n\sin nx),$$

bu yerda: $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx, (n = 0,1,2,\dots)$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx, (n = 1,2,\dots)$

Qatorni umumiy hadi $a_n\cos nx + b_n\sin nx$ ni Eyler formulasidan foydalanib quyidagi ko'rinishda yozib olamiz

$$a_n\cos nx + b_n\sin nx = a_n\frac{\ell^{inx} + \ell^{-inx}}{2} + b_n\frac{\ell^{inx} - \ell^{-inx}}{2} = \frac{a_n - ib_n}{2}\ell^{inx} + \frac{a_n + ib_n}{2}\ell^{-inx}$$

Agar bu yerda: $c_0 = \frac{a_0}{2}$ $c_n = \frac{a_n - ib_n}{2}$ $c_{-n} = \frac{a_n + ib_n}{2}$ va $c_{-n} = \bar{c}_n$ ekanligini e'tiborga olsak qatorni N – nchi qismiy yig'indisini quyidagicha yozish mumkin:

$$\frac{a_0}{2}+\sum_{i=1}^{N}(a_n cosnx+b_n sinnx)=c_0+\sum_{n=1}^{N}\left(c_n e^{inx}+c_{-n}e^{-inx}\right)=\sum_{n=-N}^{N}c_{ne^{inx}}$$

Bu yerda $N\to+\infty$ da limitga o‘tamiz va $\lim\limits_{N\to+\infty}\sum_{n=-N}^{N}c_n e^{inx}=\sum_{-\infty}^{+\infty}c_n e^{inx}$ deb belgilasak, u hol da

$$f(x)=\lim_{N\to+\infty}\left[\frac{a_0}{2}+\sum\nolimits_{n=1}^{N}a_n cosnx+b_n sinnx\right]=\lim_{N\to\infty}\sum\nolimits_{-\infty}^{+\infty}c_n e^{inx},\qquad yani\ f(x)=\sum\nolimits_{-\infty}^{+\infty}c_n e^{inx}$$

$c_n$ - ni hisoblash uchun quyidagi formulalarni keltirib chiqaramiz ($a_n$ *va* $в_n$ ni hisoblash formulalarini e’tiborga olamiz)

$$c_0=\frac{a_0}{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$c_{-n}=\frac{a_n+iв_n}{\alpha}=\frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxd+i\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxd\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{\ell nx}dx,\ (n=1,2,...)$$

U holda barcha butun n lar uchun $c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\ell^{-\ell nx}dx(n=0,\pm1,\pm2)$

Quyidagi qator$\sum_{-\infty}^{+\infty}c_n e^{\ell nx}$ - $f(x)$ funktsiyaning kompleks ko‘rinishdagi Fur’e qatoridir.

Misol. $f(x)=e^x$ , $x\in(0,2\pi)$Fur’e qatoriga yoying. Kompleks ko‘rinishdan hususiy ko‘rinishdagi Fur’e qatoriga yoying.

$$C_n=\frac{1}{2\pi}\int_0^{2n}f(x)e^{-inx}dx=\frac{1}{2\pi}\int_0^{2n}e^{(1-in)x}dx=\frac{1}{2\pi}\frac{1}{1-in}e^{(1-in)x}\Big|_0^{2n}=\frac{1}{2\pi}\frac{1}{1-in}(e^{2n}-1),n=0,\pm1,\pm2,...$$

yoki

$c_n=\frac{e^{2n}-1}{2\pi}\cdot\frac{1+in}{1+n^2},n=0,\pm1,\pm2,...$ u holda Fur’e qatori

$$e^x=\frac{e^{2\pi}-1}{2\pi}\sum_{-\infty}^{+\infty}\frac{1+in}{1+n^2}e^{inx},\ 0<x<2\pi$$

Agar biz bu yerda

$a_n=2\operatorname{Re}(C_n),n=0,1,...$

$в_n=-2I_m(C_n),n=1,2,...$

bo‘lgani uchun $a_n$ va $в_n$ larni topamiz

$$a_n=\frac{e^{2n}-1}{\pi}\quad\frac{1}{1+n^2},n=0,1,..$$

$b_n=-\frac{e^{2n}-1}{\pi}\quad\frac{1}{1+n^2},n=1,2,..$ , Fur’e qatorini

$$e^x = \frac{e^{2n}-1}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{\cos nx}{1+n^2}-\frac{n\sin nx}{1+n^2}\right)\right], 0 < x < 2\pi \text{ kelibchiqadi}$$

$Davri T = 2\ell\ \ bo'lgan f(x)$funktsiya uchun Fur'e qatori

$f(x) = \sum_{-\infty}^{+\infty} C_n e^{i\frac{n\pi}{e}x}$ bu yerda $C_n = \frac{1}{2e}\int_{-e}^{e} f(x)e^{-i\frac{n\pi}{e}2i}dx$

## 3.9.Sterjenda issiqlik tarqalish tenglamasi.

Fur'e qatori yordamida sterjenda issiqlik tarqalish tenglamasini yechimini qurish masalasini ko'ramiz.$f(x) = \ell^{2n} - 1$

$\ell$ - uzunlikdagi sterjenda ma'lum shartlarda issiqlik tarqalish tenglamasi quyidagicha yoziladi $\frac{\partial u}{\partial x} = a^2\frac{\partial^2 u}{\partial x^2}$ (3.9.1)

Noma'lum $u\,(x,t)$ funktsiya sterjenni $x = x$nuqtasini $t = t$ momentdagi temperaturasi. Sterjenni uchlarida, temperaturani o'zgarmas deb qabul qilamiz:

$$u\,(x,o) = u(\ell,0) = 0 \qquad (3.9.2)$$

Dastlabkimomentda sterjenni temperaturasi ma'lum bo'lsin: $u(x,0) = f(x)$(3.3)

Quyidagi masalani ko'ramiz (3.9.1) tenglamani (3.9.3) boshlang'ich va (3.9.2) chegaraviy shartni qanoatlantiruvchi yechimini toping.

Yechimni

$u\,(x,t) = X(x)\,T(t)$ ko'rinishda axtaramiz

U holda $XT' = a^2X''T$ ya'ni $\frac{X''}{X} = \frac{T'}{a^2T} = \mu$ deb belgilaymiz.

Demak

$$X'' - \mu X = 0 \qquad (3.9.4)$$

$$T' = \mu a^2 T = 0 \qquad (3.9.5)$$

(4) – tenglama $X(0) = X(\ell) = 0$ shartni qanoatlantirishi kerak ( $\mu = 0,\ \mu > 0$ bo'lishi mumkin emas) $\mu < 0$ bo'lsin. $\mu = -\mu^2$ deb belgilaymiz.

U holda $X = S\sin\lambda x\,, \lambda = \frac{n\pi}{\ell}n = 1{,}2, \ldots$

Buni e'tiborga olsak $X_n = C_n sin = \frac{n\pi x}{\ell}$ $, n = 1{,}2, \ldots$

$C_n$ - ixtiyoriy o'zgarmas son.

$\frac{T'}{a^2T} = -\frac{n^2\pi^2}{\ell^2}$ tenglamani olamiz.$\frac{dT}{T} = -\frac{a^2n^3\pi^2}{\ell^2}$

U holda bu tenglamani yechimi T= $M\,\ell^{-\frac{a^2\pi^2n^2}{\ell^2}t}$

M- ixtiyoriy o‘zgarmas. $C_n M = A_n$ deb belgilasak

$$u_n(x,t) = A_n \ell^{-\frac{a^2\pi^2 n^2}{\ell^2}t} \sin\frac{n\pi x}{\ell} \qquad (3.9.6)$$

Tenglamani chiziqli, birjinsliligini e’tiborga olsak

$$u(x,t) = \sum_{n=1}^{\infty} A_n \ell^{-\frac{a^2\pi^2 n^2}{\ell^2}t} \sin\frac{n\pi x}{\ell} \qquad (3.9.7)$$

yechimni quramiz (bu yechimdan hosilalar olib tenglamaga qo‘ysak, tenglamani qanoatlantiradi). Qatorni yaqinlashish masalasi bilan Shug‘ullanmaymiz.
$A_n$ni Shunday topishimiz kerakki, $u\,(x,o) = f(x)$ shart bajarilsin.
Demak

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{\ell} = f(x)$$

bu yerdan

$$A_n = \frac{2}{\ell}\int_0^l f(x)\sin\frac{n\pi x}{\ell}\,dx$$

masalani yechimi esa,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \ell^{-\frac{a^2\pi^2 n^3}{\ell^2}t} \sin\frac{n\pi x}{\ell} \qquad \text{bo‘lib,}$$

$A_n = \frac{2}{\ell}\int_0^l f(x)\sin\frac{n\pi x}{\ell}\,dx$ dan shart

## 3-bobga doir mashqlar

1. $f(x)=\begin{cases}2x, & 0\le x\le\pi,\\ x, & -\pi\le x\le 0\end{cases}$ funksiyani $(-\pi;\pi)$oraliqda Fur`e qatoriga yoying.

Javob: $\frac{\pi}{4}-\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}+3\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin kx$

2. $f(x)=1$ funksiyani $(0;\pi)$ oraliqda sinuslar bo‘yicha yoyilmasidan foydalanib, $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$ qatorining yig‘indisini hisoblang.

Javob: $\frac{\pi}{4}$

3. $f(x)=\frac{\pi^2}{12}-\frac{x^2}{4}$ funksiyani $(-\pi;\pi)$ oraliqda Fur`e qatoriga yoying.

Javob: $\cos x-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+\ldots.$

4. $f(x)=\begin{cases} -\dfrac{\pi+x}{2}, & -\pi\le x<0, \\ \dfrac{\pi-x}{2}, & 0\le x\le\pi \end{cases}$ funksiyani $(-\pi;\pi)$ oraliqda Fur`e qatoriga yoying.

Javob: $\sum_{n=1}^{\infty}\frac{\sin nx}{n}$

5. $f(x)=\begin{cases} 1, & -\pi<x\le 0, \\ -2, & 0<x<\pi \end{cases}$ funksiyani $(-\pi;\pi)$ oraliqda Fur`e qatoriga yoying.

Javob: $-\frac{1}{2}-\frac{6}{\pi}\sum_{n}^{\infty}\frac{\sin(2n+1)x}{2n+1}$

6. $f(x)=x$ funksiyani $(0;1)$ oraliqda kosinuslar bo‘yicha qatorga yoying.

Javob: $1-\frac{2}{\pi}\sum_{n}^{\infty}\frac{\cos 2\pi nx}{n}$

7. $f(x)=x$ funksiyani $(0,l)\,(l>0)$ intervalda sinuslar bo‘yicha qatorga yoying.

Javob: $\frac{2l}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin\frac{n\pi x}{\ell}}{n}$

8. $f(x)=\begin{cases} x, & 0<x\le 1, \\ 2-x, & 1<x<2 \end{cases}$ funksiyani $(0;2)$ oraliqda: a) sinuslar bo‘yicha; b) kosinuslar bo‘yicha qatorga yoying.

Javob: a) $\frac{8}{\pi^2}\sum_{n=0}^{\infty}(-1)^n\frac{\sin\frac{(2n+1)\pi x}{2}}{(2n+1)^2}$; b) $\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi x}{(2n+1)^2}$.

## 3-bob bo‘yicha bilimingizni sinab ko‘ring

1. Fur`ening trigonometrik qatoriga berilgan funksiyani $(-\pi;\pi)$ oraliqqa yoyish formulalarini yozing.
2. Fur`e qatoriga yoyish mumkin bo‘lishi uchun funksiyadan qanday shartlar talab qilinadi?
3. $(-\ell;\ell)$ oraliqda Fur`e qatoriga yoyish formulalarini yozing.
4. Sinuslar va kosinuslar bo‘yicha qatorga yoyish formulalarini yozing.
5. $[a;b]$ oraliqda Fur`e qatoriga yoyish formulalarini yozing.
6. Bessel tengsizligi, Lyapunov tengligi haqida nimalarni bilasiz?
7. Funksiyaga o‘rtacha yaqinlashish tushunchasi qanday kiritiladi?

# 4-bob. $n$ o‘lchovli integral

Bu bobda $R^n$ fazoga tegishli bo‘lgan $n$ o‘lchovli sohaning o‘lchovi hamda bunday sohada aniqlangan $n$ o‘zgaruvchili funksiyaningintegrali tushunchalarini umumiy tarzda bayon qilib, ularning ba’zi bir xossalarini aniqlab olamiz.

## 4.1. O‘lchanuvchi soha va uning o‘lchovi.

$R^n$ fazoda ochiq $n$ o‘lchovli to‘g‘riburchakli parallelepiped

$$P_n = \{(x_1, x_2, ..., x_n): a_i < x_i < b_i \ \ (i = \overline{1,n})\}$$

to‘plam vositasida aniqlanishini (13.1-bandga qarang), bu yerda $a_i < b_i \ \ (i = \overline{1,n})$ berilgan sonllar bo‘lib, $b_i - a_i = \Delta_i$ son uning i – koordinata bo‘yicha *o‘lchovi* deb yuritilishini va

$$\overline{P}_n = \{(x_1, x_2, ..., x_n): a_i \le x_i \le b_i \ \ (i = \overline{1,n})\}$$

to‘plam vositasida esa yopiq $n$ o‘lchovli to‘g‘ri burchakli parallelepiped aniqlanishini aytamiz. Bunday to‘g‘riburchakli parallelepiped uchun (ochiq yoki yopiq bo‘lsa ham) $b_i - a_i \ \ (i = \overline{1,n})$ son uning $i$ - *koordinatasi bo‘yicha o‘lchovi* deb ataladi.

**4.1.1-ta’rif.** Berilgan ochiq $P_n$ (yoki yopiq $\overline{P}_n$) $n$ o‘lchovli to‘g‘ri burchakli parallelepiped koordinatalar bo‘yicha o‘lchovlarining ko‘paytmasidan iborat bo‘lgan sonni uning *o‘lchovi* deb ataymiz va uni $\mu(P_n)$ (yoki $\mu(\overline{P}_n)$) kabi belgilaymiz.

Bu ta’rifga ko‘ra

$$\mu(P_n) = \mu(\overline{P}_n) = \prod_{i=1}^{n}(b_i - a_i)$$

formulaga ega bo‘lamiz.

Faraz qilaylik, $R^n$ fazoda $\{P_{nk}(k = \overline{1,m})\}$ ochiq $n$ o‘lchovli to‘g‘ri burchakli parallelepipedlar sistemasi berilgan bo‘lib, unga tegishli to‘g‘riburchakli parallelepipedlarning har bir jufti umumiy ichki nuqtaga ega bo‘lmasin, ya’ni

$$P_{nk} \cap P_{nq} = \varnothing \ \ (k \ne q) \qquad (4.1.1)$$

hamda $\{\overline{P}_{nk} \ \ (k = \overline{1,m})\}$ $n$ o‘lchovli yopiq to‘g‘ri burchakli parallelepipedlar sistemasi uchun

$$\overline{Q}_m = \bigcup_{k=1}^{m} \overline{P}_{nk} \qquad (4.1.2)$$

to‘plam bog‘liq sohadan iborat bo‘lsin.

**4.1.2-ta’rif.** $R^n$ fazoda (4.1.1) va (4.1.2) shartlarni qanoatlantiruvchi $\{P_{nk} \ \ (k = \overline{1,m})\}$, ochiq $n$ o‘lchovli to‘g‘ri burchakli parallelepipedlar sistemasi berilgan bo‘lsa, (4.1.2) bilan aniqlangan $\tilde{Q}_m$ sohani *yopiq zinasimon* (yoki *pog‘onasimon*) *soha* deb, mos ravishda, $\overline{Q}_m$ ning chegarasini uning o‘zidan chiqarib tashlash bilan hosil qilingan $Q_m$ ni *ochiq zinasimon* (yoki *pog'onasimon*) *soha* deb ataymiz.

Bunday kiritilgan zinasimon soha uchun uning o'lchovi deb,

$$\mu(\overline{Q}_m) = \mu(Q_m) = \sum_{K=1}^{m} \mu(P_{nk})$$

ni qabul qilish tabiiy bo'lib, uning quyidagi xossalarini ko'rish oson (A va B lar zinasimon sohalar deb faraz qilamiz):

**1⁰.** $\mu(\varnothing) = 0$ deb qabul qilamiz;

**2⁰.** $A \subset B \Rightarrow \mu(A) \leq \mu(B)$**;**

**3⁰.** $A \cap B = \varnothing \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$;

**4⁰.** $\mu(A \cup B) \leq \mu(A) + \mu(B)$**.**

Endi $R^n$ fazoga tegishli bo'lgan chekli diametrli, $D$ bog'liq sohani olaylik. Bunday holda $D$ sohani qoplovchi $\{Z_m\}$ va u bilan qoplanuvchi $\{Q_{m_0}\}$ zinasimon sohalarning sistemalari mavjudligini hamda ular uchun

$$\mu(Q_{m_0}) \leq \mu(Z_m)$$

o'rinli ekanligini, Shuningdek, $Sup\{\mu(Q_{m_0})\}$ va $\inf\{\mu(Z_m)\}$ larning mavjudligi hamda ular uchun

$$Sup\{\mu(Q_{m_0})\} \leq \inf\{\mu(Z_m)\}$$

ekanligini ko'rish osondir. $Sup\mu(Q_{m_0})$ ni $D$ sohaning *quyi*, $\inf \mu(Z_m)$ ni esa *yuqori* o'lchovi deb ataymiz.

**4.1.3-ta'rif.** Agar berilgan chekli diametrli bog'liq $D$ soha bilan qoplanuvchi $\{Q_{m_0}\}$ va uni qoplovchi $\{Z_m\}$ zinasimon sohalarning sistemalari uchun

$$Sup\{\mu(Q_{m_0})\} = \inf\{\mu(Z_m)\}$$

tenglik bajarilsa, bu qiymatni *D sohaning o'lchovi* deb ataymiz va uni $\mu(D)$ kabi belgilaymiz. Bu holda $D$ ni *o'lchanuvchi soha* deb ataymiz.

Bunday kiritilgan soha o'lchovi ham yuqorida zinasimon soha uchun keltirilgan xossalarga ega ekanligini aytamiz, buning uchun A va B lar o'lchanuvchi sohalar deb faraz qilish kerak bo'ladi.

Bundan buyon maqsadimiz qanday shartda soha o'lchanuvchi bo'ladi degan savolga javob izlash bo'lmay, balki qaralayotgan soha o'lchanuvchi degan faraz asosida ish ko'rishimizni aytamiz.

## 4.2. $n$ o'lchovli (karrali) integralning ta'rifi

Faraz qilaylik, $R^n$ fazoda chekli diametrli bog'liq va o'lchanuvchi $\overline{D}$ yopiq sohada chegaralangan $f(M)$ funksiya berilgan bo'lsin.

Agar $\overline{D}$ sohani har ikkitasi umumiy ichki nuqtaga ega bo'lmagan va har biri o'lchanuvchi bo'lgan $\Delta D_i$ $(i = \overline{1, m})$ sohalar vositasida $\overline{D} = \bigcup_{i=1}^{m} \overline{\Delta D_i}$ kabi ifodalangan bo'lsa, $\overline{\Delta D_i}$ ni $\overline{D}$ ning tashkil *etuvchisi* deb ataymiz. Bu yerda $\Delta D_i$ ning diametrini

$d_i$ bilan va $\lambda = \max_i d_i$ deb belgilab, $\lambda \to 0 (m \to +\infty)$ holni qarasak, $\overline{\Delta D_i}$ ni $\overline{D}$ ning *elementar tashkil etuvchisi* deb ataymiz.

Agar chekli diametrli, o‘lchanuvchi $\overline{D}$ soha yuqoridagidek qilib, $\overline{\Delta D_i}$ $\left(i = \overline{1,m}\right)$ elementar tashkil etuvchilarga ajratilgan bo‘lsa,

$$\mu\left(\overline{D}\right) = \sum_{i=1}^{m} \mu\left(\overline{\Delta D_i}\right)$$

bo‘lishi ravshandir.

Endi $f(M)$ funksiyaning aniqlanish sohasi bo‘lgan $\overline{D}$ sohani yuqoridagidek elementar tashkil etuvchilarga ajratilgan deb faraz qilib, ularning $k$-sida ixtiyoriy $M_k$ nuqtani olib, bu nuqtadagi funksiya qiymatining Shu elementar tashkil etuvchi o‘lchoviga ko‘paytmasini hisoblaylik. Bu ishni barcha elementar tashkil etuvchilar uchun bajarib olingan barcha natijalarning

$$\sum_{k=1}^{m} f(M_k) \Delta\mu_k \qquad (4.2.1)$$

yig‘indisini $f(M)$ funksiyaning $\overline{D}$ soha bo‘yicha *integral yig‘indisi* deb ataymiz va uni $\sigma_m$ orqali belgilaymiz, unda $\Delta\mu_k = \mu\left(\overline{\Delta D_k}\right)$ dir.

**4.2.1-ta’rif.** Agar chekli diametrli, o‘lchanuvchi $\overline{D}$ yopiq sohada chegaralangan $f(M)$ funksiya berilgan bo‘lib, $\overline{D}$ soha yuqoridagidek qilib, ixtiyoriy usulda elementar tashkil etuvchilarga ajratilgan va ularning $k$-sida $M_k$ nuqtani ixtiyoriycha tanlangan taqdirda ham $\lambda \to 0$ da (4.2.1) integral yig‘indi bitta chekli limitga ega bo‘lsa, bu limitni $f(M)$ funksiyaning $\overline{D}$ soha bo‘yicha $n$ *o‘lchovli (karrali) integrali* deb ataymiz va

$$\iint\limits_{D} \ldots \int f(M) d\mu \quad \text{yoki} \quad \iint\limits_{D} \ldots \int f(x_1, x_2, \ldots, x_n) dx_1\, dx_2 \ldots dx_n$$

kabi belgilaymiz (bu belgidagi integrallar soni $n$ tadir). Bu holda *funksiyani* $\overline{D}$ *sohada integrallanuvchi* deymiz.

Demak, bu ta’rif bo‘yicha

$$\iint\limits_{D} \ldots \int f(\mu) d\mu = \lim_{\lambda \to 0} \sum_{k=1}^{m} f(M_k) \Delta\mu_k$$

ga egamiz.

Shu bilan birga bu o‘rinda $f(M)$ *integral osti funksiyasi* $\overline{D}$ *integrallash sohasi*, $x_i$ $\left(i = \overline{1,n}\right)$ *integrallash o‘zgaruvchilari* deyilishini ham aytamiz.

**4.2.1-eslatma.** Aytaylik, $f(M)$ funksiya $R^n$ fazoga tegishli chekli diametrli va o‘lchanuvchi $\overline{D}$ yopiq sohada berilgan va chegaralangan hamda $\overline{D}$ soha yuqorida aytilgandek, ixtiyoriycha qilib elementar tashkil etuvchilarga ajratilgan bo‘lsin. U holda har bir $\overline{\Delta D_k}$ $\left(k = \overline{1,m}\right)$ tashkil etuvchi uchun

$$\underset{M \in \Delta \overline{D_k}}{Sup} f(M) = T_k \quad \text{va} \quad \inf_{M \in \Delta \overline{D_k}} f(M) = t_k$$

qiymatlarning mavjudligi aniqdir. Bulardan foydalanib,

$$\overline{S}_m = \sum_{k=1}^{m} T_k \Delta\mu_k \quad \text{va} \quad \underline{S_m} = \sum_{k=1}^{m} t_k \Delta\mu_k$$

yig‘indilarni tuzsak, ularni mos ravishda, *Darbuning yuqori va quyi yig‘indilari* deb atalishini aytamiz. Bunday aniqlangan Darbuning yig‘indilari va $f(M)$ funktsiyaning $\sigma_m$ integral yigindisi uchun

$$\underline{S}_m \leq \sigma_m \leq \overline{S}_m$$

munosabat o‘rinli bo‘lishi aniqdir. Shuningdek, agar $\lim\limits_{\lambda\to 0} \overline{S}_m = \overline{S}$ va $\lim\limits_{\lambda\to 0} \underline{S}_m = \underline{S}$ chekli limitlar mavjud bo‘lib, ular tegn $\left(\overline{S} = \underline{S}\right)$ bo‘lsa $f(M)$ funktsiya $\overline{D}$ sohada integrallanuvchi bo‘lishi aniqdir. Bu tasdiqning aksinchasi ham o‘rinlidir (mustaqil bajaring). Bulardan foydalanib, $f(M)$ funktsiya $\overline{D}$ sohada integrallanuvchi bo‘lishi uchun

$$\lim_{\lambda\to 0}\left(\overline{S}_m - \underline{S}_m\right) = 0$$

bo‘lishi zarur va yetarli ekanligiga ishonch hosil qilish mumkin (mustaqil bajaring).

Qilingan bu mulohazalar yordamida quyidagi teoremani isbotlash oson.

**4.2.1-teorema.** Agar $f(M)$ funksiya $R^n$ fazoga tegishli chekli diametrli, bog‘liq va o‘lchanuvchi bo‘lgan $\overline{D}$ yopiq sohada uzluksiz bo‘lsa, u integrallanuvchidir.

**Isbot.** $f(M)$ funksiya $\overline{D}$ yopiq sohada uzluksiz ekanligidan Kantor teoremasiga ko‘ra u tekis uzluksiz bo‘lib, $\overline{D}$ sohani Shunday $m$ ta elementar tashkil etuvchilarga ajratish mumkin bo‘ladiki, ularning har birida funksiyaning tebranishi oldindan berilgan ixtiyoriy musbat $\varepsilon$ sonidan kichik bo‘ladi. Bularni e’tiborga olsak, Darbu yig‘indilari uchun

$$0 \leq \overline{S_m} - \underline{S_m} = \sum_{k=1}^{m}(T_k - t_k)\Delta\mu_k < \varepsilon\mu(D)$$

ni olamiz. Bundan $\lim\limits_{\lambda\to 0}\left(\underline{S_m} - \overline{S_m}\right) = 0$ bo‘lishi kelib chiqadi. Teorema isbotlandi.

Bu teorema yopiq o‘lchanuvchi va chekli diametrli bog‘liq sohada uzluksiz bo‘lgan funksiyalar integrallanuvchi funksiyalar sinfiga tegishli ekanligini ko‘rsatadi. Ammo, integrallanuvchi funksiyalar sinfi bu bilan cheklanib qolmasligini aytamiz.

Chekli diametrli bog‘liq va o‘lchanuvchi $D$ sohada integrallanuvchi bo‘lgan funksiyalar uchun quyidagi xossalar o‘rinlidir.

**1⁰.** $\iint\limits_{D}\ldots\int d\mu = \mu(D)$;

**2⁰.** $\iint\limits_{D}\ldots\int 0 d\mu = 0$;

**3⁰.** Agar $f(M)$ funksiya $D$ sohada integrallanuvchi bo‘lib, $A = const$ bo‘lsa, $\iint\limits_{D}\ldots\int Af(M)d\mu = A\iint\limits_{D}\ldots\int f(M)d\mu$;

**4⁰.** Agar $f_k(M)$ $\left(k = \overline{1,m}\right)$ funksiyalar o‘lchanuvchining har biri $\overline{D}$ sohada integrallanuvchi bo‘lsa, $\sum\limits_{k=1}^{m} f_k(M)$ ham $\overline{D}$ sohada integrallanuvchi va

$\iint\limits_{D}\ldots\int\sum\limits_{k=1}^{m} f_k(M)d\mu = \sum\limits_{k=1}^{m}\iint\limits_{D}\ldots\int\sum\limits_{k=1}^{m} f_k(M)d\mu$ o‘rinlidir.

**5⁰.** Agar o‘lchanuvchi $\overline{D}$ yopiq sohani har biri o‘lchanuvchi bo‘lgan chekli sondagi $\overline{D}_j$ $\left(j=\overline{1,q}\right)$ tashkil etuvchilarga ajratilgan bo‘lib, ularning har birida $f(M)$ funksiya integrallanuvchi va har qanday juftligi umumiy ichki nuqtaga ega bo‘lmasa hamda

$$\overline{D}=\bigcup_{j=1}^{q}\overline{D}_j$$

o‘rinli bo‘lsa, $f(M)$ funksiya $\overline{D}$ sohada integrallanuvchi bo‘lib,

$$\iint\limits_{D}\dots\int f(M)d\mu=\sum_{j=1}^{q}\iint\limits_{D_j}\dots\int f(M)d\mu$$

tenglik o‘rinli bo‘ladi.

**$6^0$.** Agar chekli diametrli, bog‘liq va o‘lchanuvchi bo‘lgan $\overline{D}$ yopiq sohada integrallanuvchi bo‘lgan $f(M)$ va $\varphi(M)$ funksiyalar uchun

$$\varphi(M)\le f(M),\quad \forall M\in\overline{D}$$

tengsizlik bajarilsa,

$$\iint\limits_{D}\dots\int\varphi(M)d\mu\le\iint\limits_{D}\dots\int f(M)d\mu$$

o‘rinli bo‘ladi.

**$7^0$.** Agar $f(M)$ funksiya o‘lchanuvchi $D$ sohada integrallanuvchi bo‘lsa, $\mu(D)\cdot\inf\limits_{M\in D} f(M)\le\iint\limits_{D}\dots\int f(M)d\mu\le\mu(D)\sup\limits_{M\in D} f(M)$ o‘rinlidir.

**$8^0$** Agar chekli diametrli, bog‘liq va o‘lchanuvchi bo‘lgan $\overline{D}$ yopiq sohada uzluksiz bo‘lgan $f(M)$ hamda integrallanuvchi va ishorasini o‘zgartirmovchi $\varphi(M)$ funksiyalar berilgan bo‘lsa, $f(M)\cdot\varphi(M)$ ko‘paytma ham integrallanuvchi bo‘lib, $\exists M_0\in D$, $M_0$ nuqta uchun

$$\iint\limits_{D}\dots\int f(M)\varphi(M)d\mu=f(M_0)\iint\limits_{D}\dots\int\varphi(M)d\mu \qquad (4.2.2)$$

tenglik bajariladi.

Bu xossani *o‘rta qiymat haqidagi teorema* deb ham yuritilishini eslatamiz.

Xususiy holda $\varphi(x)\equiv 1$ bo‘lsa, (4.2.2) tenglik $1^0$ xossaga ko‘ra

$$\iint\limits_{D}\dots\int f(M)d\mu=f(M_0)\mu(D)$$

ko‘rinishni oladi. Bundan $\mu(D)>0$ bo‘ladi deb faraz qilsak,

$$f(M_0)=\frac{1}{\mu(D)}\iint\limits_{D}\dots\int f(M)d\mu$$

ni olamiz. Bu qiymatni uzluksiz bo‘lgan $f(M)$ funksiyaning $\overline{D}$ sohadagi *o‘rta qiymati* deb ataladi.

Bu $1^0$-$8^0$ xossalar ko‘rilgan aniq integral xossalari bilan bir xil ekanligiga ishonch hosil qilish qiyin emas.

Bayon qilingan xossalardan $1^0$-$7^0$ – larini bevochita $n$ o‘lchovli integralning ta’rifi yordamida isbotlash mumkin. Bu yerda $8^0$ – xossa isbotini keltiramiz. $f(M)$ funksiya $\overline{D}$ o‘lchanuvchi yopiq sohada uzluksiz bo‘lganligi sababli 4.2.1-teoremaga ko‘ra, u $\overline{D}$ sohada integrallanuvchidir. Shu bilan birga $\overline{D}$ sohada Shunday ikkita nuqta topilib, ulardagi funksiyaning qiymatlari mos ravishda $m_1$ - eng kichik va $m_2$ -eng katta qiymatlardan iborat bo‘ladi, ya’ni

$$m_1 \le f(M) \le m_2, \quad \forall M \in \overline{D}$$

Bundan, $\varphi(M) \ge 0$ deb faraz qilsak ($\varphi(M) \le 0$ hol ham Shunga o‘xshash bo‘ladi), $m_1\varphi(M) \le f(M)\varphi(M) \le m_2\varphi(M)$ o‘rinli bo‘lib, $3^0$ va $6^0$ – xossalarga ko‘ra

$$m_1 \iint_{\overline{D}} \ldots \int \varphi(M) d\mu \le \iint_{\overline{D}} \ldots \int f(M)\varphi(M) d\mu \le m_2 \iint_{\overline{D}} \ldots \int \varphi(M) d\mu$$

ga ega bo‘lamiz. Agar $\iint_{\overline{D}} \ldots \int \varphi(M) d\mu > 0$ deb faraz qilib, $\mu = \dfrac{\iint_{\overline{D}} \ldots f(M)\varphi(M) d\mu}{\iint_{\overline{D}} \ldots \int \varphi(M) d\mu}$ desak, $m_1 \le \mu \le m_2$ bo‘libб $f(M)$ funksiya $\overline{D}$ yopiq sohada uzluksiz ekanligidan, Koshi teoremasi asosida bu sohada Shunday $M_0$ nuqta topiladiki, $\mu = f(M_0)$ bo‘ladi. bundan $8^0$-xossaning isboti kelib chiqadi.

# 5-bob. Ikki karrali (ikkilangan) integral

Mazkur bobda oliy matematikaning amaliy masalalarni hal qilishda keng qo‘llaniladigan hamda nazariy jihatdan ahamiyatga mole bo‘lgan ikki karrali (ikkilangan) integral tushunchasi bilan tanishamiz. Buning uchun oldingi bobda bayon qilingan $n$ o‘lchovli integral tushunchasini $n = 2$ bo‘lgan holini qarash yyetarlidir. Shu sababli ikki karrali integralning ta’rifi va xossalarini bu yerda takrorlab o‘tirmay (bu ish bilan o‘quvchining o‘zi mustaqil Shug‘ullanishini tavsiya qilib), bu tushunchaga olib keladigan ikkita masalani va uni hisoblash usullarini keltirish hamda uning ba’zi bir tatbiqlarini ko‘rish bilan cheklanamiz.

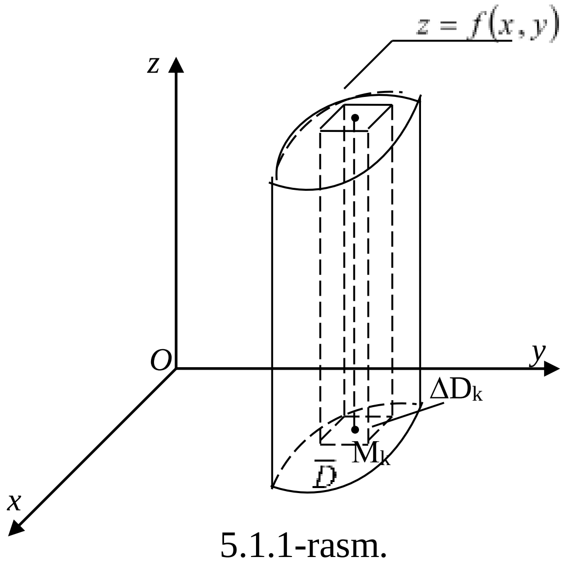
## 5.1. Ikki karrali integral tushunchasiga olib keladigan ba’zi bir masalalar

Hayotning amaliy jabhalarida, ayniqsa texnikaviy masalalarni oliy matematikaning ikkilangan va uchlangan integrallar tushunchalari orqali hal qilishga keltiriladiganlari yetarlicha ko‘pdir. Bu bandda bunday masalalardan ikkitasini keltirishni lozim topdik.

### 5.1.1. Silindrsimon jismning hajmini hisoblash

Avvalo silindrsimon jism tushunchasini aniqlab olaylik. Faraz qilaylik, $R^2$ fazoda chekli diametrli, bog‘liq va o‘lchanuvchi bo‘lgan $\overline{D}$ yopiq sohada uzluksiz hamda manfiy bo‘lmagan, ikki o‘zgaruvchili $f(x, y)$ funksiya berilgan bo‘lsin. U holda bu funksiya $\overline{D}$ sohada integrallanuvchi bo‘lishi aniqdir (4.2.1-teoremaga qarang). Shu bilan birga $z = f(x, y)$ ikki o‘zgaruvchili funksiyaning geometrik ma’nosiga ko‘ra koordinatalar fazosida ($R^3$ da) gorizontal tekislikdagi proyeksiyasi $\overline{D}$ sohadan iborat bo‘lgan biror sirtga ega bo‘lamiz .

Endi quyidan gorizontal koordinatalar ($Oxy$) tekisligi ($\overline{D}$ soha deyish ham mumkin), yondan yasovchisi $Oz$ o‘qqa parallel va yo‘naltiruvchisi $\overline{D}$ sohaning chegarasidan iborat bo‘lgan silindrik sirt hamda yuqoridan $z = f(x, y)$ funksiya grafigi bo‘lgan sirt bilan chegaralangan jismni *silindrsimon* deb ataymiz (4.1.1-rasmga qarang).




5.1.1-rasm.

Shunday aniqlangan silindrsimon jismning hajmini topish masalasini qo‘yaylik. Buning uchun $\overline{D}$ sohani, ixtiyoriycha qilib, elementar tashkil etuvchilarga ajratamiz va $k$-elementar tashkil etuvchini $\Delta D_k$ $\left(k = \overline{1, m}\right)$ uning o‘lchovini (yuzini) $\Delta S_k$, diametrini esa $d_k$ bilan belgilaymiz hamda $\lambda = \max d_k$ deb olamiz. Bu $k$-elementar tashkil etuvchiga tegishli bo‘lgan ixtiyoriy bitta $M_k(x_k, y_k)$ nuqtani olib (5.1.1-rasmga qarang), undagi $f(x_k, y_k)$ funksiya qiymatini hisoblab, uni balandligi $\Delta D_k$ ni asosi deb qabul qilingan to‘g‘ri silindr hajmi bo‘lgan $f(x_k, y_k)\Delta S_k$ ni hisoblaymiz. Bu ish jarayonini barcha elementar tashkil etuvchilar bo‘yicha bajargach, silindrsimon jism hajmining taqribiy qiymati sifatida

$$V \approx \sum_{k=1}^{m} f(x_k, y_k)\Delta S_k$$

ni qabul qilamiz. Bu taqribiy tenglik o‘ng tomonidagi ifoda $f(x, y)$ funksiyaning $\overline{D}$ soha bo‘yicha integral yig‘indisi ekanligi bizga ma’lum. Agar $\lambda \to 0$ da limitga o‘tsak, uzluksiz funksiya integrallanuvchi bo‘lishiga asosan

$$V = \iint_D f(x, y)dxdy \qquad (5.1.1)$$

ni olamiz. (5.1.1) ni *silindrsimon jism hajmining formulasi* deb qabul qilamiz. Demak, bu yerda qo‘yilgan masalaning hali ikki karrali integralni hisoblashga keltirildi.

### 5.1.2. Tekis shakl bo‘yicha tarqalgan massa

Aytaylik, biror $m$ massa tekis, chekli diametrli, o‘lchanuvchi va bog‘liq bo‘lgan $\overline{D}$ yopiq sohada tarqalgan bo‘lib, uning $(x, y)$ nuqtasidagi zichligi

$\rho = \rho(x, y)$ uzluksiz va manfiy bo‘lmagan funksiya vositasida berilgan bo‘lsin. Shu $m$ massani hisoblash masalasini qo‘yaylik. Buni hal qilish uchun oldingi masalada qilingan ish jarayonini bu yerda ham bajarib,

$$m \approx \sum_{k=1}^{m} \rho(x_k , y_k)\Delta S_k$$

taqribiy formulaga va unda $\lambda \to 0$ dagi limitga o‘tib,

$$m = \iint_D \rho(x, y)dxdy \qquad (5.1.2)$$

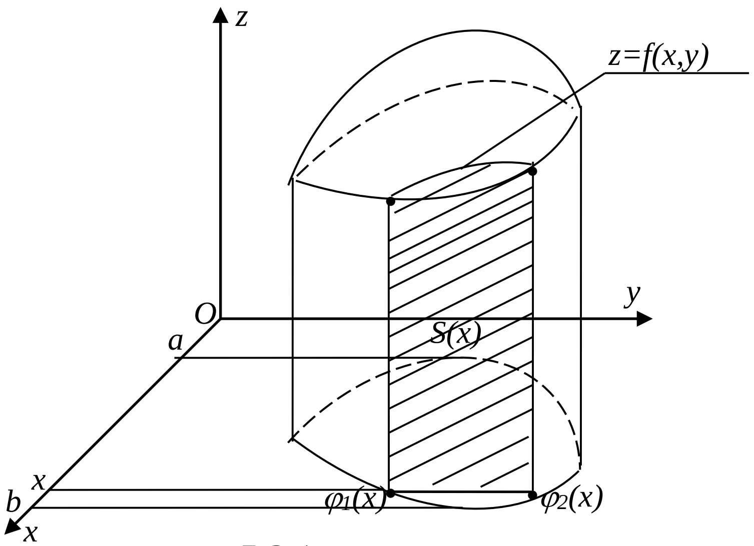
ga kelamiz. *(4.1.2) ni tekis jism massasini hisoblash formulasi* deb qabul qilamiz. Bu masalaning hali ham ikki karrali integralni hisoblashga keltirilganini aytamiz.

## 5.2. Ikki karrali integralni hisoblash

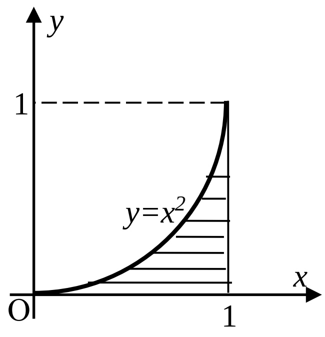
Oldingi bandda ko‘rilgan silindrsimon jism hajmini hisoblash masalasini $\overline{D}$ sohani

$$\begin{cases} a \le x \le b, \\ \varphi_1(x) \le y \le \varphi_2(x) \end{cases} \qquad (5.2.1)$$

tengsizliklar sistemasi vositasida aniqlangan holda qaraylik (bu yerda $a < b$ berilgan sonlar, $\varphi_1(x)$ va $\varphi_2(x)$ lar $[a, b]$ kesmada uzluksiz hamda $\varphi_1(x) \le \varphi_2(x)$ bo‘lgan berilgan funksiyalardir). Bundan ko‘rinadiki, $Oxy$ tekislikda joylashgan $\overline{D}$ sohaning chegarasini $Oy$ o‘qqa parallel bo‘lgan to‘g‘ri chiziq ikkitadan ortiq nuqtada kesmaydi (5.2.1-rasm). Bu holda $\overline{D}$ ni $Oy$ *o‘qqa nisbatan to‘g‘ri soha* deb ataladi.



5.2.1-rasm.



5.2.2-rasm.

Agar bu $\overline{D}$ sohada uzluksiz va manfiy bo‘lmagan $f(x, y)$ ikki o‘zgaruvchili funksiya berilgan bo‘lsa, 5.1.1-bandda aniqlangan silindrsimon jismning hajmi uchun (5.1.1) formula o‘rinlidir.

Endi, bu hajmni bizga ma’lum bo‘lgan aniq integral (bir o‘lchovli integral) vositasida hisoblashga harakat qilaylik. Buning uchun $x \in [a, b]$ tayinlangan deb faraz qilib, jismni bu nuqta orqali absissalar o‘qiga perpendikulyar bo‘lgan tekislik $(x = const)$ bilan kessak, kesimda 5.2.1-rasmdagidek, egri chiziqli trapetsiya hosil bo‘ladi va uning yuzi uchun bizga ma’lum formuladan foydalanib,

$$S(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy$$

formulani yoza olamiz. Parallel kesimlar yuzasi bo‘yicha jism hajmini hisoblash formulasiga asosan .

$$V = \int_a^b S(x)dx = \int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \right) dx$$

ni olamiz. Bu formulaning o‘ng tomonidagi ifodani

$$\int_a^b \left( \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \right) dx = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy$$

ko‘rinishda yozish qabul qilingan bo‘lib, uni *takroriy ikki karrali integral* deb yuritiladi va uni hisoblash jarayonida, avvalo ichki integral deb ataluvchi

$$\int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy$$

da $x$ ni tayinlangan (ya’ni $x = const$ ) deb faraz qilib, uni hisoblash natijasida $x$ ning $[a, b]$ kesmada aniqlangan uzluksiz funksiyasiga kelamiz va uni integrallab, ikki karrali integralning qiymatini olamiz, ya’ni bu hol uchun

$$\iint_D f(x, y)dxdy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dy \qquad (5.2.2)$$

formula o‘rinlidir.

Xuddi Shunga o‘xshash $\overline{D}$ soha

$$\begin{cases} c \le y \le d\,, \\ \psi_1(y) \le x \le \psi_2(y) \end{cases} \qquad (5.2.3)$$

sistema vositasida (bu yerda $c < d$ berilgan sonlar $\psi_1(y)$ va $\psi_2(y)$ lar $[c, d]$ kesmada uzluksiz hamda $\psi_1(y) \le \psi_2(y)$ bo‘lgan berilgan uzluksiz funksiyalar) bo‘lsa, $\overline{D}$ *sohani* $Ox$ *o‘q bo‘yicha to‘g‘ri soha* deyilib,

$$\iint_D f(x, y)dxdy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dx \qquad (5.2.4)$$

formulani olamiz. (5.2.4) ning o‘ng tomoni ham takroriy ikki karrali integral bo‘lib, uni hisoblash jarayonida, dastlab $y$ ni tayinlangan (ya’ni $y = const$ ) deb faraz qilib, ichki integralni hisoblash natijasida ning funksiyasini olamiz va uni $[c, d]$ oraliq bo‘yicha integrallab natijaga ega bo‘lamiz.

Bu yerda $f(x, y) \ge 0$ degan shartdan voz kyechish mumkinligini aytamiz. Bunday talab qilinishiga sabab ikki karrali integralning geometrik ma’nosidan foydalanganimizdir.

Agar $\overline{D}$ soha $Oy$ o‘q bo‘yicha ham $Ox$ o‘q bo‘yicha ham to‘g‘ri soha bo‘lsa, uni (5.2.1) hamda (5.2.3) sistemalardan ixtiyoriy biri vositasida aniqlash mumkin bo‘lib, (5.2.2) va (5.2.4) formulalar o‘rinli bo‘ladi va ulardan

$$\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx$$

ekanligini olamiz. Buni *takroriy ikki karrali integrallarda integrallash tartibini o'zgartirish formulasi* deb yuritiladi.

**1-misol.** $\iint_D (1+x^2+y^2)$ ikki karrali integralni hisoblang. Bu yerda $\overline{D}$ $x=0,\ x=1$ va $y=x^2$ chiziqlar bilan chegaralangan soha.

**Yechish.** 5.2.2-rasmdan ko'rinadiki, $\overline{D}$ sohani (5.2.1) ko'rinishdagi

$$\begin{cases} 0 \le x \le 1, \\ 0 \le y \le x^2 \end{cases}$$

sistema vositasida aniqlash mumkin. (5.2.2) formuladan foydalanamiz:

$$\iint_D (1+x^2+y^2)dxdy = \int_0^1 dx \int_0^{x^2} (1+x^2+y^2)dy =$$

$$= \int_0^1 dx\left(y + x^2 y + \frac{1}{3}y^3\right)\Bigg|_{y=0}^{y=x^2} = \int_0^1 \left(x^2 + x^4 + \frac{1}{3}x^6\right)dx =$$

$$= \left(\frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{21}x^7\right)\Bigg|_0^1 = \frac{1}{3} + \frac{1}{5} + \frac{1}{21} = \frac{61}{105}$$

Xuddi Shunga o'xshash $\overline{D}$ sohani (5.2.3) ko'rinishda bo'lgan

$$\begin{cases} 0 \le y \le 1, \\ \sqrt{y} \le x \le 1 \end{cases}$$

sistema vositasida ham aniqlash mumkin. Demak, (5.2.4) formulaga asosan

$$\iint_D (1+x^2+y^2)dxdy = \int_0^1 dy \int_{\sqrt{y}}^1 (1+x^2+y^2)dx =$$

$$= \int_0^1 dy\left(x + \frac{1}{3}x^3 + y^2 x\right)\Bigg|_{x=\sqrt{y}}^{x=1} = \int_0^1 \left(1 + \frac{1}{3} + y^2 - \sqrt{y} - \frac{1}{3}\left(\sqrt{y}\right)^3 - y^2\sqrt{y}\right)dy =$$

$$= \int_0^1 \left(\frac{4}{3} + y^2 - y^{\frac{1}{2}} - \frac{1}{3}y^{\frac{3}{2}} - y^{\frac{5}{2}}\right)dy = \left(\frac{4}{3}y + \frac{1}{3}y^3 - \frac{2}{3}y\sqrt{y} - \frac{2}{15}y^2\sqrt{y} - \frac{2}{7}y^3\sqrt{y}\right)\Bigg|_0^1 =$$

$$= \frac{4}{3} + \frac{1}{3} - \frac{2}{3} - \frac{2}{15} - \frac{2}{7} = \frac{61}{105}$$

Har ikkala holda ham bitta natijaga keldik.

Demak,

$$\iint_D (1+x^2+y^2)dxdy = \frac{61}{105}.$$

## 5.3. Ikki karrali integralda o'zgaruvchilarni almashtirish

Aytaylik, $Oxy$ koordinatalar tekisligida uzluksiz L yopiq chiziq (kontur) bilan chegaralangan va o‘lchanuvchi $\overline{D}$ yopiq soha berilgan bo‘lib, unda uzluksiz bo‘lgan $f(x,y)$ funksiya aniqlangan bo‘lsin. U holda

$$\iint_D f(x,y)ds$$

ikki karrali integral mavjuddir (5.2.1-teorema).

Endi

$$x=\varphi(u,v);\ \ y=\psi(u,v) \qquad (5.3.1)$$

sistema vositasida $x$ va $y$ koordinatalar yangi $u$ va $v$ koordinatalarga almashtirilgan bo‘lib, unda $\varphi(u,v)$ va $\psi(u,v)$ larni quyidagicha aniqlanadigan o‘lchanuvchi $\overline{D}'$ yopiq sohada uzluksiz diferensiallanuvchi funksiyalar deb faraz qilamiz. Undan tashqari (5.3.1) o‘zaro bir qiymatli teskarilanuvchi almashtirish deb talab qilamiz (buning uchun almashtirish Yakobianining noldan farqli bo‘lishini talab qilishga to‘g‘ri keladi). Bu qilingan farazlar asosida $Oxy$ tekislikdagi $L$ yopiq chiziqni (5.3.1) almashtirish $Ouv$ tekislikdagi biror $L'$ yopiq chiziqqa $D$ sohaning ichki nuqtalarini esa $L'$ bilan chegaralangan $D'$ sohaning ichki nuqtalariga o‘zaro bir qiymatli akslantiradi.

Agar $D$ sohani biror usul bilan elementar tashkil etuvchilarga ajratilgan deb faraz qilsak, $f(x,y)$ funksiyaning $D$ soha bo‘yicha integral yig‘indisi uchun (5.3.1) ni hisobga olgan holda

$$\sum_{k=1}^{m} f(x_k,y_k)\Delta S_k=\sum_{k=1}^{m} f(\varphi(u_k,v_k),\psi(u_k,v_k))\Delta S_k$$

tenglikni yoza olamiz. Bu yerda $\Delta S_k$ $\overline{D}$ soha $k$ - elementar tashkil etuvchisi $\overline{\Delta D_k}$ ning o‘lchovi (yuzi) bo‘lib, $(x_k,y_k)\in\overline{\Delta D_k}$ hamda $(u_k,v_k)$ bu nuqtaning $\overline{D'}$ dagi aksidir.

Endi, $\overline{D}$ sohaning elementar tashkil etuvchisini $Ouv$ tekislikda $u=const,\ u+\Delta u=const$ va $v=const, v+\Delta v=const$ to‘g‘ri chiziqlar bilan chegaralangan to‘rtburchakning (5.3.1-rasmda $p_1'\ p_2'\ p_3'\ p_4'$ to‘rtburchak) $Oxy$ tekislikdagi asli bo‘lgan egri chiziqli to‘rtburchak (5.3.2-rasmdagi $p_1\ p_2\ p_3\ p_4$ shakl) deb qarasak, uning uchlarining koordinatalarini yozaylik:

$$\begin{cases} P_1(x_1,y_1); & x_1=\varphi(u,v), & y_1=\psi(u,v);\\ P_2(x_2,y_2); & x_2=\varphi(u+\Delta u,v), & y_2=\psi(u+\Delta u,v);\\ P_3(x_3,y_3); & x_3=\varphi(u+\Delta u,v+\Delta v), & y_3=\psi(u+\Delta u,v+\Delta v);\\ P_4(x_4,y_4); & x_4=\varphi(u,v+\Delta v), & y_4=\psi(u,v+\Delta v). \end{cases} \qquad (5.3.2)$$

Bu o‘rinda $p_1\ p_2\ p_3\ p_4$ egri chiziqli to‘rtburchakning yuzi $\Delta S$ $p_1\,p_2\,p_3$ uchburchakning ikkilangan yuzi bilan ekvivalent bo‘lishini va bu qiymatni hisoblash uchun analitik geometriyadan ma’lum bo‘lgan

$$|(x_3-x_1)(y_3-y_2)-(x_3-x_2)(y_3-y_1)|$$

ifodadan foydalanish mumkinligini aytamiz.

Yuqorida $\varphi(u,v)$ va $\psi(u,v)$ funksiyalarga qo‘yilgan talab asosida (5.3.2) dagi koordinatalarni yuqori tartibli cheksiz kichik qo‘shiluvchilarni hisobga olmagan holda quyidagicha yozamiz:

$$\begin{cases} x_1 = \varphi(u,v), \quad y_1 = \psi(u,v); \\ x_2 = \varphi(u,v) + \dfrac{\partial\varphi}{\partial u}\Delta u, \quad y_2 = \psi(u,v) + \dfrac{\partial\psi}{\partial u}\Delta u; \\ x_3 = \varphi(u,v) + \dfrac{\partial\varphi}{\partial u}\Delta u + \dfrac{\partial\varphi}{\partial v}\Delta v, \quad y_3 = \psi(u,v) + \dfrac{\partial\psi}{\partial u}\Delta u + \dfrac{\partial\psi}{\partial v}\Delta v; \\ x_4 = \varphi(u,v) + \dfrac{\partial\varphi}{\partial v}\Delta v, \quad y_4 = \psi(u,v) + \dfrac{\partial\psi}{\partial v}\Delta v; \end{cases}$$

Yuqorida $\Delta S$ ga nisbatan qilingan mulohazadagi va oxirgi koordinatalarga asoslanib,

$$\Delta S \approx \left| \frac{\partial\varphi}{\partial u} \cdot \frac{\partial\psi}{\partial v} - \frac{\partial\varphi}{\partial v} \cdot \frac{\partial\psi}{\partial u} \right| \Delta u \Delta v$$

ni olamiz. Bunda absolyut qiymat belgisi ostidagi ifoda (5.3.1) almashtirish Yakobiani

$$J = \begin{vmatrix} \dfrac{\partial\varphi}{\partial u} & \dfrac{\partial\varphi}{\partial v} \\ \dfrac{\partial\psi}{\partial u} & \dfrac{\partial\psi}{\partial v} \end{vmatrix}$$

ekanligi ma’lumdir .Demak,

$$\Delta S = |J| \Delta S'$$

ekanligini (bunda $\Delta S' = \Delta u \Delta v$ ) va

$$\sum_{k=1}^{m} f(x_k, y_k)\Delta S_k \approx \sum_{k=1}^{m} f(\varphi(u_k, v_k)\psi(u_k, v_k))|J|\Delta S'_k$$

ga ega bo‘lamiz. Oxirgida $\lambda \to 0$ dagi limitga o‘tib,

$$\iint_D f(x,y)dxdy = \iint_{D'} f(\varphi(u,v), \psi(u,v))|J(u,v)|dudv \qquad (5.3.3)$$

ni olamiz. Bu ikki karrali integralda o‘zgaruvchilarni almashtirish formulasidir.

Bu o‘rinda (5.3.1) ning maxsus holi bo‘lgan qutb koordinatalariga o‘tish almashtirishini qaraylik ($u = \rho$ qutb radiusi, $v = \theta$ qutb burchagi):

$$x = \rho\cos\theta, y = \rho\sin\theta. \qquad (5.3.4)$$

Almashtirish Yakobiani:

$$J(\rho,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial\rho} & \dfrac{\partial x}{\partial\theta} \\ \dfrac{\partial y}{\partial\rho} & \dfrac{\partial y}{\partial\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho.$$

Buni va (5.3.4) ni (5.3.3) ga qo‘yib,

$$\iint_D f(x,y)dxdy = \iint_{D'} f(\rho\cos\theta, \rho\sin\theta)\rho d\rho d\theta \qquad (5.3.5)$$

ni olamiz. (5.3.5) ikki karrali integralda qutb koordinatalariga o‘tish formulasidir.

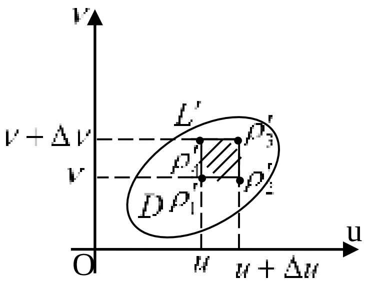
**2-misol.** $\iint_D e^{-x^2-y^2}dxdy$ iki karrali integralni $D$ - markazi koordinatalar boshida va radiusi $R$ ga teng bo‘lgan doiradan iborat bo‘lgan holda hisoblang.

**Yechish.** Agar $\overline{D}$ sohani $-R \le x \le R,\ -\sqrt{R^2 - x^2} \le y \le \sqrt{R^2 - x^2}$ sistema vositasida yozish mumkinligini e’tiborga olsak,

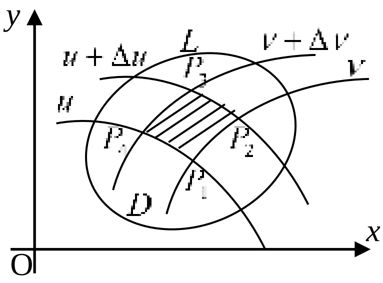
$$\iint_D e^{-x^2-y^2}dxdy = \int_{-R}^{R} e^{-x^2}dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} e^{-y^2}dy$$

ni yozaolamiz. Bundan ko‘rinadiki, amaliy jihatdan olinmaydigan integralga keldik.

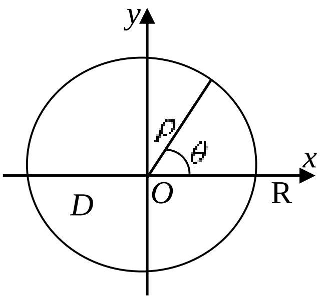
Endi uni qutb koordinatalariga o‘tib hisoblaylik. Bu holda 5.3.3-rasmdan ko‘rinadiki, $\overline{D}$ doiraning nuqtalari uchun $0 \le \rho \le R,\ 0 \le \theta \le 2\pi$ bo‘lishi aniqdir. Ana Shu $O\rho\theta$ koordinatalar sistemasidagi to‘g‘ri to‘rtburchak $\overline{D'}$ sohadir (5.3.4-rasm).



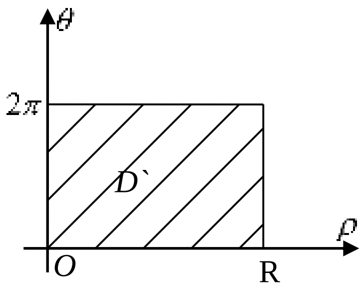

5.3.1-rasm.




5.3.2-rasm.




5.3.3-rasm.




5.3.4-rasm.

(5.3.5) formulaga asosan

$$\iint_D e^{-x^2-y^2}dxdy = \iint_D e^{-\rho^2\cos^2\theta-\rho^2\sin^2\theta}\rho d\rho d\theta =$$

$$= \iint_D e^{-\rho^2}\rho d\rho d\theta = \int_0^R e^{-\rho^2}\rho d\rho \int_0^{2\pi} d\theta = \pi\left(1 - e^{-R^2}\right)$$

Bu misolda berilgan ikki karrali integralni bevosita takroriy karrani integralga o‘tish amaliy jihatdan olinmaydigan integralga keltirdi, lekin unda qutb koordinatalariga o‘tib hisoblash ishni birmuncha osonlashtirdanini ko‘ramiz.

## 5.4. Sirt yuzini hisoblash

Aytaylik, chekli diametrli va o‘lchanuvchi $\overline{D}_{xy}$ yopiq sohada $z = f(x, y)$ ikki o‘zgaruvchili uzluksiz differensiallanuvchi funksiya berilgan bo‘lsin. Bu funksiya vositasida aniqlangan sirtning yuzini hisoblash masalasini qo‘yamiz.

Bu masalani hal qilish maqsadida $\overline{D}_{xy}$ sohani ixtiyoriycha qilib, $m$ ta elementar tashkil etuvchilarga ajratamiz. Ulardan $i$ -sini va uning o‘lchovini (yuzini) ham $\Delta S_i$ $\left(i=\overline{1,m}\right)$ bilan unga mos keluvchi sirtning bo‘lagini esa va uning yuzini ham $\Delta\sigma_i$ bilan belgilaylik. U holda aytilganlar asosida hisoblanishi talab qilingan sirtning yuzi $\sigma$ uchun

$$\sigma=\sum_{i=1}^{m}\Delta\sigma_i \qquad (5.4.1)$$

ni yozaolamiz. Endi $\Delta\sigma_i$ ning taqribiy bo‘lsa ham qiymatini hisoblash maqsadida $\Delta S_i$ elementar bo‘lakchada ixtiyoriy bitta $P_i(x_i,y_i)$ nuqtani olib, unga mos sirtning $\Delta\sigma_i$ elementar bo‘lagida $M_i(x_i,y_i,f(x_i,y_i))$ nuqtaga ega bo‘lamiz. $M_i$ nuqta orqali sirtga urinma tekislik o‘tkazamiz va $\Delta\sigma_i$ elementar bo‘lakcha Shu urinma tekislikda yotadi deb faraz qilamiz.

Urinma tekislik $Oxy$ gorizontal koordinatalar tekisligi bilan $\gamma_i$ burchak hosil qiladi desak, bu burchak $M_i$ nuqtada sirtga o‘tkazilgan normalning $Oz$ o‘q bilan hosil qilgan burchagiga teng bo‘ladi. U holda

$$\Delta S_i\approx\Delta\sigma_i\cos\gamma_i$$

va

$$\cos\gamma_i=\frac{1}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}$$

ekanligidan

$$\Delta\sigma_i\approx\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\cdot\Delta S_i$$

ni olamiz. Endi buni (5.4.1) ga qo‘yib, so‘ngra $\lambda\to 0$ dagi, bu yerda $\lambda=\max\limits_i d_i$, $d_i$ esa $\Delta S_i$ ning diametri, limitga o‘tib,

$$\sigma=\iint\limits_{D_{xy}}\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dxdy \qquad (5.3.2)$$

ni olamiz. Bu $z=f(x,y)$ sirtning yuzini hisoblash formulasidir.

Xuddi Shunga o‘xshash sirt $\overline{D}_{yz}-Oxz$ koordinatalar tekisligidagi sohada $x=f(x,y)$ funksiya vositasida aniqlangan bo‘lsa,

$$\sigma=\iint\limits_{D_{xz}}\sqrt{1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2}\,dydz \qquad (5.3.3)$$

formula va $\overline{D}_{yz}-Oyz$ koordinatalar tekisligidagi sohada berilgan $y=f(x,z)$ funksiya vositasida aniqlanganda

$$\sigma=\iint\limits_{D_{yz}}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2}\,dydz \qquad (5.3.4)$$

formula o‘rinlidir.

**3-misol.** $x^2+y^2+z^2=R^2$ sferaning yuzini hisoblang.

**Yechish.** Avvalo, yarim sfera yuzini hisoblaymiz. Buning uchun sferaning gorizontal koordinatalar tekisligidan yuqorida joylashgan qismini olsak,

$$z = \sqrt{R^2 - x^2 - y^2}, \quad x^2 + y^2 \le R^2$$

bo‘lishi aniqdir. Xususiy hosilalarni topamiz:

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}, \quad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}$$

$\overline{D}$ soha $x^2 + y^2 \le R^2$ doiradan iboratdir. Bularni (5.3.2) formulaga qo‘yamiz:

$$\frac{1}{2}\sigma = \iint_{D_{xy}} \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}}\,dxdy = \iint_{D_{xy}} \frac{R}{\sqrt{R^2 - x^2 - y^2}}\,dxdy.$$

Bunda qutb koordinatalariga o‘tib, hamda $0 \le \rho \le R$, $0 \le \theta \le 2\pi$ ekanligini e’tiborga olib, (5.3.5) formulani qo‘llab,

$$\frac{1}{2}\sigma = \iint_{D_{xy}} \frac{R}{\sqrt{R^2 - \rho^2}}\,\rho d\rho d\theta = R\int_0^{2\pi} d\theta \int_0^R \frac{\rho d\rho}{\sqrt{R^2 - \rho^2}}$$

$$= R \cdot 2\pi \cdot \left(-\sqrt{R^2 - \rho^2}\right)\Big|_0^R = 2\pi R^2.$$

Bundan sfera sirtining yuzi uchun

$$\sigma = 4\pi R^2$$

formulani olamiz.

## 5.5. Tekis shakl yuzining inersiya momenti

Nazariy mexanika kursidan ma’lumki, $m$ massaga ega bo‘lgan $M$ moddiy nuqtaning biror O nuqtaga nisbatan *inersiya momenti* deb

$$I = mr^2$$

ga aytiladi, bu yerda $r = |OM|$.

Agar mos ravishda $m_i$ $\left(i = \overline{1, n}\right)$ massaga ega bo‘lgan $M_i$ moddiy nuqtalarning sistemasi berilgan bo‘lsa, bu sistemaning biror O nuqtaga nisbatan inersiya momenti deb

$$I = \sum_{i=1}^{n} m_i r_i^2$$

ni qabul qilinadi, bu yerda $r_i = |OM_i|$.

Aytaylik, biror tekis shakl chekli diametrli, o‘lchanuvchi $\overline{D}$ yopiq soha ko‘rinishida berilgan bo‘lib, uning har bir $(x; y)$ nuqtasidagi zichligi $\rho = \rho(x, y)$ uzluksiz va manfiy bo‘lmagan funksiya vositasida aniqlangan bo‘lsin. Bu tekis shaklning koordinata boshiga nisbatan inersiya momentini hisoblash maqsadida uni ixtiyoriy usulda $n$ ta elementar tashkil etuvchilarga ajratamiz va ulardan $i$-sining $\left(i = \overline{1, n}\right)$ (yuzini ham) $\Delta S_i$ bilan belgilab, unga tegishli bo‘lgan ixtiyoriy bitta $M_i(x_i; y_i)$ nuqtani olamiz. Uni moddiy nuqta sifatida qarab, massasi $m_i$ sifatida $\rho(x_i, y_i) \cdot \Delta S_i$ ni qabul qilamiz. Bu ishni bajarib, $D$ tekis shakl koordinatalar boshiga nisbatan inersiya momentining taqribiy qiymati deb, har birining massasi $m_i$ dan iborat bo‘lgan $M_i(x_i; y_i)$ $\left(i=\overline{1,n}\right)$ moddiy nuqtalar sistemasining inersiya momentini qabul qilamiz:

$$I_0 \approx \sum_{i=1}^{n} r_i^2 \rho(x_i; y_i)\Delta S_i \ ,$$

bu yerda $r_i = \sqrt{x_i^2 + y_i^2}$ ekanligi ma’lumdir.

Agar oxirgi taqribiy tenglikda $\lambda \to 0$ dagi limitga o‘tib, bu yerda $\lambda = \max\limits_{i=\overline{1,n}} d_i$ ( $d_i - \Delta S_i$ ning diametri) va

$$I_0 = \iint_D \left(x^2 + y^2\right)\rho(x; y)dxdy \qquad (5.5.1)$$

deb qabul qilib, $D$ *tekis shaklning koordinata boshiga nisbatan inersiya momentini hisoblash uchun formulani* olamiz.

Bu o‘rinda

$$I_x = \iint_D y^2 \rho(x; y)dxdy,$$

$$I_y = \iint_D x^2 \rho(x; y)dxdy$$

lar mos ravishda, $D$ *tekis shaklning* $Ox$ *yoki* $Oy$ *o‘qqa nisbatan inersiya momenti* deb yuritilishini aytamiz.

## 5.6. Tekis shakl og‘irlik markazining koordinatalari

$D$ tekis shakl oldingi bandda aytilgandek $\rho = \rho(x, y)$ zichlik funksiyasi bilan berilgan bo‘lib, uni elementar tashkil etuvchilarga ajratilib, massasi $m_i = \rho(x_i; y_i)\Delta S_i$ bo‘lgan $M_i(x_i; y_i)$ $\left(i=\overline{1,n}\right)$ moddiy nuqtalarning sistemasi hosil qilingan bo‘lsin. Bunday qurilgan moddiy nuqtalar sistemasi og‘irlik markazining koordinatalari $C\left(\overline{x}_c; \overline{y}_c\right)$ uchun

$$\overline{x}_c = \frac{1}{m}\sum_{i=1}^{n} x_i m_i \ , \quad \overline{y}_c = \frac{1}{m}\sum_{i=1}^{n} y_i m_i$$

formulalar o‘rinli ekanligi mexanikadan ma’lum, bu yerda $m = \sum\limits_{i=1}^{n} m_i$ - sistemaning massasidir.

Endi $\lambda \to 0$ dagi limitga o‘tish bilan $D$ tekis shakl og‘irlik markazi $C(x_c; y_c)$ ning koordinatalari uchun

$$x_c = \frac{1}{m}\iint_D x\rho(x; y)dxdy, \quad y_c = \frac{1}{m}\iint_D y\rho(x; y)dxdy$$

formulalarni olamiz, bu yerda $m = \iint\limits_D \rho(x; y)dxdy$ -tekis shaklning massasidir.

Bu o‘rinda

$$M_x = \iint_D y\rho(x;y)dxdy,$$
$$M_y = \iint_D x\rho(x;y)dxdy$$
lar mos ravishda, $D$ tekis shaklning $Ox$ yoki $Oy$ o‘qqa nisbatan *statik momenti* (*turg‘unlik lahzasi*) deb yuritilishini aytamiz.

**4-misol.** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellips bilan chegaralangan va har bir nuqtasidagi zichligi $\rho=1$ bo‘lgan bir jinsli tekis shakl birinchi chorakda joylashgan qismining og‘irlik markazini toping.

**Yechish.** $m=\iint_D \rho(x;y)dxdy=\iint_D dxdy$

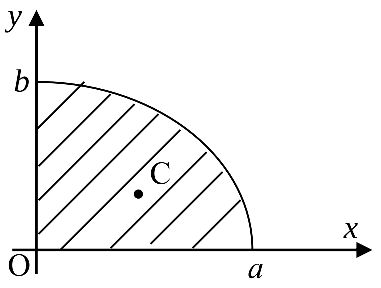
5.6.1-rasmdan $D$ tekis sohani
$$0\le x\le a,\ \ 0\le y\le \frac{b}{a}\sqrt{a^2-x^2}$$
sistema vositasida aniqlash mumkinligini ko‘ramiz. Demak,
$$m=\int_0^a dx\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}dy=\frac{b}{a}\int_0^a\sqrt{a^2-x^2}dx$$
Bunda $x=a\sin t$, $t\in\left[0;\frac{\pi}{2}\right]$ almashtirib qilib,
$$m=\frac{b}{a}\int_0^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin^2 t}\,a\cos tdt=ab\int_0^{\frac{\pi}{2}}\cos^2 tdt=$$
$$=\frac{ab}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2t)dt=\frac{\pi ab}{4}.$$
$\iint_x x\rho(x;y)dxdy=\iint_D xdxdy.$

Bunda $x=a\rho\cos\theta$, $y=b\rho\sin\theta$ almashtirish qilsak, $0\le\theta\le\frac{\pi}{2}$, $0\le\rho\le 1$ va Yakobian uchun
$$J(\rho;\theta)=\begin{vmatrix} a\cos\theta & -a\rho\sin\theta \\ b\sin\theta & b\rho\cos\theta \end{vmatrix}=ab\rho$$
bo‘lib,
$$\iint_D xdxdy=\int_0^{\frac{\pi}{2}}da\int_0^1 a\rho\cos\varphi ab\rho d\rho=\int_0^{\frac{\pi}{2}}\cos\varphi d\varphi\int_0^1 a^2b\rho^2 a\rho=\frac{a^2b}{3};$$
$$\iint_D y\rho(x;y)dxdy=\iint_D ydxdy=\int_0^{\frac{\pi}{2}}da\int_0^1 b\rho\sin\theta\cdot ab\rho d\rho=$$
$$=\int_0^{\frac{\pi}{2}}\sin\theta d\theta\int_0^1 ab^2\rho^2d\rho=\frac{ab^2}{3}$$
Demak,

$$x_c=\frac{\frac{a^2b}{3}}{\frac{\pi ab}{4}}=\frac{4a}{3\pi},\quad y_c=\frac{\frac{a^2b}{3}}{\frac{\pi ab}{4}}=\frac{4b}{3\pi};$$

Ya’ni

$$C\left(\frac{4a}{3\pi};\frac{4b}{3\pi}\right).$$




5.6.1-rasm.

## 5.7. Ikkilangan xosmas integral

Yuqorida ko‘p o‘lchovli integral (jumladan ikkilangan integral) tushunchasini kiritish uchun integrallash sohasini chekli diametrli va o‘lchanuvchi, unda aniqlangan funksiyani esa chegaralangan bo‘lishini talab qilgan edik. Xuddi bir o‘zgaruvchili funksiya uchun kiritilgan xosmas integral tushunchasi singari ko‘p o‘lchovli integral (jumladan ikkilangan integral) uchun ham integrallash sohasi cheksiz diametrli yoki funksiya chegaralanmagan holda ham limit tushunchasi vositasida ko‘p o‘lchovli xosmas integral tushunchasini kiritish mumkin. Bu ishni takroriy karrali integral uchun bajarish ancha qulaydir. Masalan, integrallash sohasi $D$ koordinata tekisligidan iborat bo‘lsa, ikki karrali xosmas integralni

$$\iint_D f(x;y)ds=\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}f(x;y)dy=\int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}f(x;y)dx=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x;y)dxdy$$

ko‘rinishlarda yozish qulaydir.

**5-misol.** $\int_{-\infty}^{+\infty}e^{-x^2}dx$ - Puasson integralini hisoblang.

**Yechish.** $\iint_D e^{-x^2-y^2}dxdy$ - ikkilangan integralni olaylik. Agar $D$ soha markazi koordinata boshida, tomoni $2a$ $(a>0)$ ga teng va koordinata o‘qiga parallel bo‘lgan kvadratdan iborat bo‘lsa (5.7.1-rasmga qarang),

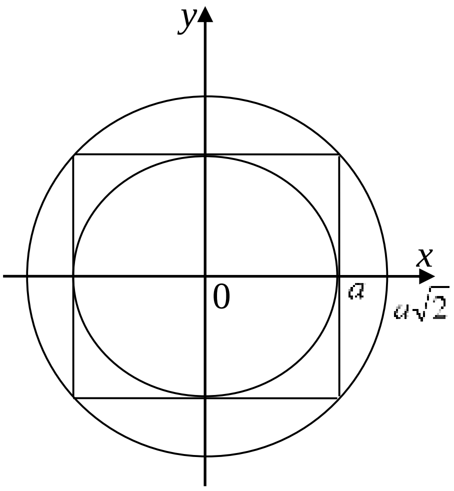
$$I_{k\vartheta}=\iint_D e^{-x^2-y^2}ds=\int_{-a}^{a}dx\int_{-a}^{a}e^{-x^2-y^2}dy=\int_{-a}^{a}e^{-x^2}dx\int_{-a}^{a}e^{-y^2}dy=\left(\int_{-a}^{a}e^{-x^2}dx\right)^2$$

ni olamiz.

Agar $D$ soha markazi koordinata tekisligining boshida radiusi $R\,(R>0)$ ga teng doiradan iborat bo‘lsa, u holda ikkilangan integralda qutb koordinatalariga o‘tib,

$$I_R = \iint_D e^{-x^2-y^2}dxdy = \int_0^{2\pi} d\varphi \int_0^R e^{-\rho^2}\rho d\rho = \pi\left(1-e^{-R^2}\right)$$

ni olamiz.




5.7.1-rasm.

5.7.1-rasmdan ko‘rinadiki, yuqorida aytilgan kvadratga ichki va tashqi chizilgan doiralarning radiuslari, mos ravishda $a$ va $a\sqrt{2}$ ga tengdir. Integral osti funksiyasi koordinata tekisligida uzluksiz va musbat ekanligidan

$$I_a < I_{k\vartheta} < I_{a\sqrt{2}}$$

Munosabatni yozaolamiz. Agar $a\to+\infty$ dagi limitga o‘tsak, yuqorida $I_R$ uchun olingan natijada $R=a$ va $R=a\sqrt{2}$ deb faraz qilib,

$$\lim_{a\to+\infty} I_a = \lim_{a\to+\infty} I_{a\sqrt{2}} = \pi$$

ni hamda $I_{kb}$ uchun olingan ifodadan

$$\lim_{a\to+\infty} I_{k\vartheta} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}dxdy = \left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2$$

ni olamiz va bulardan so‘nggi tengsizlikka asosan

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}dxdy = \left(\int_{-\infty}^{+\infty} e^{-x^2}dx\right)^2 = \pi$$

ekanligini ko‘ramiz. Bundan esa Puasson integrali uchun

$$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$$

natijaga kelamiz.

## 5.8. Parametrga bog‘liq integralni hisoblash

Aytaylik, $D=[a;b)\times[c;d]$ iki o‘lchovli sohada aniqlangan $f(x;y)$ ikki o‘zgaruvchili funksiya berilgan bo‘lib, $\forall\, y\in[c;d]$ uchun

$$I(y)=\int_a^b f(x;y)dx \qquad (5.8.1)$$

integral mavjud bo‘lsin. Agar $b$ chekli bo‘lib, $f(x;y)$ funksiya $D=[a;b]\times[c;d]$ yopiq to‘g‘ri to‘rtburchakda uzluksiz bo‘lsa, (5.8.1) integral mavjud bo‘lib, uni $y$ *parametrga bog‘liq integral* deb ataymiz hamda bu integral $[c;d]$ kesmada uzluksiz bo‘lishini aytamiz. U holda

$$\int_c^d I(y)dy=\int_c^d dy\int_a^b f(x;y)dx \qquad (5.8.2)$$

integral mavjud ekanligi ravshandir. (5.8.2) tenglikning o‘ng tomonidagi ifoda takroriy ikki karrali integral ekanligini ko‘ramiz va unda integrallash tartibini o‘zgartirib,

$$\int_c^d I(y)dy=\int_a^b dx\int_c^d f(x;y)dy \qquad (5.8.3)$$

formulani olamiz. (5.8.3) dan parametrga bog‘liq bo‘lgan integralni integrallash uchun integral osti funksiyasini Shu parametr bo‘yicha integrallash yetarli ekanligini ko‘ramiz.

Bu o‘rinda (5.8.3) formula $y$ parametr yoki $x$ bo‘yicha integral xosmas bo‘lgan holda ham o‘rinli bo‘lishini aytamiz. Buni quyidagi misolda ko‘ramiz.

**6-misol.** $\int_0^{+\infty}\frac{e^{-ax}-e^{-bx}}{x}dx$ xosmas integralni hisoblang $(0<a<b)$.

**Yechish.** Berilganga mos aniqmas integral olinmaydigan turga kirishini ko‘rish oson. Bu integralni olinishi qiyin bo‘lmagan

$$\int_0^{+\infty} e^{-\alpha x}dx=\frac{1}{\alpha} \qquad (\alpha>0)$$

integral vositasida hisoblaymiz. Oxirgining ikki tomonini $\alpha\in[a;b]$ oraliq bo‘yicha integrallab,

$$\int_a^b d\alpha\int_0^{+\infty} e^{-\alpha x}dx=\ln\frac{b}{a}$$

ni olamiz. Bu yerda takroriy ikki karali integralda integrallash tartibini o‘zgartirib,

$$\int_0^{+\infty} dx\int_a^b e^{-\alpha x}dx=\ln\frac{b}{a},$$

$$\int_0^{+\infty}\frac{e^{-\alpha x}-e^{-bx}}{x}dx=\ln\frac{b}{a}$$

ni olamiz. Bu talab qilingan natijadir.

## 5-bobga doir mashqlar

Ikkilangan integralni hisoblang.

1. $\iint_D (x^2+y^2)dxdy$, bu yerda $D=[0;1]\times[1;2]$

Javob. $\frac{8}{3}$.

2. $\iint_D \frac{dxdy}{(x+y)^2}$, bu yerda $D=[1;4]\times[1;2]$

Javob. $\frac{25}{24}$.

3. $\iint_D xydxdy$, bu yerda $D$ $x=1,\ x=2,\ y=x,\ y=x\sqrt{3}$ to‘g‘ri chiziqlar bilan chegaralangan soha.

Javob. $\frac{15}{4}$

Takroriy ikki karrali integralda integrallash tartibini o‘zgartiring.

4. $\int_1^2 dx\int_3^4 f(x;y)dy$.

5. $\int_0^1 dx\int_{x^3}^{\sqrt{x}} f(x;y)dy$

6. $\int_0^a dy\int_0^{\sqrt{2ay-y^2}} f(x;y)dx$

7. $\int_{-1}^1 dx\int_0^{\sqrt{1-x^2}} f(x;y)dy$

8. $\int_0^1 dy\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x;y)dx$

9. $\int_0^1 dy\int_{-\sqrt{1-y^2}}^{1-y} f(x;y)dx$

Qutb koordinatalariga o‘tib ikkilangan integralni hisoblang.

10. $\iint_D \sqrt{a^2-x^2-y^2}\,dxdy$, bu yerda $D$- markazi koordinata boshida radiusi $a$ ga teng bo‘lgan doiraning koordinata tekisligining birinchi choragiga joylashgan qismi.

Javob $\frac{\pi}{6}a^3$.

11. $\iint_D (x^2+y^2)dxdy$, bu yerda $D$ - 10-mashqdagi soha.

Javob. $\frac{\pi}{8}a^4$.

12. $\iint_D e^{-x^2-y^2}dxdy$, bu yerda $D$- koordinata tekisligining birinchi choragidan iborat soha.

Javob. $\frac{\pi}{4}$.

13. $\iint_D dxdy$, bu yerda $D$- markazi $(a;0)$ nuqtada radiusi $a\ (a>0)$ ga teng bo‘lgan doira.

Javob. $\pi a^2$.

14. $x^2 + y^2 = z^2$ konusning $x^2 + y^2 = 2ax$ silindr bilan kesilgan qismidan iborat bo‘lgan sirt yuzini hisoblang.

Javob. $2\pi a^2\sqrt{2}$.

15. $x + y + z^2 = 2a$ sirtning koordinata fazosining birinchi oktantasiga joylashgan va $x^2 + y^2 = a^2$ silindr bilan chegaralangan qismining yuzini hisoblang.

Javob. $\frac{\pi a^2}{4}\sqrt{3}$.

16. Ikkita $x^2 + y^2 = a^2$ va $y^2 + z^2 = a^2$ silindrik sirtlarning kesishishidan hosil bo‘lgan sirt yuzini hisoblang.

Javob. $16a^2$.

## 5-bob bo‘yicha bilimingizni sinab ko‘ring

1. Ikkilangan integral haqida tushuncha bering.
2. Takroriy ikki karrali integral haqida tushuncha bering.
3. Ikkilangan integralni hisoblashning asosiy formulasini yozing.
4. Ikkilangan integralda o‘zgaruvchilarni almashtirish formulasini yozing va uni tuShuntiring.
5. Ikkilangan integralda qutb koordinatalariga o‘tishni tuShuntiring va formulasini yozing.
6. Ikkilangan integral yordamida jism hajmini hisoblashni tuShuntiring.
7. Sirt yuzini ikkilangan integral yordamida hisoblash formulasini yozing va uni tuShuntiring.
8. Tekis shakl yuzining inersiya momentlari haqida nimalarni bilasiz.
9. Tekis shakl yuzining og‘irlik markazini topish formulalarini yozing.
10. Ikkilangan xosmas integral haqida tushuncha bering.
11. Parametrga bog‘liq integral haqida tushuncha bering.

# 6-bob. Uch karrali (uchlangan) integral

Agar 4-bobda bayon qilingan $n$ o‘lchovli integralning ta’rifida $n=3$ deb faraz qilsak, uch o‘lchovli integral tushunchasiga kelamiz. Uni *uch karrali (uchlangan) integral* deb atash va

$$\iiint_D f(\mu)dV \quad \text{yoki} \quad \iiint_D f(x;y;z)dxdydz$$

kabi belgilash qabul qilingan.

Bu bobda uchlangan integralning ta’rifini, mavjudligi haqidagi shartni va xossalarini takrorlamay (buni o‘quvchi 4-bobdagi $n$ o‘rniga 3 ni qo‘yish bilan keltirib chiqaroladi deb ishonamiz), uni hisoblashning asosiy usulini, unda o‘zgaruvchilarni almashtirish formulasini va uning ba’zi bir tatbiqlarini keltiramiz.

## 6.1. Uchlangan integralni hisoblash

Odatda $R^3$-uch o‘lchovli fazoga tegishli bo‘lgan chekli diametrli va o‘lchanuvchi $D$ sohani, geometrik jihatdan, uning chegarasidan iborat bo‘lgan biror $\gamma$ yopiq sirt bilan o‘ralgan jism deb qaralishi ma’lumdir. Faraz qilaylik, $Oxyz$ uch o‘lchovli koordinata fazosida chekli diametrli va o‘lchanuvchi $D$ soha berilgan bo‘lib, uning chegarasidan iborat bo‘lgan yopiq $\gamma$ sirt $Oz$ koordinata o‘qiga parallel bo‘lgan ixtiyoriy to‘g‘ri chiziq bilan ikkitadan ortiq umumiy nuqtaga ega bo‘lmasin (6.1.1-rasmga qarang) yoki ularning umumiy nuqtalarining to‘plami yasovchisi $Oz$ o‘qiga parallel bo‘lgan silindrik sirtdan iborat (6.1.2-rasmga qarang), hamda uning $Oxy$ koordinata tekisligidagi proyeksiyasi ikki o‘lchovli to‘g‘ri $D_{xy}$ sohadan iborat bo‘lsin (6.1.1- va 6.1.2-rasmga qarang), u holda $D$ ni $Oz$ *o‘qqa nisbatan uch o‘lchovli to‘g‘ri soha* deb ataladi. Xuddi Shunga o‘xshash $Ox$ (yoki $Oy$) o‘qqa nisbatan uch o‘lchovli to‘g‘ri soha tushunchasi ham kiritiladi.

$Oz$ o‘qqa nisbatan uch o‘lchovli to‘g‘ri $D$ sohani

$$\begin{cases} a \le x \le b, \\ \varphi_1(x) \le y \le \varphi_2(x), \\ \psi_1(x;y) \le z \le \psi_2(x;y) \end{cases} \tag{6.1.1}$$

yoki

$$\begin{cases} c \le y \le d, \\ \varphi_1(y) \le x \le \varphi_2(y), \\ \psi_1(x;y) \le z \le \psi_2(x;y) \end{cases} \tag{6.1.2}$$

sistema vositasida aniqlash mumkinligi ravshandir. Bu holda (6.1.1) o‘rinli bo‘lganda

$$\iiint_D f(x;y;z)dxdydz = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} dy \int_{\psi_1(x;y)}^{\psi_2(x;y)} f(x;y;z)dz \tag{6.1.3}$$

tenglik to‘g‘ri bo‘lib, u uchlangan integralni hisoblashning asosiy formulalaridan biridir. (6.1.3) formulaning o‘ng tomonidagi ifodani *takroriy uch karrali integral* deb atalib, uni hisoblash uchun, dastavval,

$$\int_{\psi_1(x;y)}^{\psi_2(x;y)} f(x;y;z)dz = \Phi(x;y)$$

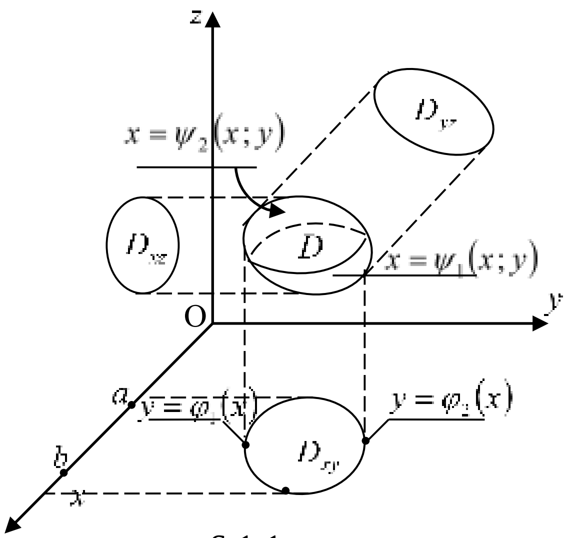
ifodada $x$ va $y$ o‘zgaruvchilar tayinlangan (o‘zgarmas) deb faraz qilib, $z$ bo‘yicha integrallash amalini bajarib, qandaydir ikki o‘zgaruvchili $\Phi(x;y)$ funksiyaga ega bo‘lamiz, so‘ngra unda $x$ ni tayinlangan va $\varphi_1(x) \le y \le \varphi_2(x)$ deyilgan faraz asosida $y$ bo‘yicha integrallab,

$$\int_{\varphi_1(x)}^{\varphi_2(x)} \Phi(x;y)dy = F(x)$$

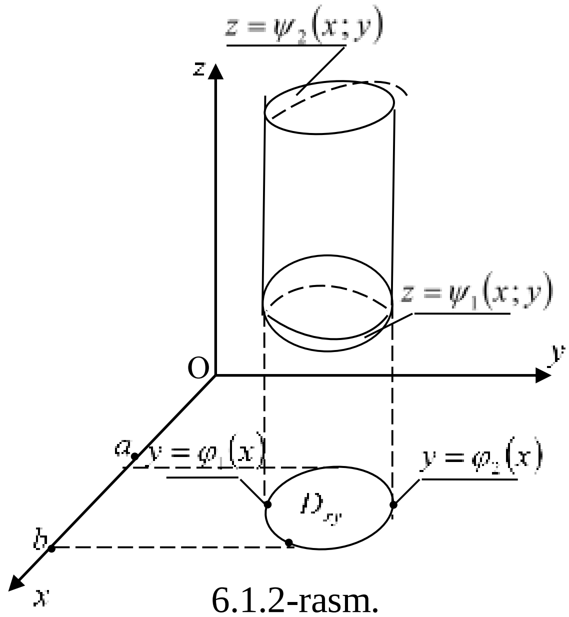
$x$ ning funksiyasini hamda $F(x)$ ni $[a;b]$ oraliq bo‘yicha integrallab,

$$\int_a^b F(x)dx = A$$

takroriy uch karrali integralning, demak, uchlangan integralning ham qiymatiga ega bo‘lamiz.




6.1.1-rasm.




6.1.2-rasm.

Agar $D$ soha (6.1.2) sistema vositasida aniqlangan bo‘lsa,

$$\iiint_D f(x;y;z)dxdydz = \int_c^d dy \int_{\varphi_1(y)}^{\varphi_2(y)} dx \int_{\psi_1(x;y)}^{\psi_2(x;y)} f(x;y;z)dz \qquad (6.1.4)$$

formula o‘rinlidir. (6.1.3) va (6.1.4) formulalarni

$$\iiint_D f(x;y;z)dxdydz = \iint_{D_{xy}} dxdy \int_{\psi_1(x;y)}^{\psi_2(x;y)} f(x;y;z)dz \qquad (6.1.5)$$

ko‘rinishda ham yozish mumkin. (6.1.5) dan ko‘rinadiki, $D$ $Oz$ o‘qqa nisbatan uch o‘lchovli to‘g‘ri soha bo‘lsa, $x$ va $y$ o‘zgaruvchilar tayinlangan deb faraz qilib,

$$\int_{\psi_1(x;y)}^{\psi_2(x;y)} f(x;y;z)dz = \Phi(x;y)$$

da $z$ bo‘yicha integrallash amalini bajarib,

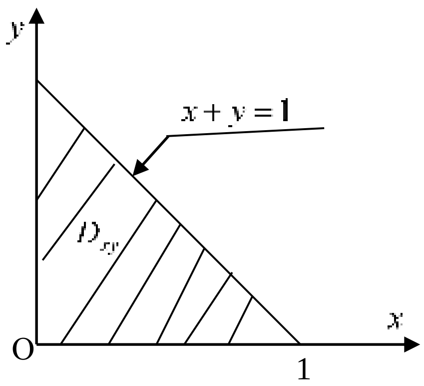
$$\iint\limits_{D_{xy}} \Phi(x;y)dxdy$$

ikkilangan integralga kelamiz.

Bu o‘rinda (6.1.3) formulani qo‘llash mumkin bo‘lgan holda, uning o‘ng tomonidagi *takroriy uch karrali integralni hisoblash tartibi (integrallash ketma-ketligi)* $z,y,x$ *bo‘yicha,* (6.1.4) formulani qo‘llash mumkin bo‘lgan holda esa $z,x,y$ bo‘yicha bo‘ladi. boshqa tartibda integrallash formulalarini ham yozish mumkin (mustaqil bajaring).

**1-misol.** $\iiint\limits_{D} xyzdxdydz$ ni hisoblang, bu yerda $D$ $x=0$, $y=0$, $z=0$, $x+y+z=1$ tekisliklar bilan chegaralangan piramidadan iborat sohadir.

**Yechish.** Bu misoldagi integrallash sohasi $D$ koordinata o‘qlarining uchchalasiga nisbatan ham to‘g‘ri sohadir. Agar $D$ ning $Oxy$ koordinata tekisligidagi proyeksiyasi bo‘lgan $D_{xy}$ ikki o‘lchovli sohani olsak, $u$ $x=0$, $y=0$, $x+y=1$ to‘g‘ri chiziqlar bilan chegaralangandir (6.1.3-rasmga qarang).




6.1.3-rasm.

Bu holda (6.1.5) formulani, $D$ sohada $z$ o‘zgaruvchi uchun $0 \le z \le 1-x-y$ bo‘lishini e’tiborga olib, qo‘llasak,

$$\iiint\limits_{D} xyz\,dxdydz = \iint\limits_{D_{xy}} xy\,dxdy \int\limits_{0}^{1-x-y} z\,dz$$

ni olamiz. Buni (6.1.3) ko‘rinishga keltirish uchun $D_{xy}$ ni $0 \le x \le 1$, $0 \le y \le 1-x$ sistema vositasida aniqlasak, $z,y,x$ tartibda integrallashga kelamiz:

$$\iiint\limits_{D} xyz\,dxdydz = \int\limits_{0}^{1} x\,dx \int\limits_{0}^{1-x} y\,dy \int\limits_{0}^{1-x-y} z\,dz =$$

$$= \int\limits_{0}^{1} x\,dx \int\limits_{0}^{1-x} \frac{y(1-x-y)^2}{2}dy = \frac{1}{24}\int\limits_{0}^{1}(1-x)^4 dx = \frac{1}{120}$$

## 6.2. Uchlangan integralda o‘zgaruvchilarni almashtirish

Aytaylik,

$$\begin{cases} x=\varphi(u;v;w), \\ y=\psi(u;v;w), \\ z=\chi(u;v;w) \end{cases} \tag{6.2.1}$$

sistema $x;y;z$ Dekart koordinatalari vositasida berilgan uch o‘lchovli chekli diametrli va o‘lchanuvchi bo‘lgan $D$ sohani egri chiziqli $u,v,w$ koordinatalar vositasida aniqlanuvchi $D'$ sohaga o‘zaro bir qiymatli ravishda akslantirsin (6.2.1-rasm). Buning uchun (6.2.1) almashtirish Yakobiani

$$J(u,v,w)=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

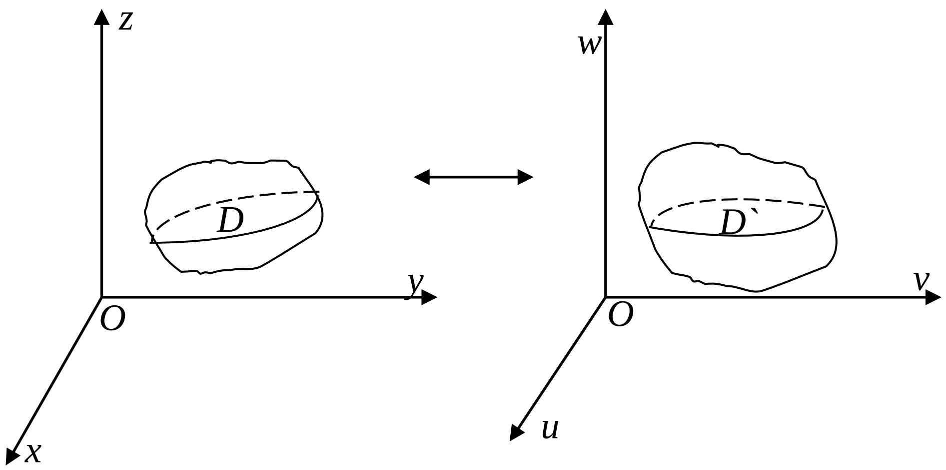
uzluksiz va noldan farqli bo‘lishini talab qilish yetarli bo‘lib, bu holda (6.2.1) almashtirish $D$ sohaning har bir elementar bo‘lagini $D'$ sohaning biror elementar bo‘lagiga akslantirishini va ularning mos ravishda o‘lchovlari bo‘lgan $\Delta\mu$ va $\Delta\mu'$ lar uchun

$$\lim_{\Delta\mu'\to 0}\frac{\Delta\mu}{\Delta\mu'}=|J(u;v;w)|$$

munosabat bajarilishini aytamiz (isbotsiz). Bu holda

$$\iiint_D f(x;y;z)dxdydz=\iiint_{D'} \varphi(u;v;w);\psi(u;v;w);\chi(u;v;w)|J(u;v;w)|dudvdw \tag{6.2.2}$$

tenglik o‘rinli ekanligi isbotlangandir. (6.2.2) *uchlangan integralda o‘zgaruvchilarni almashtirish formulasi* deb yuritiladi. Bu yerda (6.2.2) formulaning amaliy jihatdan ko‘p qo‘llaniladigan ikkita holini qarash bilan cheklanamiz.




6.2.1-rasm.

### 6.2.1. Uchlangan integralda silindrik koordinatalarga o‘tish

Agar $Oxyz$ Dekart koordinata fazosida $M(x;y;z)$ nuqtani olsak, uning $Oxy$ koordinata tekisligidagi proyeksiyasi $M_0(x;y;0)$ nuqtadan iborat bo‘lib, uning Shu tekislikdagi qutb koordinatalarini $\rho$ va $\theta$ desak (6.2.2-rasmga qarang), u holda $\rho,\theta,z$ larni $M$ nuqtaning *silindrik koordinatalari* deyiladi va $M(\rho;\theta;z)$ kabi yoziladi. 6.2.2-rasmdan ko‘rinadiki, $M$ nuqtaning Dekart va silindrik koordinatalari orasida

$$\begin{cases} x = \rho\cos\theta, \\ y = \rho\sin\theta, \\ z = z \end{cases} \qquad (6.2.3)$$

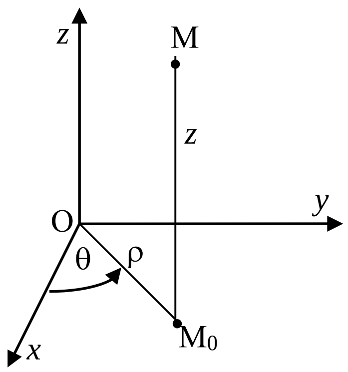
bog‘lanish bo‘lib, bu sistemani *Dekart koordinatalaridan silindrik koordinatalarga o‘tish almashtirishi* deb yuritiladi. Bu almashtirish uchlangan integralda $D$ sohani biror $D'$ sohaga akslantiradi deb faraz qilgan holda uning Yakobianini hisoblaylik

$$J(\rho;\theta;z) = \begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho .$$

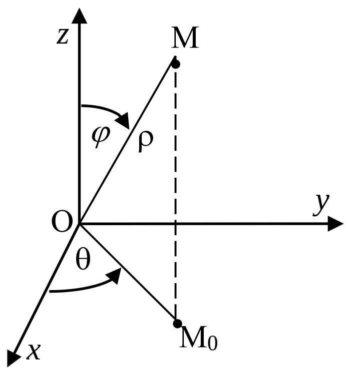
Bundan ko‘rinadiki, (6.2.3) almashtirish Yakobiani uzluksiz va $\rho \neq 0$ bo‘lganda noldan farqlidir va (6.2.3) natijasida (6.2.2) formula

$$\iiint_D f(x;y;z)dxdydz = \iiint_{D'} f(\rho\cos\theta;\rho\sin\theta;z)\rho d\rho d\theta dz \qquad (6.2.4)$$

ko‘rinishni oladi. (6.2.4) *uchlangan integralda silindrik koordinatalarga o‘tish formulasi* deyiladi.



6.2.2-rasm.



6.2.3-rasm.

### 6.2.2. Uchlangan integralda sferik koordinatalarga o‘tish

Faraz qilaylik, $Oxyz$ Dekart koordinata fazosida $M(x;y;z)$ nuqta berilgan bo‘lsin. 6.2.3-rasmdan ko‘rinadiki, bu nuqtaning $R^3$ fazodagi o‘rnini Dekart koordinatalaridan farq qiluvchi $|\overrightarrow{OM}| = \rho$, $(\overrightarrow{OM_0} \hat{;} Ox) = \theta$ va $(\overrightarrow{OM} \hat{;} Oz) = \varphi$ qiymatlar (sonlar) vositasida ham aniqlash mumkin, bu yerda $M_0M$ nuqtaning $Oxy$

tekislikdagi proyeksiyasidir. Bu holda $\rho;\theta;\varphi$ larni $M$ nuqtaning *sferik koordinatalari* deb ataladi va ular Dekart koordinatalari bilan quyidagicha bog‘lanishga ega ekanligiga 6.2.3-rasm asosida ishonch hosil qilish oson:

$$\begin{cases} x = \rho\cos\theta\sin\varphi, \\ y = \rho\sin\theta\sin\varphi, \\ z = \rho\cos\varphi \end{cases} \tag{6.2.5}$$

(6.2.5) sistemani *Dekart koordinatalaridan sferik koordinatalarga o‘tish almashtirishi* deb yuritiladi. Uning Yakobianini hisoblaylik:

$$J(\rho;\theta;\varphi) = \begin{vmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} = -\rho^2\sin\varphi$$

(6.2.5) va yuqorida olingan Yakobian qiymatini, (6.2.2) formulaga, $u=\rho,\ v=\theta,\ w=\varphi$ deb faraz qilib qo‘ysak,

$$\iiint\limits_D f(x;y;z)dxdydz = \iiint\limits_{D'} f(\rho\cos\theta\sin\varphi;\rho\sin\theta\sin\varphi;\rho\cos\varphi)\rho^2\sin\varphi d\rho d\theta d\varphi \tag{6.2.6}$$

ni olamiz. *(6.2.6) uchlangan integralda sferik koordinatalarga o‘tish formulasi* bo‘lib, undagi $D'$ (6.2.5) almashtirish natijasidagi $D$ sohaning aksidir.

## 6.3. Uchlangan integralning ba’zi bir tatbiqlari

Mazkur bandda uchlangan integral yordamida hal qilinadigan mexanikaning bir qator masalalarini keltiramiz.

### 6.3.1. Jismning massasini hisoblash

Aytaylik, biror uch o‘lchovli, chekli diametrli va o‘lchanuvchi bo‘lgan $D$ soha vositasida, har bir $(x;y;z)$ nuqtasidagi zichligi uzluksiz va manfiy bo‘lmagan $\rho=\rho(x;y;z)$ funksiya orqali aniqlangan jism berilgan bo‘lsin. Shu jism massasini hisoblash masalasini qo‘yaylik. Bu masalani hal qilish maqsadida $D$ sohani $i$ - bo‘lagining o‘lchovi $\Delta v_i$ dan iborat bo‘lgan $(i=\overline{1,m})$ elementar tashkil etuvchilarga ajratamiz va Shu $i$ - bo‘lakga tegishli bo‘lgan ixtiyoriy bitta $M_i(x_i;y_i;z_i)$ nuqtani olib, u nuqtadagi jism zichligini hisoblab, bo‘lakchani zichligi Shu qiymatdan iborat bo‘lgan bir jinsli jism deb faraz qilib, $i$ - bo‘lak massasining taqribiy qiymati sifatida $\rho(x_i;y_i;z_i)\Delta v_i$ ni qabul qilsak, butun jism massasining taqribiy qiymati uchun

$$m \approx \sum_{i=1}^{m} \rho(x_i;y_i;z_i)\Delta v_i$$

ni olamiz. Bu taqribiy formulaning o‘ng tomonidagi ifoda $\rho(x;y;z)$ funksiyaning $D$ soha bo‘yicha integral yig‘indisi ekanligidan, unda $\lambda \to 0$ dagi ( $\lambda = \max_i d_i$ , $d_i$ – $i$ - bo‘lak diametri) limitga o‘tib,

$$m = \iiint_D \rho(x;y;z)dxdydz \qquad (6.3.1)$$

ni olamiz. *(6.3.1) ni jism massasini hisoblash formulasi* deb qabul qilamiz.

### 6.3.2. Jismning inersiya momentlari

Aytaylik, $R^3$ fazoda chekli diametrli va o‘lchanuvchi $D$ soha vositasida biror jism berilgan bo‘lib, uning ixtiyoriy $(x;y;z)$ nuqtasidagi zichligi uzluksiz va manfiy bo‘lmagan $\rho = \rho(x;y;z)$ funksiya orqali aniqlangan bo‘lsin. Mexanikadan ma’lumki, $m$ massali $M(x;y;z)$ moddiy nuqtaning $Ox, Oy, Oz$ koordinata o‘qlariga va koordinata boshiga nisbatan inersiya momentlari, mos ravishda,

$$I_{xx} = (y^2 + z^2)m,$$
$$I_{yy} = (x^2 + z^2)m,$$
$$I_{zz} = (x^2 + y^2)m,$$
$$I_0 = (x^2 + y^2 + z^2)m,$$

formulalar bilan aniqlanadi. Buni e’tiborga olgan holda oldingi bandda qilingan ish jarayonini bu yerda ham bajarib, $D$ jismni $m_i = \rho(x_i;y_i;z_i)\Delta v_i$ massali $M_i(x_i;y_i;z_i)$ $(i = \overline{1,m})$ moddiy nuqtalar sistemasidan iborat deb faraz qilib, $\lambda \to 0$ dagi limitga o‘tish natijasida aytilgan jismning $Ox, Oy, Oz$ koordinata o‘qlariga va koordinata boshiga nisbatan inersiya momentlari uchun, mos ravishda

$$I_{xx} = \iiint_D (y^2 + z^2)\rho(x;y;z)dxdydz,$$
$$I_{yy} = \iiint_D (x^2 + z^2)\rho(x;y;z)dxdydz,$$
$$I_{zz} = \iiint_D (x^2 + y^2)\rho(x;y;z)dxdydz,$$
$$I_0 = \iiint_D (x^2 + y^2 + z^2)\rho(x;y;z)dxdydz$$

formulalarni keltirib chiqarish oson.

### 6.3.3. Jism og‘irlik markazining koordinatalari

Faraz qilaylik, oldingi bandda aytilganidek, biror $D$ jism massasi $\rho = \rho(x;y;z)$ zichlik funksiyasi bilan berilgan bo‘lsin. U holda bu jismni elementar tashkil etuvchilarga ajratib, 5.6-bandda tekis shakl yuzining og‘irlik markazini topish uchun qilinganga o‘xshash mulohazalarni bu yerda ham bajarib, jism $C(x_c\,;y_c\,;z_c)$ og‘irlik markazining koordinatalarini hisoblash uchun

$$x_c = \frac{1}{m}\iiint\limits_D x\rho(x;y;z)dxdydz,$$

$$y_c = \frac{1}{m}\iiint\limits_D y\rho(x;y;z)dxdydz, \qquad (6.3.2)$$

$$z_c = \frac{1}{m}\iiint\limits_D z\rho(x;y;z)dxdydz,$$

formulalar o‘rinli bo‘lishiga ishonch hosil qilish oson, bu yerda $m$ jism massasi bo‘lib, uni (6.3.1) formula bilan hisoblanadi.

Bu o‘rinda

$$M_{Oxy} = \frac{1}{m}\iiint\limits_D z\rho(x;y;z)dxdydz,$$

$$M_{Oyz} = \frac{1}{m}\iiint\limits_D x\rho(x;y;z)dxdydz,$$

$$M_{Oxz} = \frac{1}{m}\iiint\limits_D y\rho(x;y;z)dxdydz$$

munosabatlar $D$ jismning, mos ravishda $Oxy, Oyz, Oxz$ koordinata tekisligiga nisbatan statik momentlarini (turg‘unlik lahzalarini) hisoblash formulalari ekanligini aytamiz.

Agar $D$ jism bir jinsli (ya’ni $\rho = const$) bo‘lsa, u holda $C(x_c\,;y_c\,;z_c)$ og‘irlik markazini topish uchun (6.3.2) dan

$$x_c = \frac{1}{v}\iiint\limits_D xdxdydz, \quad y_c = \frac{1}{v}\iiint\limits_D ydxdydz, \quad z_c = \frac{1}{v}\iiint\limits_D zdxdydz, \qquad (6.3.3)$$

formulalarga ega bo‘lamiz, bu yerda $\vartheta = \mu(D)$, ya’ni

$$\vartheta = \iiint\limits_D dxdydz$$

jism hajmidir.

**2-misol.** Markazi koordinata boshida, radiusi $R$ ga teng bo‘lgan bir jinsli sharning $Oxy$ koordinata tekisligidan yuqorida joylashgan qismning og‘irlik markazini toping.

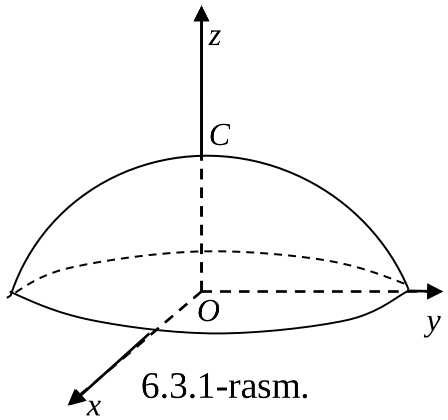
**Yechish.** Aytilgan yarim shar $z = \sqrt{R^2 - x^2 - y^2}$, $z = 0$ sirtlar bilan chegaralangandir (6.3.1-rasmga qarang). Masalani hal qilish uchun quyidagi uchlangan integrallarni hisoblaymiz:

1) $\vartheta = \iiint\limits_D dxdydz$ - sferik koordinatalarga o‘tamiz.

6.3.1-rasmdan uni 6.2.2-rasm bilan taqqoslash natijasida sferik koordinatalar uchun

$$0 \le \theta \le 2\pi, \quad 0 \le \varphi \le \frac{\pi}{2}, \quad 0 \le \rho \le R$$

bo‘lishini ko‘ramiz. Bu (6.2.6) formuladagi  ni aniqlovchi sistemadir.




6.3.1-rasm.

Demak, $\vartheta = \iiint\limits_D dxdydz = \int\limits_0^{2\pi} d\theta \int\limits_0^{\frac{\pi}{2}} \sin\varphi d\varphi \int\limits_0^R \rho d\rho = \frac{2\pi}{3} R^3 = \frac{2\pi R^3}{3}$.

2)

$$\iiint\limits_D xdxdydz = \iiint\limits_D \rho\cos\theta\sin\varphi\cdot\rho^2\sin\varphi d\rho d\theta d\varphi =$$

$$= \int\limits_0^{2\pi}\cos\theta d\theta \int\limits_0^{\frac{\pi}{2}} \sin^2\varphi d\varphi \int\limits_0^R \rho^3 d\rho = 0\cdot\frac{\pi}{4}\cdot\frac{R^4}{4} = 0;$$

3)

$$\iiint\limits_D ydxdydz = \iiint\limits_D \rho\sin\theta\sin\varphi\cdot\rho^2\sin\varphi d\rho d\theta d\varphi =$$

$$= \int\limits_0^{2\pi}\sin\theta d\theta \int\limits_0^{\frac{\pi}{2}} \sin^2\varphi d\varphi \int\limits_0^R \rho^3 d\rho = 0\cdot\frac{\pi}{4}\cdot\frac{R^4}{4} = 0;$$

4)

$$\iiint\limits_D zdxdydz = \iiint\limits_D \rho\cos\varphi\cdot\rho^2\sin\varphi d\rho d\theta d\varphi =$$

$$= \int\limits_0^{2\pi} d\theta \int\limits_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi d\varphi \int\limits_0^R \rho^3 d\rho = 2\pi\cdot\frac{1}{2}\cdot\frac{R^4}{4} = \frac{\pi}{4}R^4.$$

Olingan natijalarni (6.3.3) ga qo‘yib,

$$x_c = y_c = 0,\quad z_c = \frac{3}{8}R,$$

ya’ni og‘irlik markazi $C\left(0;0;\frac{3}{8}R\right)$ ekanligini topamiz (6.3.1-rasmga qarang).

## 6-bobga doir mashqlar

1. $\iiint\limits_D \frac{dxdydz}{(x+y+z+1)^3}$ ni hisoblang, bu yerda $D$ koordinata tekisliklari hamda $x+y+z=1$ tekislik bilan chegaralangan soha.

Javob. $\frac{1}{2}\ln 2-\frac{5}{16}$.

2. $\int_0^a xdx\int_0^x ydy\int_0^y zdz$ takroriy uch karrali integralni hisoblang.

Javob. $\frac{a^6}{48}$.

3. $x^2+y^2+z^2=4$ sfera va $x^2+y^2=3z$ paraboloid sirti bilan chegaralangan jismning hajmini hisoblang.

Javob. $\frac{19}{6}\pi$.

4. $x=0,\ y=0,\ z=0,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ tekisliklar bilan chegaralangan birjinsli piramida og‘irlik markazining koordinatalarini va koordinata o‘qlariga nisbatan inersiya momentlarini hisoblang.

Javob. $x_c=\frac{a}{4},\ y_c=\frac{b}{4},\ z_c=\frac{c}{4};\quad I_{xx}=\frac{a^3bc}{60},\ I_{yy}=\frac{ab^3c}{60},\ I_{zz}=\frac{abc^3}{60}$.

5. Balandligi h ga, asosining radiusi esa r ga teng bo‘lgan birjinsli to‘g‘ri doiraviy konusning o‘z o‘qiga nisbatan inersiya momentini hisoblang.

Javob. $\frac{1}{10}\pi hr^4$.

6. $\left(x^2+y^2+z^2\right)^2=a^3x$ sirt bilan chegaralangan jismning hajmini hisoblang.

Javob. $\frac{1}{3}\pi a^3$.

7. Birjinsli to‘g‘ri doiraviy konusning balandligi h ga, asosining radiusi r ga teng bo‘lsa, asosining diametriga nisbatan inersiya momentini hisoblang.

Javob. $\frac{\pi hr^4}{60}\left(2h^2+3r^2\right)$.

## 6-bob bo‘yicha o‘z bilimingizni sinab ko‘ring

1. Uchlangan integral haqida tushuncha bering.
2. Uchlangan integralni hisoblashning asosiy usulini tuShuntiring.
3. Takroriy uch karrali integral nima?
4. Takroriy uch karrali integralda integrallash tartibi haqida tushuncha bering.
5. Uchlangan integralda o‘zgaruvchilarni almashtirish formulasini yozing.
6. Uchlangan integralda silindrik koordinatalarga o‘tishni tuShuntiring.
7. Uchlangan integralda sferik koordinatalarga o‘tishni tuShuntiring.
8. Jism hajmini uchlangan integral yordamida hisoblashni tuShuntiring.
9. Jism og‘irlik markazini topish formulalarini yozing.
10. Jismning inersiya momentlarini hisoblash formulalarini yozing.

# 7-bob. Egri chiziqli integrallar.

Mazkur bobda oliy matematikaning ham nazariy ham amaliy jihatdan muhim ahamiyatga ega tushunchalaridan biri bo‘lgan egri chiziqli integralni bayon qilinadi.

## 7.1. Egri chiziqli integralning ta’rifi

Aytaylik, $R^3$ fazoga tegishli bo‘lgan o‘lchanuvchi AB egri chiziq berilgan va uning har bir $(x;y;z)$ nuqtasida

$$\vec{F} = P(x;y;z)\vec{i} + Q(x;y;z)\vec{j} + R(x;y;z)\vec{k} \qquad (7.1.1)$$

vektor funktsiya aniqlangan bo‘lsin. Berilgan AB egri chiziqni A nuqtasidan B nuqtasigacha qaragan yo‘nalish bo‘yicha olingan $A, A_1, ..., A_{m-1}, B$ nuqtalari vositasida $m$ ta bo‘laklarga ajrataylik va ulardan *i*-siga mos vektorni $\overrightarrow{A_{i-1}A_i} = \overrightarrow{\Delta S_i}$ kabi belgilasak (7.1.1-rasmga qarang), $\overrightarrow{\Delta S_i} = \{\Delta x_i; \Delta y_i; \Delta z_i\}$ bo‘ladi, bu yerda $A_i(x_i; y_i; z_i)$ $\left(i = \overline{0, m}\right)$ bo‘lib, $\Delta x_i = x_i - x_{i-1}$, $\Delta y_i = y_i - y_{i-1}$, $\Delta z_i = z_i - z_{i-1}$. Endi egri chiziqning *i*-bo‘lagiga tegishli bo‘lgan ixtiyoriy bitta $(\xi_i; \eta_i; \zeta_i)$ nuqtani olib, (7.1.1) vektor funktsiyaning undagi qiymati bo‘lgan $\vec{F_i}$ ni hisoblab, $\left(i = \overline{1, m},\ A_0 = A,\ A_m = B\right)$ , so‘ngra

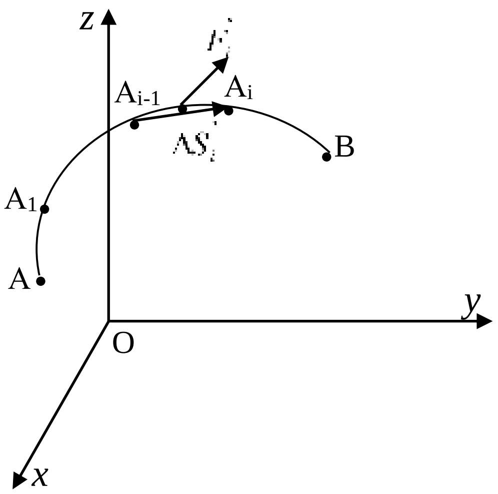
$$\sum_{i=1}^{m} \vec{F_i} \cdot \overrightarrow{\Delta S_i} = \sum_{i=1}^{m} \left[P(\xi_i; \eta_i; \zeta_i)\Delta x_i + Q(\xi_i; \eta_i; \zeta_i)\Delta y_i + R(\xi_i; \eta_i; \zeta_i)\Delta z_i\right] \qquad (7.1.2)$$

yig‘indini tuzamiz va uni (7.1.1) vektor funktsiyaning AB egri chiziq bo‘yicha *integral yig‘indisi* deb ataymiz.

Agar $\lambda = \max\limits_i |\Delta S_i|$ deb belgilab, $\lambda \to 0$ holni qarasak, AB egri chiziqni yuqoridagicha qilib $m$ ta bo‘lakka ajratishni *elementar tashkil etuvchilarga ajratish* deb ataymiz. Bu holda (7.1.2) yig‘indi chekli limitga ega va u egri chiziqni elementar tashkil etuvchilarga ajratish usuliga hamda har bir bo‘lakdan olinadigan nuqtaning o‘rniga bog‘liq bo‘lmasa, bu limitni $P(x;y;z)$, $Q(x;y;z)$, $R(x;y;z)$ fnutsiyalarning *AB egri chiziq bo‘yicha ikkinchi jins (tur) egri chiziqli integrali* deb ataymiz va

$$\int_{AB} P(x;y;z)dx + Q(x;y;z)dy + R(x;y;z)dz$$

kabi belgilaymiz. Bunda AB egri chiziqni *integrallash yo'li* deb ham yuritiladi.




7.1.1-расм.

(7.1.2) yig'indi uchta qismdan iborat ekanligini ko'rish oson. Agar AB egri chiziq o'zining parametrik $x=\varphi(t),\ \ y=\psi(t),\ \ z=\chi(t),\ \ t\in[\alpha;\beta]$ tenglamalari bilan berilgan bo'lib, $\varphi(t),\ \psi(t),\ \chi(t)$ lar $[\alpha;\beta]$ oraliqda uzluksiz differensiallanuvchi hamda parametrning $t=\alpha$ qiymatiga egri chiziqning boshi bo'lgan A nuqtasi, $t=\beta$ ga esa oxiri bo'lgan V nuqtasi mos keladi deb faraz qilsak, yuqorida aytilgan yig'indining uchchala qismlarining har biri $\lambda\to 0$ bo'lgan holda, mos ravishda

$$\int_{\alpha}^{\beta} P(\varphi(t);\psi(t);\chi(t))\varphi'(t)dt,$$

$$\int_{\alpha}^{\beta} Q(\varphi(t);\psi(t);\chi(t))\psi'(t)dt,$$

$$\int_{\alpha}^{\beta} R(\varphi(t);\psi(t);\chi(t))\chi'(t)dt$$

aniq integrallarga olib kelishini ko'rish qiyin emas. Demak, bu hol uchun

$$\int_{AB} P(x;y;z)dx+Q(x;y;z)dy+R(x;y;z)dz=$$
$$=\int_{\alpha}^{\beta}\left[P(\varphi(t);\psi(t);\chi(t))\varphi'(t)+Q(\varphi(t);\psi(t);\chi(t))\psi'(t)+R(\varphi(t);\psi(t);\chi(t))\chi'(t)\right]dt \qquad (7.1.3)$$

ni olamiz. *(7.1.3) ikkinchi jins egri chiziqli integralni parametrik tenglamalar vositasida hisoblash formulasi*dir.

Agar AB $Oxy$ Dekart koordinata tekisligidagi tekis chiziqdan iborat va uning parametrik tenglamalari $x=\varphi(t),\ \ y=\psi(t),\ \ t\in[\alpha;\beta]\ (t=\alpha\to A,\ \ t=\beta\to B)$ bo'lib, $\varphi(t)$ va $\psi(t)$ $[\alpha;\beta]$ oraliqda uzluksiz differensiallanuvchi deb faraz qilsak, (7.1.3) ni

$$\int_{AB} P(x;y)dx+Q(x;y)dy=\int_{\alpha}^{\beta}\left[P(\varphi(t);\psi(t))\varphi'(t)+Q(\varphi(t);\psi(t))\psi'(t)\right]dt \qquad (7.1.4)$$

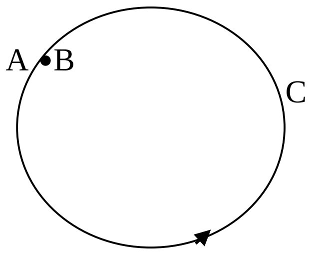
ko'rinishda yozaolamiz. (7.1.4) ning chap tomonidagi ifodani *tekis egri chiziqli integral (ikkinchi tur)* deb ataladi va *(7.1.4) tekis egri chiziqli integralni parametrik tenglamalar vositasida hisoblash formulasi*dir.

Bu o‘rinda ikkinchi jins egri chiziqli integralni $d\vec{S}=\{dx;dy;dz\}$ vektorni kiritib va (7.1.1) ni e’tiborga olib,

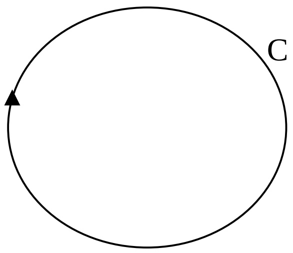
$$\int_{AB}\vec{F}\cdot d\vec{S} \qquad (7.1.5)$$

ko‘rinishda yozish mumkinligini va nazariy jihatdan undan foydalanish qulay ekanligini aytamiz.

Agar yo‘nalishli AB egri chiziqda uning boshi hisoblangan A nuqtasi bilan uning oxiri hisoblangan B nuqtasi ustma-ust tushgan (yaoni A=B) holni qarasak (7.1.2-rasm), bu holda yo‘nalishli yopiq AB egri chiziqqa ega bo‘lamiz va uni *kontur* (*yo‘nalishli kontur*) deb ataymiz. CHuni ham aytamizki, tekis kontur uchun, maxsus eslatilmagan holda uning yo‘nalishini musbat deb faraz qilamiz va uni bitta harf bilan belgilashni qabul qilamiz, masalan, C kabi (7.1.2-rasm). manfiy yo‘nalishli kontur uchun $C^-$ kabi belgilashni ishlatamiz (7.1.3-rasm).



7.1.2-rasm



7.1.3-rasm

Agar (7.1.5) da AB egri chiziq biror S konturdan iborat bo‘lsa, uni

$$\oint_{c}\vec{F}\cdot d\vec{S}$$

kabi yozishni qabul qilamiz.

**Eslatma.** Bu o‘rinda birinchi (tur) egri chiziqli integral tushunchasi ham mABjudligini va u quyidagicha kiritilishini aytamiz. Faraz qilaylik, o‘lchanuvchi AB egri chiziq yoyida biror uzluksiz bo‘lgan $f(x;y;z)$ funktsiya aniqlangan bo‘lsin. AB egri chiziqni yuqoridagi usul bilan elementar tashkil etuvchilarga ajratib, har bir i-bo‘lakdan olingan $(\xi_i;\eta_i;\zeta_i)$ nuqta hamda $\Delta S_i=\left|\Delta\vec{S_i}\right|$ lar vositasida

$$\sum_{i=1}^{m} f(\xi_i;\eta_i;\zeta_i)\Delta S_i$$

yig‘indini tuzamiz. Agar $\lambda\to 0$ da bu yig‘indi uchun AB ni elementar tashkil etuvchilarga ajratish usuliga va har bir i-bo‘lakdan olingan $(\xi_i;\eta_i;\zeta_i)$ nuqtaning o‘rniga bog‘liq bo‘lmagan chekli limit mavjud bo‘lsa, uni $f(x;y;z)$ *funktsiyaning AB egri chiziq bo‘yicha birinchi jins (tur) egri chiziqli integrali* deymiz va $\int_{AB} f(x;y;z)dS$ kabi belgilaymiz.

Agar AB chiziq o‘zining $x=\varphi(t),\ y=\psi(t),\ z=\chi(t),\ (\alpha\le t\le\beta)$ parametrik tenglamalari bilan berilgan bo‘lib, $\varphi(t),\ \psi(t),\ \chi(t)$ lar uzluksiz differensiallanuvchi bo‘lsa, birinchi tur egri chiziqli integralni hisoblash uchun

$$\int_{AB} f(x;y;z)dS=\int_{\alpha}^{\beta} f(\varphi(t);\psi(t);\chi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)+\chi'^2(t)}dt$$

formulani olamiz.

## 7.2. Egri chiziqli integralning xossalari

Yuqorida bayon qilingan ikkinchi jins egri chiziqli integralning taorifidan uning quyidagi xossalari kelib chiqishni ko‘rish oson:

**1⁰.** $\int\limits_{AB} \vec{F}\cdot d\vec{S} = -\int\limits_{BA} \vec{F}\cdot d\vec{S}$; $\oint\limits_{C} \vec{F}\cdot d\vec{S} = -\int\limits_{C^-} \vec{F}\cdot d\vec{S}$;

**2⁰.** $\int\limits_{AB} \vec{O}\cdot d\vec{S} = O$; $\oint\limits_{C} \vec{O}\cdot d\vec{S} = O$;

**3⁰.** AB egri chiziq o‘zining yo‘nalishi bo‘yicha chekli sondagi $A_{i-1}A_i$ $\left(i=\overline{1,m};\ A_0 = A,\ A_m = B\right)$ tashkil etuvchi bo‘laklarga ajratilgan bo‘lib, ularning har biri bo‘yicha (7.1.1) vektor funktsiyaning ikkinchi jins egri chiziqli integrali mABjud bo‘lsa, $\int\limits_{AB} \vec{F}\cdot d\vec{S} = \sum\limits_{i=1}^{m} \int\limits_{A_{i-1}A_i} \vec{F} d\vec{S}$ o‘rinli bo‘ladi;

**4⁰.** $\int\limits_{AB} P(x;y;z)dx + Q(x;y;z)dy + R(x;y;z)dz = \int\limits_{AB} P(x;y;z)dx + \int\limits_{AB} Q(x;y;z)dy + \int\limits_{AB} R(x;y;z)dz$ o‘rinlidir.

Birinchi jins egri chiziqli integral uchun $2^0$- va $3^0$-xossalarga o‘xshash xossalar o‘rinlidir (mustaqil Shug‘ullaning).

## 7.3. O‘zgaruvchi kuchning egri chiziqli yo‘lda bajargan ishni va yoymassasini hisoblash

Avvalo, ikkinchi jins egri chiziqli integral vositasida

$$\vec{F} = \{P(x;y;z); Q(x;y;z); R(x;y;z)\}$$

kuchning biror AB egri chiziqli yo‘l bo‘yicha harakatlanishi natijasida bajargan ishining miqdorini hisoblash masalasini qaraylik. Bu ish miqdorining taqribiy qiymati sifatida AB egri chiziqni $m$ ta elementar tashkil etuvchi bo‘laklarga ajratib, ulardan i-siga tegishli ixtiyoriy olingan bitta nuqtadagi vektor funktsiyaning $\vec{F_i}$ qiymati bo‘lgan va Shu bo‘lakka mos $\Delta\vec{S_i}$ vektorlar vositasida (7.1.1-rasmga qarang) tuzilgan (7.1.2) yig‘indida $\lambda \to 0$ ($\lambda = \max\limits_i \left|\Delta\vec{S_i}\right|$) dagi limitga o‘tib, hisoblashi talab qilingan ish miqdori uchun

$$W = \int\limits_{AB} \vec{F}\cdot d\vec{S}$$

formulani qabul qilamiz. Demak, qo‘yilgan masalaning yechimi

$$W = \int\limits_{AB} P(x;y;z)dx + Q(x;y;z)dy + R(x;y;z)dz$$

formula vositasida ikkinchi jins egri chiziqli integral orqali ifodalanishini ko‘ramiz.

Xuddi Shunga o‘xshash, AB egri chiziq yoyining har bir $(x;y;z)$ nuqtasida $\rho = \rho(x;y;z)$ zichlik bilan tarqalgan $m$ massani hisoblash masalasini qo‘ysak, yuqoridagidek ishlarni bu holda ham bajarib, aytilgan massani hisoblash uchun quyidagi birinchi jins egri chiziqli integralga kelamiz:

$$m=\int_{AB}\rho(x;y;z)dS\ .$$

## 7.4. Tekis egri chiziqli integralni hisoblash usullari

Bu bandda tekis egri chiziq biror funktsiyaning grafigidan iborat bo‘lgan holda y bo‘yicha egri chiziqli integralni hisoblash usullarini keltiramiz.

### 7.4.1. Tekis egri chiziqli integralni biror koordinata o‘zgaruvchisi bo‘yicha hisoblash

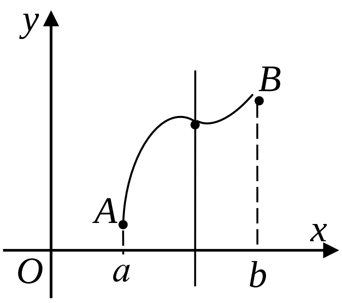
Agar AB tekis egri chiziq uzluksiz va $Oy$ o‘qiga parallel bo‘lgan to‘g‘ri chiziq bilan bittadan ortiq umumiy nuqtaga ega bo‘lmasa, uni biror $[a;b]$ oraliqda aniqlagan, uzluksiz $y=f(x)$ funktsiyaning grafigidan iborat bo‘lishi maolumdir (7.4.1-rasmga qarang). Bu yerda $f(x)$ funktsiyani uzluksiz differensiallanuvchi deb faraz qilamiz. $A(a;f(a))$, $B(b;f(b))$ bo‘lgan holda AB yo‘nalishli egri chiziqning parametrik tenglamalarini

$$\begin{cases} x=x, \quad a\le x\le b, \\ y=f(x) \end{cases} \tag{7.4.1}$$

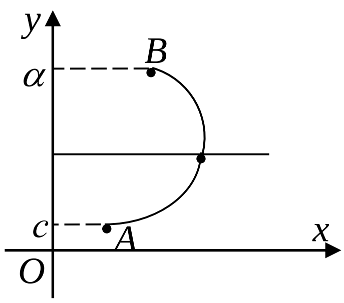
ko‘rinishda yozish mumkin bo‘lib, (7.1.4) formulada $t=x,\ \varphi(t)=x,\ \psi(t)=f(t),\ \alpha=a,\ \beta=b$ deb faraz qilsak,

$$\int_{AB}P(x;y)dx+Q(x;y)dy=\int_a^b[P(x,f(x))+Q(x;f(x))f'(x)]dx$$

ni olamiz. Buni *tekis egri chiziqli integralni x bo‘yicha integrallash formulasi* deb yuritiladi. Undan ko‘rinadiki, bu holda egri chiziqli integralni hisoblash x o‘zgaruvchi bo‘yicha aniq integralga keltiriladi.




7.4.1-расм.




7.4.2-расм.

Xuddi Shunga o‘xshash, agar AB tekis egri chiziq $Ox$ o‘qiga parallel to‘g‘ri chiziq bilan bittadan ortiq umumiy nuqtaga ega bo‘lmasa (7.4.2-rasmga qarang), u holda bu egri chiziqning parametrik tenglamalarini

$$\begin{cases} x = \varphi(y), \quad c \le y \le d, \\ y = f(x) \end{cases} \qquad (7.4.2)$$

ko‘rinishda ifodalash mumkin bo‘lib, $A(\varphi(c);c)$, $B(\varphi(d);d)$ va $\varphi(y)$ funktsiya $[c;d]$ oraliqda uzluksiz differensiallanuvchi degan faraz asosida (7.1.4) formuladan

$$\int_{AB} P(x;y)dx + Q(x;y)dy = \int_c^d [P(\varphi(y);y)\varphi'(y) + Q(\varphi(y);y)]dy$$

ni olamiz. Bu *tekis egri chiziqli integralni y bo‘yicha integrallash formulasi*dir.

Agar AB tekis egri chiziq monoton va uzluksiz bo‘lsa, uning parametrik tenglamalarini (7.4.1) ko‘rinishda ham (7.4.2) ko‘rinishda ham ifodalash mumkin bo‘lib, bularda $f(x)$ va $\varphi(y)$ lar o‘zaro teskari funktsiyalardan iborat bo‘ladi. bu holda

$$\int_{AB} P(x;y)dx + Q(x;y)dy = \int_a^b P(x;f(x))dx + \int_c^d Q(\varphi(y);y)dy$$

formulani olish qiyin emas. Bu formula AB tekis egri chiziq monoton bo‘lganda yaroqli bo‘lib, uni *tekis egri chiziqli integralni ham x ham y bo‘yicha integrallash formulasi* deb yuritiladi.

Bu o‘rinda yuqoridagi biror koordinata o‘zgaruvchisi bo‘yicha hisoblash formulalari ikkinchi jins egri chiziqli integral uchun berliganligini va bunga o‘chshash formulalar birinchi jins egri chiziqli integral uchun ham o‘rinli bo‘lishini aytamiz (ularni chiqarish bilan mustaqil Shug‘ullaning).

## 7.5. Grin formulasi

Bu bandda biror tekis chekli diametrli va o‘lchanuvchi bo‘lgan D soha bo‘yicha ikkilangan integral bilan bu sohaning chegarasidan iborat bo‘lgan S kontur bo‘yicha egri chiziqli integral orasidagi bog‘lanishni aniqlovchi formulani (Grin formulasini) bayon qilamiz.

Aytaylik, D $Ox$ va $Oy$ koordinata o‘qlarining ikkalasiga nisbatan ham to‘g‘ri sohadan iborat bo‘lsin (7.5.1$_a$-rasmga qarang). U holda bu sohaning chegarasidan iborat bo‘lgan S kontur musbat yo‘nalishli deb faraz qilib, uni qandaydir

$$y = \varphi_1(x),\ a \le x \le b;\ \ y = \varphi_2(x),\ b \ge x \ge a;\ \ (\varphi_1(x) \le \varphi_2(x))$$

funktsiyalar grafiklarining yoki

$$x = \psi_1(y),\ m \ge y \ge k;\ \ x = \psi_2(y),\ k \le y \le m;\ \ (\psi_1(y) \le \psi_2(y))$$

funktsiyalar grafiklarining birlashmasi deb qarash mumkin bo‘ladi.

Endi $P(x;y)$ funktsiya D sohada uzluksiz va $y$ argument bo‘yicha uzluksiz xususiy hosilaga ega deb faraz qilib, quyidagilarni bajaramiz:

$$\oint_c P(x;y)dx = \int_{AKB} P(x;y)dx + \int_{BMA} P(x;y)dx =$$
$$= \int_a^b P(x;\varphi_1(x))dx = \int_b^a P(x;\varphi_2(x))dx =$$
$$= \int_a^b [P(x;\varphi_2(x)) - P(x;\varphi_1(x))]dx =$$
$$= -\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y}dy = -\iint_D \frac{\partial P}{\partial y}dxdy.$$

Bu yerda $AKB$ yoy $y=\varphi_1(x)$ $(a \le x \le b)$ funktsiyaning $BMA$ esa $y=\varphi_2(x)$ $(b \ge x \ge a)$ funktsiyaning grafigidan iborat deb faraz qilindi.

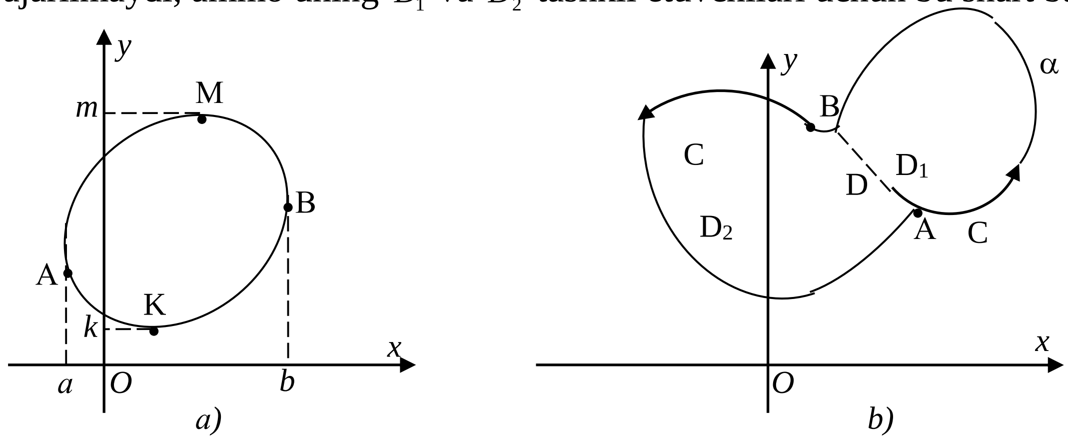
Xuddi Shunga o‘xshash, $Q(x;y)$ funktsiyani $D$ sohada uzluksiz va $x$ argument bo‘yicha uzluksiz xususiy hosilaga ega deb faraz qilib,

$$\oint_c Q(x;y)dy = \iint_D \frac{\partial Q}{\partial x}dxdy$$

munosabatni olish mumkin. Shunday qilib, yuqorida olingan natijalarga asosan

$$\oint_c P(x;y)dx + Q(x;y)dy = \iint_D \left[\frac{\partial Q(x;y)}{\partial x} - \frac{\partial P(x;y)}{\partial y}\right]dxdy \qquad (7.5.1)$$

ni olamiz. (7.5.1) formulani keltirib chiqarish davomida $D$ ni koordinata o‘qlarining ikkalasiga nisbatan ham to‘g‘ri soha bo‘lishi talab qilingan edi. Bu formula $D$ sohani chekli sondagi har ikkala koordinata o‘qlariga nisbatan ham to‘g‘ri bo‘lgan tashkil etuvchi $D_i$ $(i=\overline{1,m})$ sohalarga ajratish mumkin bo‘lgan holda ham to‘g‘ri bo‘lishini aytamiz. Masalan, 7.5.1b-rasmdagi $D$ soha uchun yuqorida aytilgan shart bajarilmaydi, ammo uning $D_1$ va $D_2$ tashkil etuvchilari uchun bu shart bajariladi.




7.5.1-расм.

Bu holda

$$\oint_{A\alpha BA} Pdx + Qdy = \iint_{D_1} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]dxdy, \qquad (7.5.2)$$

.

Endi $A\alpha BA$ va $B\beta AB$ konturlarni tashkil etuvchilarga ajratib,

$$\oint\limits_{A\alpha BA} Pdx+Qdy=\int\limits_{A\alpha B} Pdx+Qdy+\int\limits_{BA} Pdx+Qdy,$$

$$\oint\limits_{B\beta AB} Pdx+Qdy=\int\limits_{B\beta A} Pdx+Qdy+\int\limits_{AB} Pdx+Qdy=\int\limits_{B\beta A} Pdx+Qdy-\int\limits_{BA} Pdx+Qdy.$$

Bu tengliklarni hadlab qo‘shsak,

$$\oint\limits_{A\alpha BA} Pdx+Qdy+\oint\limits_{B\beta AB} Pdx+Qdy=\int\limits_{A\alpha B} Pdx+Qdy+\int\limits_{B\beta A} Pdx+Qdy=\oint\limits_{c} Pdx+Qdy.$$

Oxirgi tenglikni e’tiborga olib, (7.5.2) va (7.5.3) tengliklarni hadlab qo‘shib,

$$\oint\limits_{c} Pdx+Qdy=\iint\limits_{D_1}\left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]+\iint\limits_{D_2}\left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]=\iint\limits_{D}\left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]dxdy.$$

Bu hol uchun Grin formulasi to‘g‘ri bo‘lishini ko‘ramiz.

## 7.6. Tekis ikkinchi jins egri chiziqli integralning integrallash yo‘liga bog‘liq bo‘lmaslik sharti

Aytaylik,

$$\int\limits_{L} P(x;y)dx+Q(x;y)dy \qquad (7.6.1)$$

tekis ikkinchi jins egri chiziqli integral berilgan bo‘lib, unda $P(x;y)$ va $Q(x;y)$ funktsiyalar biror o‘lchanuvchi, yopiq $D$ bog‘liq sohada uzluksiz differensiallanuvchi hamda integrallash yo‘lidan iborat bo‘lgan L tekis egri chiziq $D$ sohaga joylashgan holda uzluksiz va bo‘lakli silliq bo‘lsin deb faraz qilaylik. Quyida tekis ikkinchi jins egri chiziqli integral uchun muhim nazariy asoslardan biri hisoblangan uning integrallash yo‘liga bog‘liq bo‘lmaslik shartini aniqlovchi teoremalarni keltiramiz.

**7.6.1-teorema.** Agar (7.6.1) tekis ikkinchi jins egri chiziqli integral uchun yuqorida qilingan faraz o‘rinli bo‘lsa, u integrallash yo‘liga bog‘liq bo‘lmasligi uchun D sohada yotuvchi ixtiyoriy kontur bo‘yicha ikkinchi jins chiziqli intgeralning nolga teng bo‘lishi zarur va yetarlidir.

**Isbot. Yetarliligi.** Bu yerda ikkinchi jins egri chiziqli integral integrallash yo‘liga bog‘liq emas deyilganda, uning qiymati egri chiziqning boshlang‘ich va oxirgi nuqtalarigagina bog‘liq bo‘lib, ularni tutashtiruvchi egri chiziqning shakliga bog‘liq emasligini va bu holda egri chiziqli integralni

$$\int\limits_{AB}\vec{F}\cdot d\vec{s}=\int\limits_{A}^{B}\vec{F}\cdot d\vec{s}$$

ko‘rinishda ham yozilishini aytamiz.

Faraz qilaylik, D sohaga tegishli bo‘lgan ixtiyoriy kontur bo‘yicha ikkinchi jins egri chiziqli integral nolga teng bo‘lsin. U holda D sohaga tegishli bo‘lgan A va V nuqtalarni tutashtiruvchi ikkita umumiy ichki nuqtaga ega bo‘lmagan (kesishmaydigan) $L_1$ va $L_2$ chiziqlarni olsak (7.6.1-rasmga qarang), $c=L_1\cup L_2^-$ konturni olamiz. Yuqoridagi farazga va egri chiziqli integral xossalariga ko‘ra

$$O=\oint\limits_{c} Pdx+Qdy=\int\limits_{L_1} Pdx+Qdy+\int\limits_{L_2^-} Pdx+Qdy=\int\limits_{L_1} Pdx+Qdy-\int\limits_{L_2^-} Pdx+Qdy$$

ni olamiz. Bundan

$$\int_{L_1} Pdx + Qdy = \int_{L_2} Pdx + Qdy$$

bo‘lishi kelib chiqadi.

Agar A va V nuqtalarni tutashtiruvchi $L_1$ va $L_2$ chiziqlar kesiShuvchi, yaoni ichki umumiy nuqtaga ega bo‘lsa (7.6.2-rasmga qarang), u holda ular bilan kesishmaydigan uchinchi A va V nuqtalarni tutashtiruvchi $L_3$ chiziqni olsak,

$$\int_{L_1} Pdx + Qdy = \int_{L_3} Pdx + Qdy$$

$$\int_{L_2} Pdx + Qdy = \int_{L_3} Pdx + Qdy$$

larga ega bo‘lamiz va bulardan

$$\int_{L_1} Pdx + Qdy = \int_{L_2} Pdx + Qdy$$

bo‘lishi kelib chiqadi.

**Zarurligi.** Faraz qilaylik, D sohaga tegishli bo‘lgan ixityoriy egri chiziq bo‘yicha ikkinchi jins egri chiziqli integral integrallash yo‘liga bog‘liq bo‘lmasin. U holda D sohaga tegishli bo‘lgan S konturning ikkita nuqtasini olib, uni ikkita tashkil etuvchi qismlarga ajratish mumkin bo‘lib, ularni $L_1$ va $L_2^-$ deb belgilasak,

$$\oint_c Pdx + Qdy = \int_{L_1} Pdx + Qdy + \int_{L_2^-} Pdx + Qdy = \int_{L_1} Pdx + Qdy - \int_{L_2} Pdx + Qdy = 0$$

ni olamiz.

**7.6.2-teorema.** Aytaylik, biror tekis bog‘liq va o‘lchanuvchi $\overline{D}$ yopiq sohada uzluksiz bo‘lgan $P(x;y)$ va $Q(x;y)$ funktsiyalar berilgan bo‘lib, ular $\frac{\partial P}{\partial y}$ va $\frac{\partial Q}{\partial x}$ uzluksiz xususiy hosilalarga ega bo‘lsin. U holda $\overline{D}$ sohada yotuvchi ixtiyoriy kontur bo‘yicha ikkinchi jins egri chiziqli integral nolga teng bo‘lishi uchun

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \forall (x;y) \in \overline{D} \qquad (7.6.2)$$

o‘rinli bo‘lishi zarur va yetarlidir.

**Isbot. Yetarliligi.** Faraz qilaylik, (7.6.2) bajarilsin. U holda ixtiyoriy S kontur bilan chegaralangan sohani $D_c$ deb belgilasak, $D_c \subset D$ talab asosida Grin formulasini qo‘llab, (7.6.2) ga va ikkilangan (ko‘p o‘lchovli) integralning $2^0$-xossasiga asosan

$$\oint_c Pdx + Qdy = \iint_D \left[\frac{\partial Q}{x} - \frac{\partial P}{y}\right] dxdy = \iint_{D_c} 0dxdy = 0$$

ni olamiz.

**Zarurligi.** Faraz qilaylik, $D$ sohaga tegishli bo‘lgan ixtiyoriy S kontur bo‘yicha egri chiziqli integral nolga teng bo‘lsin. $D$ sohaning biror $(x_0\,;y_0)$ nuqtasida (7.6.2) bajarilmaydi deb faraz qilaylik, u vaqtda $(x_0\,;y_0)$ nuqtaning Shunday $\delta$ $(\delta > 0)$ atrofi mABjud bo‘lib, unda teorema shartiga ko‘ra

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \neq 0$$

bo‘lishi va chap tomondagi ifoda o‘z ishorasini saqlashi ravshandir. Bu $\delta$ atrofni $D_\delta$ va uning chegarasidan iborat konturni $C_\delta$ deb belgilasak, Grin formulasiga asosan

$$\oint_{C_\delta} Pdx + Qdy = \iint_{D_\delta} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dxdy$$

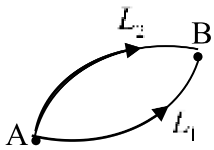
ni olamiz. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ ifoda musbat bo‘lsa, oxirgidan

$$\oint_{C_\delta} Pdx + Qdy > 0$$

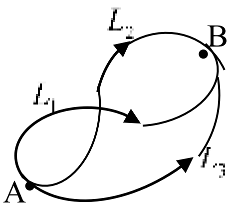
bo‘lishi, manfiy bo‘lganda esa

$$\oint_{C_\delta} Pdx + Qdy < 0$$

bo‘lishi kelib chiqadi va $C_\delta$ kontur bo‘yicha ikkinchi jins egri chiziqli integralning noldan farqli bo‘lishini ko‘ramiz. Bu teorema shartiga ziddir. Demak, (7.6.1) $D$ sohaning barcha ichki nuqtalarida bajariladi. $D$ sohaning chegarABiy nuqtalarida ham bajarilishi xususiy hosilalarning yopiq $\overline{D}$ sohada uzluksizligidan kelib chiqadi.




7.6.1-rasm.




7.6.2-rasm.

## 7-bobga doir mashqlar

Quyidagi egri chiziqli integral hisoblansin.

1. $\oint_c y^2dx + 2xydy$, bu yerda S $x = a\cos t,\ y = a\sin t$ aylana konturi

   Javob. 0

2. $\oint_c ydx - xdy$, bu yerda S $x = a\cos t,\ y = b\sin t$ aylana konturi

   Javob. $-2\pi ab$.

3. $\int_L \frac{ydx + xdy}{x^2 + y^2}$, bu yerda L (1;1) va (2;2) nuqtalarni tutashtiruvchi kesma.

   Javob. $\ln 2$

4. $\int_L yzdx + xzdy + xydz$, bu yerda L vint chizig‘idir: $x = a\cos t,\ y = a\sin t,\ z = kt,\ 0 \le t \le 2\pi$ .

   Javob. 0

5. $\int_L xdy - ydx$, bu yerda L $x = a\cos^3 t,\ y = a\sin^3 t$ gipotsikloida yoyi.

   Javob. $\frac{3}{4}\pi a^2$.

### 7-bob bo‘yicha bilimingizni sinab ko‘ring

1. Ikkinchi jins egri chiziqli integral taorifini keltiring.
2. Parametrik tenglamalar vositasida egri chiziqli integralni hisoblash formulasini yozing.
3. Kontur bo‘yicha egri chiziqli integralni tuShuntiring.
4. Egri chiziqli integralning xossalarini keltiring.
5. O‘zgaruvchi kuchning egri chiziqli yo‘lda bajargan ishi qanday hisoblanadi?
6. Tekis ikkinchi jins egri chiziqli integralni hisoblashning qanday usullarini bilasiz?
7. Grin formulasini yozing va uni tuShuntiring.
8. Tekis ikkinchi jins egri chiziqli integralning integrallash yo‘liga bog‘liq bo‘lmaslik shartini ayting.

# 8-bob. Sirt integrali

Bu bob oliy matematikaning ham nazariy ham Amaliy tatbiqlari jihatidan muhim axamiyatga ega bo‘lgan sirt integrali tushunchasini bayon qilishga bag‘ishlanadi.

## 8.1. Sirt integralining ta’rifi

Aytaylik, $Oxyz$ Dekart koordinata sistemasida bog‘liq va o‘lchanuvchi $D$ soha va unga tegishli bo‘lgan qandaydir $\sigma$ sirt berilgan bo‘lib, uning har biri $(x;y;z)$ nuqtasida

$$\vec{F} = P(x;y;z)\vec{i} + Q(x;y;z)\vec{j} + R(x;y;z)\vec{k} \qquad (8.1.1)$$

vektor funktsiya aniqlangan hamda uning proeksiyalaridan iborat $P(x;y;z)$, $Q(x;y;z)$, $R(x;y;z)$ lar uzluksiz funktsiyalar bo‘lsin. Undan tashqari $\sigma$ sirtning har bir nuqtasidagi yo‘naltiruvchi kosinuslari uzluksiz funktsiyalardan va birlik vektori $\vec{n}$ bilan belgilangan normali aniqlangan deb faraz qilamiz. Bu sirtni qandaydir usul Bilan elementar tashkil etuvchilarga ajrataylik va uning i-bo‘lagining o‘lchovini (yuzini) $\Delta\sigma_i$ $\left(i=\overline{1,m}\right)$ bilan belgilab, unga tegishli bo‘lgan ixtiyoriy bitta $M_i$ nuqtani olib,

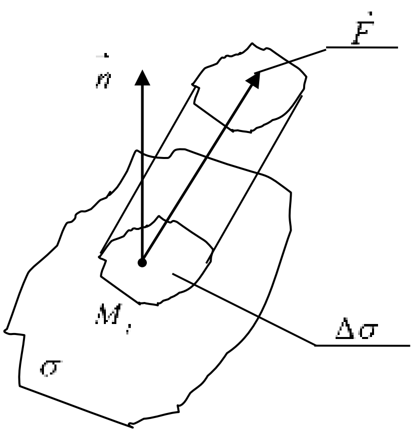
$$\sum_{i=1}^{m} \vec{F}(M_i)\cdot\vec{n}(M_i)\Delta\sigma_i \qquad (8.1.2)$$

yig‘indini tuzaylik. Endi $\lambda = \max\limits_i d_i$ deb belgilab, bu yerda $d_i$ i-bo‘lak diametri, (8.1.2) yig‘indida $\lambda \to 0$ dagi limitga mABjud bo‘lib, u $\sigma$ ni elementar tashkil etuvchilarga ajratish usuliga va har bir bo‘lakdan olinadigan nuqtaning o‘rniga bog‘liq bo‘lmasa, bu limitni $\vec{F}$ *vektor funktsiyalarining $\sigma$ bo‘yicha integrali* deymiz va

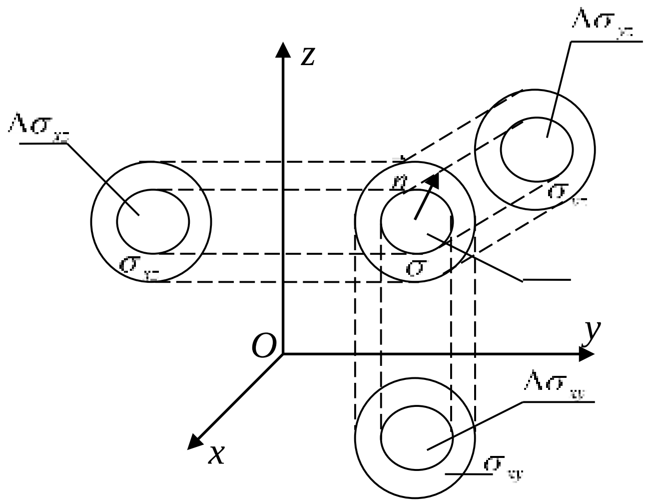
$$\iint\limits_{\sigma} \vec{F} \cdot \vec{n} d\sigma \qquad (8.1.3)$$

kabi belgilaymiz.

Bu o‘rinda sirt bo‘yicha integral iborasi o‘rnida, ko‘pincha, *sirt integrali* iborasi qo‘llanishini aytamiz.




8.1.1-расм.




8.1.2-расм.

Shunday qilib, taorifga ko‘ra

$$\iint\limits_{\sigma} \vec{F} \cdot \vec{n}\, d\sigma = \lim_{\lambda \to 0} \sum_{i=1}^{m} \vec{F_i} \cdot \vec{n_i} \Delta\sigma_i$$

ni olamiz, bu yerda $\vec{F} = \vec{F}(M_i)$, $\vec{n_i} = \vec{n}(M_i)$, $\Delta\sigma - \sigma$ sirtning i-elementar tashkil etuvchisi i ning o‘rniga m-elementar tashkil etuvchi bo‘laklarning soni bo‘lib, $\lambda \to 0 \Rightarrow m \to \infty$ bo‘lishi ravshandir.

Bu taorifning quyidagicha mexanik talqini mABjud: agar berilgan $\vec{F}$ vektorni $\sigma$ sirt $\vec{n}$ normali yo‘nalishi bo‘yicha u orqali o‘tayotgan suyuqlik oqimining tezligi (vaqt birligi davomida sirt nuqtasidan o‘tayotgan suyuqlik miqdori) deb qarasak, u holda

$$\vec{F_i} \cdot \vec{n_i} \Delta\sigma_i = \left|\vec{F_i}\right| \Delta\sigma_i \cos\left(\vec{n_i}\,; \vec{F_i}\right)$$

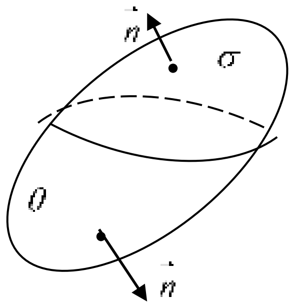
sirtning $\Delta\sigma_i$ bo‘lagidan vaqt birligi davomida oqib o‘tgan suyuqlik miqdorining (suyuqlik oqimining) taqribiy qiymati deb qarash mumkin bo‘ladi (8.1.1-rasmdagi tsilindrsimon jism hajmiga e’tibor bering). (8.1.3) sirt integrali esa, *$\sigma$ sirt orqali vaqt birligi davomida oqib o‘tgan suyuqlik oqimining miqdori (suyuqlik oqimi)* ekanligini ko‘rish qiyin emas.

Bunga asoslangan holda (8.1.3) sirt integralini mexanik jihatdan *(8.1.1) vektorning $\sigma$ sirt orqali oqimi* yoki qisqacha *vektor oqimi* deb qabul qilish mumkin.

Agar $\sigma$ sirt biror uch o‘lchovli chekli diametrli va o‘lchanuvchi $D$ bog‘liq sohaning chegarasidan iborat bo‘lsa, u yopiq bo‘lib, bu holda u *tashqi normal qoidasi bo‘yicha yo‘nalishga ega*, yaoni bu sirtning normali undan tashqariga qarab yo‘naltirilgan deb faraz qilinadi (8.1.3-rasmga qarang). Bu holda sirt integralini

$$\oiint_{\sigma} \vec{F} \cdot \vec{n} d\sigma$$

Ko‘rinishda yozish qabul qilingan.




8.1.3-расм.

Bayon qilingan sirt integralining taorifiga asoslanib, uning quyidagi xossalari mABjudligiga ishonch hosil qilish oson:

**1⁰.** Agar $\sigma$ sirt normalining yo‘nalishini qarama-qarshisiga o‘zgartirilganda uni $\sigma$-kabi belgilashga kelishsak, u holda

$$\iint_{\sigma^-} \vec{F} \cdot \vec{n} d\sigma = -\iint_{\sigma} \vec{F} \cdot \vec{n} d\sigma$$

o‘rinlidir, yaoni sirt normalining yo‘nalishini qarama-qarshisiga o‘zgartirilsa, sirt integralining qiymati qarama-qarshisiga o‘zgaradi.

**2⁰.** $\iint_{\sigma^-} \vec{0} \cdot \vec{n} d\sigma = 0$

**3⁰.** Agar $\sigma$ sirt $m$ ta tashkil etuvchilarga ajratilgan, yaoni har qanday juftligi umumiy ichki nuqtaga ega bo‘lmagan $\sigma_i \left(i = \overline{1, m}\right)$ sirtlarning birlashmasi sifatida $\sigma = \bigcup_{i=1}^{m} \sigma_i$ kabi ifodalangan, hamda har bir $\sigma_i$ bo‘yicha sirt integrali mABjud bo‘lsa, $\sigma$ bo‘yicha sirt integrali ham mABjud va

$$\iint_{\sigma^-} \vec{F} \cdot \vec{n} d\sigma = \sum_{i=1}^{m} \iint_{\sigma_i} \vec{F} \cdot \vec{n} d\sigma$$

o‘rinli bo‘ladi.

**4⁰.** $\iint_{\sigma} \vec{n} \cdot \vec{n} d\sigma = \iint_{\sigma} d\sigma = \mu(\sigma)$, bu yerda $\vec{n}$ $\sigma$ sirtning birlik normali, $\mu(\sigma) - \sigma$ sirtning o‘lchovi (yuzi).

## 8.2. Sirt integralining hisoblash

Agar (8.1.1) bilan aniqlangan $\vec{F}$ vektor funktsiyaning va $\sigma$ sirt $\vec{n}$ normalining

$$\vec{F} = \{P(x;y;z),\ Q(x;y;z),\ R(x;y;z)\},$$
$$\vec{n} = \left\{\cos\left(\vec{n};^\wedge Ox\right),\ \cos\left(\vec{n};^\wedge Oy\right),\ \cos\left(\vec{n};^\wedge Oz\right)\right\}$$

proeksiyalarini hisobga olgan holda sirt integralini yozsak,

$$\iint_{\sigma^-} \vec{F}\cdot\vec{n}d\sigma = \iint_{\sigma} \left[P\cos\left(\vec{n};^\wedge Ox\right) + Q\cos\left(\vec{n};^\wedge Oy\right) + R\cos\left(\vec{n};^\wedge Oz\right)\right]d\sigma \qquad (8.1.3)$$

ni olamiz. 8.1.2-rasmda $\sigma$ sirt $\Delta\sigma$ elementar bo‘lagining $Oxy$, $Oyz$, $Oxz$ koordinata tekisliklaridagi proeksiyalari, mos ravishda, $\Delta\sigma_{xy}$, $\Delta\sigma_{yz}$, $\Delta\sigma_{xz}$ orqali tasvirlangan bo‘lib, ular uchun $\Delta\sigma$ bilan

$$\Delta\sigma\cos\left(\vec{n};^\wedge Ox\right) \approx \Delta\sigma_{yz},$$
$$\Delta\sigma\cos\left(\vec{n};^\wedge Oy\right) \approx \Delta\sigma_{xz},$$
$$\Delta\sigma\cos\left(\vec{n};^\wedge Oz\right) \approx \Delta\sigma_{xy}.$$

bog‘liklari mABjudligiga ishonch hosil qilish oson. Bularga mos ravishda (8.2.1) ni

$$\iint_{\sigma} \vec{F}\cdot\vec{n}d\sigma = \iint_{\sigma} P(x;y;z)dydz + Q(x;y;z)dydz + R(x;y;z)dydz$$

ko‘rinishda ham yozish qabul qilingan.

Endi sirt bo‘yicha integralni hisoblash usulini keltirib chiqarishga harakat qilamiz. Buning uchun (8.2.1) ni

$$\iint_{\sigma} \vec{F}\cdot\vec{n}d\sigma = \iint_{\sigma} P\cos\left(\vec{n};^\wedge Ox\right)d\sigma + \iint_{\sigma} Q\cos\left(\vec{n};^\wedge Oy\right)d\sigma + \iint_{\sigma} R\cos\left(\vec{n};^\wedge Oz\right)d\sigma \qquad (8.2.2)$$

ko‘rinishda yozib olib, uning o‘ng tomonidagi ifodaning har bir qo‘shiluvchisini alohida hisoblashga harakat qilamiz. Masalan, ulardan uchinchisini olsak, u

$$\iint_{\sigma} R(x;y;z)\cos\left(\vec{n};^\wedge Oz\right)d\sigma \qquad (8.2.3)$$

ko‘rinishda bo‘lib, $\sigma$ sirt $Ox$ o‘qqa parallel to‘g‘ri chiziq bilan bittadan ortiq umumiy nuqtaga ega emas deb faraz qilsak, uni qandaydir

$$z = f(x;y),\quad \forall(x;y)\in\sigma_{xy} \qquad (8.2.4)$$

Tenglama vositasida aniqlash mumkinligi aniqdir (8.2.1-rasmga qarang), bu yerda $\sigma_{xy}$ orqali $\sigma$ sirtning $Oxy$ koordinata tekisligidagi proeksiyasi belgilangan.

Agar (8.2.2) ning o‘ng tomonidagi (8.2.3) ko‘rinishdagi qo‘shiluvchisiga mos kelgan (8.2.4) ni hisobga olgan holda tuzilgan integral yig‘indini yozsak,

$$\sum_{i=1}^{m} R(x_i;y_i;f(x_i;y_i))\cos\left(\vec{n_i};^\wedge Oz\right)\Delta\sigma_i = \pm\sum_{i=1}^{m} R(x_i;y_i;f(x_i;y_i))(\Delta\sigma)_i$$

ni olamiz, bundagi «+» ishora $\vec{n}$ normal $Oz$ o‘q bilan o‘tkir, «-» ishora esa, o‘tmas burchak tashkil qilgan holga mosdir.

Oxirgida $\lambda\to 0$ dagi limitga o‘tib ($\lambda = \max\limits_i d_i$ $d_i - \left(\Delta\sigma_{xy}\right)_i$ bo‘lgan diametri), (8.2.3) ni hisoblash uchun

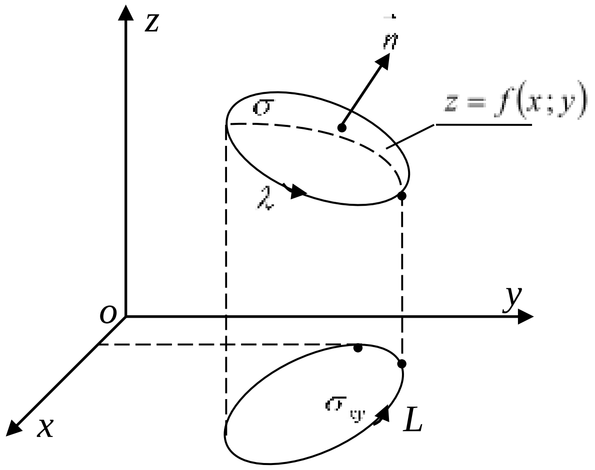
$$\iint_{\sigma} R(x;y;z)\cos\left(\vec{n};^\wedge Oz\right)d\sigma = \pm\iint_{\sigma_{xy}} R(x;y;f(x;y))dxdy$$

ikkilangan integralga kelamiz.

Xuddi Shunga o‘xshash, (8.2.2) ning o‘ng tomonidagi yig‘indining qolgan birinchi va ikkinchi qo‘shiluvchilarini hisoblash uchun ularga mos ikkilangan integrallarni ham olamiz:

$$\pm\iint\limits_{\sigma_{yz}} P(\varphi(y;x);y;z)dxdy, \quad \pm\iint\limits_{\sigma_{xz}} Q(x;\psi(x;z);z)dxdz$$

Bulardagi ishorani tanlash ham yuqoridagiga o‘xshash. $\sigma$ sirt $\vec{n}$ normalining mos $Ox$ yoki $Oy$ koordinata o‘qi bilan qanday burchak hosil qilishiga bog‘liq ravishda tanlanadi.




8.2.1-расм.

Yuqorida qilingan mulohazalardan sirt integralini hisoblash oxir oqibatda ikkilangan integral orqali bajarilishini ko‘ramiz.

## 8.3. Stoks formulasi

Aytaylik, $R^3$ fazoga ($Oxyz$ Dekart koordinata sistemasiga) tegishli biror $\sigma$ sirt berilgan bo‘lib, uni $Oz$ o‘qqa parallel bo‘lgan to‘g‘ri chiziq bittadan ortiq nuqtada kesmasin. Bu $\sigma$ sirt chegarasini $\lambda$ bilan belgilaylik va uning birlik normali bo‘lgan $\vec{n}$ ni $Oz$ o‘q bilan o‘tkir burchak hosil qiladigan qilib yo‘naltirib, $\lambda$ kontur yo‘nalishini esa, normal yo‘nalishni tomonidan qaralganda soat ko‘rsatkichi harakati yo‘nalishiga qarama-qarshi (yaoni musbat) qilib tanlaylik (8.2.1-rasmga qarang). Bu holda $\sigma$ sirtni (8.2.4) ko‘rinishdagi qandaydir tenglama vositasida aniqlash mumkinligi ravshan bo‘lib, unda $f(x;y)$ funktsiya $\sigma_{xy}$ sohada uzluksiz differensiallanuvchi deb faraz qilsak, $\sigma$ sirt $\vec{n}$ birlik normalining yo‘naltiruvchi

kosinuslari
$$\cos\left(\vec{n};\,^\wedge Ox\right)=-\frac{\dfrac{\partial f}{\partial x}}{\sqrt{1+\left(\dfrac{\partial f}{\partial x}\right)^2+\left(\dfrac{\partial f}{\partial y}\right)^2}},$$
$$\cos\left(\vec{n};\,^\wedge Oy\right)=-\frac{\dfrac{\partial f}{\partial y}}{\sqrt{1+\left(\dfrac{\partial f}{\partial x}\right)^2+\left(\dfrac{\partial f}{\partial y}\right)^2}},\qquad (8.3.1)$$
$$\cos\left(\vec{n};\,^\wedge Oz\right)=-\frac{1}{\sqrt{1+\left(\dfrac{\partial f}{\partial x}\right)^2+\left(\dfrac{\partial f}{\partial y}\right)^2}},$$
formulalar orqali ifodalanishi aniqdir.

Endi $\sigma$ sirt $R^3$ fazoga tegishli biror o‘lchanuvchi $D$ sohaga tegishli bo‘lib, bu sohada o‘zining birinchi tartibli xususiy hosilalari Bilan birga uzluksiz bo‘lgan $P(x;y;z)$ funktsiya berilgan bo‘lsin deb faraz qilaylik va
$$\oint_\lambda P(x;y;z)dx$$
ikkinchi jins egri chiziqli integralni qaraylik. $\lambda$ konturga tegishli nuqta koordinatalari uchun (8.2.4) o‘rinli deb faraz qilaylik. $\lambda$ ning $Oxy$ koordinata tekisligidagi proeksiyasini L kontur deb olsak (8.2.1-rasmga qarang), u $\sigma$ sirtning $Oxy$ tekislikdagi proeksiyasidan iborat $\sigma_{xy}$ ikki o‘lchovli sohaning chegarasi bo‘lishi ravshandir. Aytilganlarga asosan
$$\oint_\lambda P(x;y;z)dx=\oint_\lambda P(x;y;f(x;y))dx \qquad (8.3.2)$$
ni yozaolamiz. Endi (7.5.1) Grin formulasida $c=L$, $D=\sigma_{xy}$, $P(x;y)=P(x;y;f(x;y))$, $Q(x;y)=0$ deb faraz qilib,
$$\oint_\lambda P(x;y;f(x;y))dx=-\iint_{\sigma_{xy}}\frac{\partial P(x;y;f(x;y))}{\partial y}dxdy \qquad (8.3.3)$$
ni olamiz. Murakkab funktsiyani differensiallash qoidasiga asosan (8.2.4) ni hisobga olgan holda
$$\frac{\partial P(x;y;f(x;y))}{\partial y}=\frac{\partial P(x;y;z)}{\partial y}+\frac{\partial P(x;y;z)}{\partial z}\cdot\frac{\partial f(x;y)}{\partial y} \qquad (8.3.4)$$
ga ega bo‘lamiz. (8.3.4) ni (8.3.3) ning o‘ng tomoniga qo‘yib,
$$\oint_L P(x;y;f(x;y))dx=-\iint_{\sigma_{xy}}\left[\frac{\partial P(x;y;z)}{\partial y}+\frac{\partial P(x;y;z)}{\partial z}\cdot\frac{\partial f(x;y)}{\partial y}\right]dxdy$$
ni olamiz, bunda (8.3.4) o‘rinlidir. Oxirgida (8.3.2) ni hisobga olgan holda
$$\oint_\lambda P(x;y;z)dx=-\iint_{\sigma_{xy}}\frac{\partial P(x;y;z)}{\partial y}dxdy-\iint_{\sigma_{xy}}\frac{\partial P(x;y;z)}{\partial z}\frac{\partial f(x;y)}{\partial y}dxdy \qquad (8.3.5)$$
tenglikka kelamiz. (8.3.5) ning o‘ng tomonidagi ikkilangan integrallar quyidagicha sirt integrallari orqali ifodalanadi (8.2-bandga qarang):

$$\iint\limits_{\sigma_{xy}} \frac{\partial P(x;y;z)}{\partial y} dxdy = \iint\limits_{\sigma} \frac{\partial P(x;y;z)}{\partial y} \cos(\vec{n};^\wedge Oz) d\sigma \tag{8.3.6}$$

$$\iint\limits_{\sigma_{xy}} \frac{\partial P(x;y;z)}{\partial z} \frac{\partial f}{\partial y} dxdy = \iint\limits_{\sigma} \frac{\partial P(x;y;z)}{\partial z} \frac{\partial f}{\partial y} \cos(\vec{n};^\wedge Oz) d\sigma \tag{8.3.7}$$

(8.3.1) ning ikkinchi va uchinchi formulalaridan kelib chiqadigan

$$\cos(\vec{n};^\wedge Oz)\frac{\partial f}{\partial y} = -\cos(\vec{n};^\wedge Oy)$$

tenglikdan foydalanib, (8.3.7) ni

$$\iint\limits_{\sigma_{xy}} \frac{\partial P(x;y;z)}{\partial z} \frac{\partial f}{\partial y} dxdy = -\iint\limits_{\sigma} \frac{\partial P(x;y;z)}{\partial z} \frac{\partial f}{\partial y} \cos(\vec{n};^\wedge Oy) d\sigma \tag{8.3.8}$$

ko‘rinishga keltiramiz.

(8.3.6) va (8.3.8) larni e’tiborga olib, (8.3.5) ni

$$\oint\limits_{\lambda} P(x;y;z)dx = \iint\limits_{\sigma_{xy}} \left[ \frac{\partial P(x;y;z)}{\partial x} \cos(\vec{n};^\wedge Oy) - \frac{\partial P(x;y;z)}{\partial y} \cos(\vec{n};^\wedge Oy) \right] d\sigma$$

ko‘rinishda yozish mumkin.

Xuddi Shunga o‘xshash

$$\oint\limits_{\lambda} Q(x;y;z)dy = \iint\limits_{\sigma_{xy}} \left[ \frac{\partial Q(x;y;z)}{\partial x} \cos(\vec{n};^\wedge Oz) - \frac{\partial Q(x;y;z)}{\partial z} \cos(\vec{n};^\wedge Ox) \right] d\sigma,$$

$$\oint\limits_{\lambda} R(x;y;z)dz = \iint\limits_{\sigma_{xy}} \left[ \frac{\partial R(x;y;z)}{\partial y} \cos(\vec{n};^\wedge Ox) - \frac{\partial R(x;y;z)}{\partial x} \cos(\vec{n};^\wedge Oy) \right] d\sigma$$

formulalarni ham yozish mumkin.

Oxirgi yozilgan uchchala formulalarni hadlab qo‘shib,

$$\oint\limits_{\lambda} P(x;y;z)dx + \oint\limits_{\lambda} Q(x;y;z)dy + \oint\limits_{\lambda} R(x;y;z)dz =$$

$$= \iint\limits_{\sigma_{xy}} \left[ \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cos(\vec{n};^\wedge Oz) + \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \cos(\vec{n};^\wedge Ox) + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \cos(\vec{n};^\wedge Oy) \right] d\sigma \tag{8.3.9}$$

ni olamiz. (8.3.9) ni *Stoks formulasi* deb yuritiladi.

## 8.4. Ostrogrodskiy formulasi

Aytaylik, $Oxyz$ koordinata fazosida o‘lchanuvchi $\overline{D}$ yopiq soha berilgan bo‘lib, u koordinata o‘qlarining har biriga nisbatan ham to‘g‘ri sohadan hamda uning chegarasi $\sigma$ yopiq sirtdan iborat bo‘lsin. Undan tashqari bu $\sigma$ sirtni uchta $\sigma_1, \sigma_2$ va $\sigma_3$ qismlarga ajratish mumkin bo‘lib, ulardan dastlabki ikkitasining tenglamalari mos ravishda

$$z = f_1(x;y), \quad z = f_2(x;y), \quad f_1(x;y) \le f_2(x;y), \ (x;y) \in D_{xy} \tag{8.4.1}$$

Bo‘lib, bu yerda $D_{xy}$ soha $\sigma$ ning $Oxy$ koordinata tekisligidagi proeksiyasi, $\sigma_3$ esa yasovchisi $Oz$ o‘qqa parallel bo‘lgan qandaydir tsilindrik sirt bo‘lsin deb faraz qilamiz (8.4.1-rasmga qarang).

$$I = \iiint\limits_{D} \frac{\partial R(x;y;z)}{\partial z} dxdydz$$

Uchlangan integralni yozib, unda (6.1.5) formulaga asosan shakl almashtirish bajaramiz:

$$I=\iint\limits_{D_{xy}} dxdy\int\limits_{f_1(x;y)}^{f_2(x;y)}\frac{\partial R(x;y;z)}{\partial z}dz=\iint\limits_{D_{xy}} R(x;y;f_2(x;y))dxdy-\iint\limits_{D_{xy}} R(x;y;f_1(x;y))dxdy$$

Bu tenglamaning o‘ng tomonidagi ikkilangan integrallarni esa quyidagicha sirt integrallari orqali ifodalash mumkin:

$$\iint\limits_{D_{xy}} R(x;y;f_1(x;y))dxdy=-\iint\limits_{\sigma_1} R(x;y)\cos(\vec{n};\widehat{\ } Oz)d\sigma,$$

$$\iint\limits_{D_{xy}} R(x;y;f_2(x;y))dxdy=-\iint\limits_{\sigma_2} R(x;y)\cos(\vec{n};\widehat{\ } Oz)d\sigma.$$

Bularni (8.4.1) ni e’tiborga olgan holda

$$I=\iint\limits_{\sigma_1} R(x;y;z)\cos(\vec{n};\widehat{\ } Oz)d\sigma+\iint\limits_{\sigma_2} R(x;y;z)\cos(\vec{n};\widehat{\ } Oz)d\sigma$$

ko‘rinishda yoza olamiz. Agar $\sigma_3$ da $\vec{n}\perp Oz$ ekanligidan $\cos(\vec{n};\widehat{\ } Oz)=0$ bo‘lishini hisobga olsak,

$$I=\iint\limits_{\sigma_3} R(x;y;z)\cos(\vec{n};\widehat{\ } Oz)d\sigma=\iint\limits_{\sigma_3} Od\sigma=0$$

bo‘ladi. U holda

$$I=\iint\limits_{\sigma_1} R(x;y;z)\cos(\vec{n};\widehat{\ } Oz)d\sigma+\iint\limits_{\sigma_2} R(x;y;z)\cos(\vec{n};\widehat{\ } Oz)d\sigma+$$

$$+\iint\limits_{\sigma_3} R(x;y;z)\cos(\vec{n};\widehat{\ } Oz)d\sigma=\oint\!\!\oint\limits_{\sigma} R(x;y;z)\cos(\vec{n};\widehat{\ } Oz)d\sigma$$

ga kelamiz. Shunday qilib,

$$\iiint\limits_{D}\frac{\partial R(x;y;z)}{\partial z}dxdydz=\oint\!\!\oint\limits_{\sigma} R(x;y;z)\cos(\vec{n};\widehat{\ })z)d\sigma \qquad (8.4.2)$$

tenglikni olamiz. Xuddi Shunga o‘xshash:

$$\iiint\limits_{D}\frac{\partial Q(x;y;z)}{\partial y}dxdydz=\oint\!\!\oint\limits_{\sigma} Q(x;y;z)\cos(\vec{n};\widehat{\ })y)d\sigma \qquad (8.4.3)$$

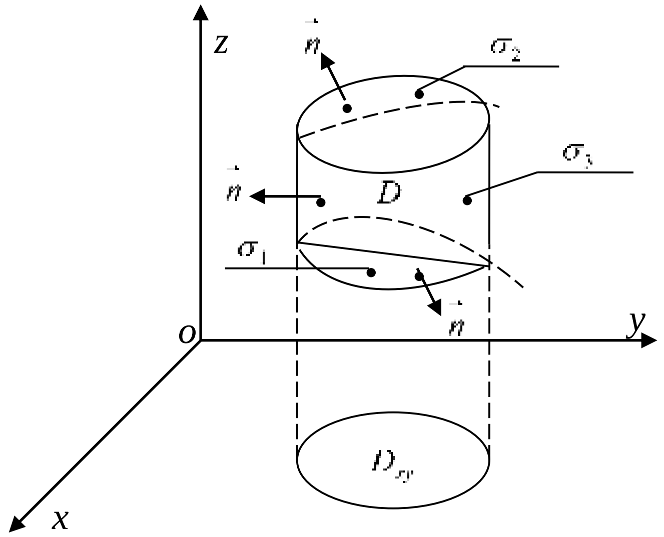
$$\iiint\limits_{D}\frac{\partial P(x;y;z)}{\partial x}dxdydz=\oint\!\!\oint\limits_{\sigma} P(x;y;z)\cos(\vec{n};\widehat{\ })x)d\sigma \qquad (8.4.4)$$

tengliklarni yozish mumkin. (8.4.2)-(8.4.4) tengliklarni hadlab qo‘shib,

$$\iiint\limits_{D}\left[\frac{\partial P(x;y;z)}{\partial x}+\frac{\partial Q(x;y;z)}{\partial y}+\frac{\partial R(x;y;z)}{\partial z}\right]dxdydz=$$

$$=\oint\!\!\oint\limits_{\sigma}\left[P(x;y;z)\cos(\vec{n};\widehat{\ })x)+Q(x;y;z)\cos(\vec{n};\widehat{\ })y)+R(x;y;z)\cos(\vec{n};\widehat{\ })z)\right]d\sigma$$

ni olamiz. Bu *Ostrogrodskiy formulasi*dir.




8.4.1-расм.

## 8-bob bo‘yicha mashqlar

1. Agar $\sigma$-yopiq sirt bo‘lib, $\vec{n}$ uning normali (tashqi normali), $\oiint\limits_{\sigma} \cos(\vec{n};\,{}^{\wedge}Oz)d\sigma = 0$ bo‘lishini isbotlang.

2. Agar $\sigma$-yopiq sirt bo‘lib, $\oiint\limits_{\sigma} \left[x\cos(\vec{n};\,{}^{\wedge}Ox) + y\cos(\vec{n};\,{}^{\wedge}Oy) + z\cos(\vec{n};\,{}^{\wedge}Oz)\right]d\sigma$ ni hisoblang.

Javob: 3V, bu yerda V-$\sigma$ yopiq sirt bilan chegaralangan jism hajmi.

3. Agar $\sigma$ $x^2 + y^2 + z^2 = R^2$ sfera sirtining tashqi tomoni bo‘lsa, $\oiint\limits_{\sigma} x^2dydz + y^2dxdz + z^2dxdy$ ni hisoblang. Javob: $\pi R^2$

Ostrogrodskiy formulasi yordamida yopiq sirt bo‘yicha integralni hisoblang.

4. $\oiint\limits_{\sigma} (x\cos\alpha + y\cos\beta + z\cos\gamma)d\sigma$, bu yerda $\sigma$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ellipsoidning tashqi sirt bo‘lib, $\alpha;\beta;\gamma$ lar uning normalining mos ravishda $Ox;Oy;Oz$ o‘qlar bilan hosil qilgan burchaklari.

5. $\oiint\limits_{\sigma} xdydz + ydxdz + zdxdy$, bu yerda $\sigma$ $x^2 + y^2 = a^2$, $-h \le z \le h$ tsilindr sirtining tashqi tomoni.

Javob: $3\pi a^2 h$.

Egri chiziqli integralni Stoks formulasidan foydalanib va bevosita hisoblang.

6. $\oint\limits_{\lambda} (y+z)dx + (z+x)dy + (x+y)dz$, bu yerda $\lambda$ kontur $x^2 + y^2 + z^2 = R^2$, $x + y + z = 0$ aylana.

Javob: 0.

7. $\oint\limits_{\lambda} x^2y^3dx + dy + zdz$, bu yerda $\lambda$ kontur $x^2 + y^2 = R^2$, $z = 0$ aylana.

Javob: $-\dfrac{\pi R^6}{8}$.

### 8-bob bo‘yicha bilimingizni sinab ko‘ring

1. Sirt integralining taorifini bayon qiling va tuShuntiring.
2. Yopiq sirt bo‘yicha integralni tuShuntiring.
3. Sirt integralining xossalarini ayting.

# 9-bob. Maydonlar nazariyasining elementlari

Bu bobda mexanika fani uchun ham nazariy ham Amaliy jihatdan muhim ahamiyatga ega tatbig‘i bo‘lgan maydonlar nazariyasining baozi bir elementlarining bayoni asosan $R^3$ fazoda keltiriladi.

## 9.1. Skalyar va vektor maydonlar

Aytaylik, $R^3$ fazoga tegishli bo‘lgan biror bog‘liq $D$ sohada qandaydir uch o‘zgaruvchili $U = f(x;y;z)$ funktsiya aniqlangan bo‘lsa, $D$ sohani *skalyar maydon* deb ataladi. Agar aytilgan $D$ sohada

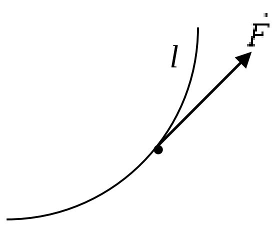
$$\vec{F} = \{P(x;y;z);Q(x;y;z);R(x;y;z)\} \qquad (9.1.1)$$

Vektor funktsiya aniqlangan bo‘lsa, $D$ ni *vektor maydon* deb ataymiz.

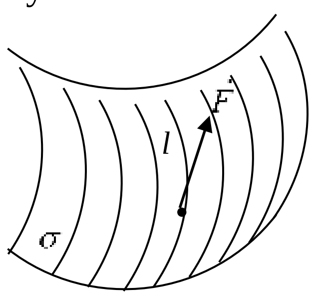
Bu o‘rinda ABvalroq funktsiya haqida aytilgan uning xossalari va baozi bir tushunchalar skalyar maydonning elementlaridan iborat ekanligini aytib, bu yerda asosan vektor maydonga tegishli bo‘lgan xossalari va tushunchalarning bayoni bilan Shug‘ullanamiz.

Agar (9.1.1) vektor funktsiya biror bog‘liq $D$ sohada aniqlangan bo‘lsa, uni vektor maydonning *vektor funktsiyasi* (yoki *vektor maydonni aniqlovchi vektor funktsiya*) deb ataymiz.

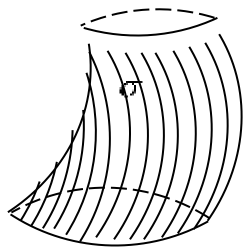
Har bir nuqtasida (9.1.1) bilan aniqlangan vektor urinma bo‘lgan chiziq *vektor maydonning vektor chizig‘i* (9.1.1-rasmga qarang) va bunday vektor chiziqlarning qandaydir to‘plami vositasida tuzilgan sirt *vektor maydonning vektor sirti* (9.1.2-rasmga qarang) hamda ko‘ndalang kesimi qandaydir koturdan iborat bo‘lgan vektor quvuri (9.1.3-rasmga qarang) deb ataymiz.



9.1.1-рacм.



9.1.2-рacм.



9.1.3-рacм.

Agar biror $l$ chiziq (9.1.1) vektor funktsiya vositasida aniqlangan $D$ vektor maydonga tegishli bo‘lib,

$$\int_l P(x;y;z)dz + Q(x;y;z)dy + R(x;y;z)dz$$

ikkinchi jins egri chiziqli integral yozilgan bo‘lsa, uni *vektor maydonning chiziqli integrali*, $l$ chiziq yopiq konturdan iborat bo‘lgan holda esa

$$\oint_l P(x;y;z)dz + Q(x;y;z)dy + R(x;y;z)dz$$

chiziqli integralni *vektor maydonning l-kontur bo‘yicha tsirkulyatsiyasi* deb ataymiz.

Agar (9.1.1) vositasida aniqlangan $D$ vektor maydonga tegishli bo‘lgan biror $\sigma$ sirt berilgan bo‘lsa, sirt integrali yordamida aniqlangan

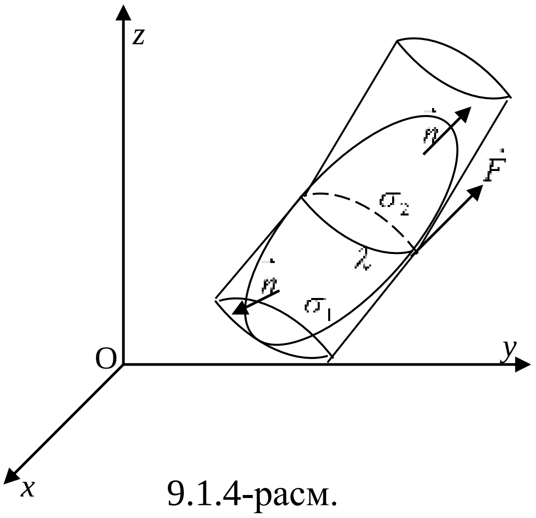
$$B = \iint_\sigma \vec{F} \cdot \vec{n}\, d\sigma$$

miqdor *vektor maydonning $\sigma$ sirt bo‘yicha vektor oqimi* deyiladi.

Endi yuqorida aytilgan $\sigma$-yopiq sirtdan iborat bo‘lgan holni qarab,

$$B = \oiint_\sigma \vec{F} \cdot \vec{n}\, d\sigma$$

yopiq sirt bo‘yicha vektor oqimini tekshiraylik. 9.1.4-rasmda ko‘rsatilganidek, $\sigma$ yopiq sirtni o‘z ichiga oluvchi biror vektor quvur mABjud va uning $\sigma$ yopiq sirt bilan urinish chizig‘i konturdan iborat bo‘lib, u $\sigma_1$ va $\sigma_2$ sirtni ikkita qismlarga ajratadi.




9.1.4-расм.

U holda sirt integralining xossasi asosida

$$B = \oiint_\sigma \vec{F} \cdot \vec{n}\, d\sigma = \iint_{\sigma_1} \vec{F} \cdot \vec{n}\, d\sigma + \iint_{\sigma_2} \vec{F} \cdot \vec{n}\, d\sigma = B_2 - B_1$$

ni olamiz, bu yerda

$$B = \iint_{\sigma_1^-} \vec{F} \cdot \vec{n}\, d\sigma = \iint_{\sigma_1} \vec{F} \cdot \left(-\vec{n}\right) d\sigma = B_2 - B_1$$

bo‘lib, u $\sigma$ sirtga kiruvchi vektor oqimining,

$$B_2 = \iint_{\sigma_2} \vec{F} \cdot \vec{n}\, d\sigma$$

bo‘lib, u sirtdan chiquvchi vektor oqimining miqdoridan iborat bo‘ladi.

Shunday qilib, yopiq sirt bo‘yicha vektor oqimi undan chiquvchi va unga kiruvchi vektor oqimlarining farqi (ayirmasi)dan iborat ekanligini ko‘ramiz. Bu miqdor musbat bo‘lganda yopiq sirt ichida vektor oqimini ortdiruvchi manba mABjudligini, manfiy bo‘lganda esa vektor oqimining kamaytiruvchi o‘lqon borligini ko‘rsatadi.

## 9.2. Solenondal vektor maydon

(9.1.1) vektor funktsiya vositasida biror $D$ vektor maydon aniqlangan bo‘lsin. U holda $di\vartheta\vec{F}$ kabi belgilanuvchi

$$di\vartheta\vec{F}=\frac{\partial P(x;y;z)}{\partial x}+\frac{\partial Q(x;y;z)}{\partial y}+\frac{\partial R(x;y;z)}{\partial z} \qquad (9.2.1)$$

bilan aniqlanuvchi miqdor *vektor maydonning divergentsiyasi* deb ataladi.

(9.2.1) dan ko‘rinadiki, $\vec{F}$ vektor funksiyaning proeksiyalaridan unga kiruvchi mos xususiy hosilalarga ega bo‘lishini talab qilinadi.

Vektor maydonga tegishli biror yopiq $\sigma_0$ sirt bilan chegaralangan $D$ sohada (9.2.1) ni hisobga olib, Ostrogrodskiy formulasini

$$\iiint\limits_{D_0} di\vartheta\vec{F}dxdydz=\oint\!\!\!\oint\limits_{\sigma_0} \vec{F}\cdot\vec{n}\,d\sigma \qquad (9.2.2)$$

ko‘rinishda yozish mumkin bo‘ladi. (9.2.2) Ostrogrodskiy formulasining vektor ko‘rinishda yozilishidir.

Agar $di\vartheta\vec{F}=0$ tenglik $D$ sohada o‘rinli bo‘lsa, vektor maydonni *solenoidal* (*quvursim*) deyiladi.

(9.2.2) formuladan ko‘rinadiki, solenoidal vektor maydonda yopiq sirt bo‘yicha vektor oqimi nolga tengdir. Shu xossa yordamida solenoidal vektor maydonda olingan vektor quvurning ko‘ndalang kesim bo‘yicha vektor oqim o‘zgarmas bo‘lishini isbotlash qiyin emas.

Solenoidal vektor maydonni manbalarsiz (bazan quvursimon) maydon deb ham yuritiladi.

## 9.3. Potensial vektor maydon

Proeksiyalari

$$\frac{\partial R(x;y;z)}{\partial y}-\frac{\partial Q(x;y;z)}{\partial z},\quad \frac{\partial P(x;y;z)}{\partial z}-\frac{\partial R(x;y;z)}{\partial z},\quad \frac{\partial Q(x;y;z)}{\partial x}-\frac{\partial P(x;y;z)}{\partial y}$$

bo‘lgan, vektor $\vec{F}=\{P(x;y;z);Q(x;y;z);R(x;y;z)\}$ vektor funktsiyaning *rotori (uyurmasi)* deb ataladi va $rot\vec{F}$ kabi belgilanadi.

Buni hisobga olgan holda (8.3.9) vtoks formulasini

$$\oint\limits_{\lambda} \vec{F}\cdot d\vec{S}=\iint\limits_{\sigma} rot\vec{F}\cdot\vec{n}\,d\sigma \qquad (9.3.1)$$

vektor ko‘rinishda yozish mumkin.

Agar $\vec{F}$ vektor funktsiyaning rotori $D$ sohada aynan nolga teng bo‘lsa (yaoni $rot\vec{P}=0$), u holda *vektor maydonni potensial* deyiladi.

(9.3.1) formula yordamida potensial vektor maydonda ixtiyoriy yopiq kontur bo‘yicha egri chiziqli integral (yaoni tsirkulyatsiya) nolga tengligini ko‘rsatish oson. Bundan potensial vektor maydonda uning chiziqli integrali integrallash yo‘liga bog‘liq bo‘lmasligi kelib chiqadi. Bu holda vektor maydon chiziqli integrali ostidagi ifoda biror uch o‘zgaruvchili $U=(x;y;z)$ funktsiyaning to‘liq differensialidan iborat bo‘ladi:

$$P(x;y;z)dx+Q(x;y;z)dy+R(x;y;z)dz=dU(x;y;z)$$

Bunday $U=(x;y;z)$ funktsiya *potensial vektor maydonning potensial funktsiyasi* deyiladi.

Vektor maydonning rotori taorifi asosida vektor maydon potensial (yaoni $rot\vec{F}=0$) bo‘lishining zaruruiy va yetarlilik shartini

$$\frac{\partial R}{\partial y}=\frac{\partial Q}{\partial z};\quad \frac{\partial P}{\partial z}=\frac{\partial R}{\partial x};\quad \frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y},\quad \forall(x;y;z)\in D \tag{9.3.2}$$

ko‘rinishda yozish mumkinligi kelib chiqadi. (9.2.2) shartning bajarilishi vektor maydon chiziqli integralining integrallash yo‘liga bog‘liq bo‘lmasligi uchun zaruriy va yetarli shartdan iborat bo‘lib, bunday holda potensial vektor maydon potensial funktsiyasini topish uchun

$$U(x;y;z)=c+\int_{x_0}^{x} P(\xi;y;z)d\xi+\int_{y_0}^{y} Q(x_0;\eta;z)d\eta+\int_{z_0}^{z} R(x_0;y_0;\varsigma)d\varsigma$$

Formulani chiqarish qiyin emas, bu yerda $c$-ixtiyoriy o‘zgarmas, $(x_0;y_0;z_0)\in D$ tayinlangan nuqtaning potensial funktsiyasini berilishi Bilan to‘liq aniqlash mumkinligini ko‘ramiz.

*Potensial vektor maydonni uyurmasiz maydon* deb ham yuritiladi. Unda chiziqli integral integrallash yo‘liga bog‘liq bo‘lmaydi va maydonning potensial funktsiyasi $U=(x;y;z)$ aniqlangan bo‘lsa, uning chiziqli integralini hisoblash uchun

$$\int_{AB} Pdx+Qdy+Rdz=\int_{A}^{B} du=U(B)-U(A)$$

formula o‘rinlidir.

## 9.4. Gamilton (nabla) operatori

Simvolik tarzda

$$\nabla=\vec{i}\,\frac{\partial}{\partial x}+\vec{j}\,\frac{\partial}{\partial y}+\vec{k}\,\frac{\partial}{\partial z}$$

ko‘rinishda kiritilgan ifodani (vektorni) *Gamil’ton* (*nabla*) *operatori* (*vektori*) deb ataladi.

1) Agar biror $D$ sohada uzluksiz xususiy xossalari mABjud bo‘lgan uch o‘zgaruvchili $U(x;y;z)$ funktsiya berilgan bo‘lsa, unga Gamilton operatorini qo‘llash natijasi

$$\nabla = \vec{i}\,\frac{\partial u}{\partial x} + \vec{j}\,\frac{\partial u}{\partial y} + \vec{k}\,\frac{\partial u}{\partial z} = gradU$$

dan iborat ekanligini ko‘rish oson.

2) Agar biror $D$ sohada proeksiyasining mos uzluksiz xususiy hosilasi mABjud bo‘lgan (9.1.1) vektor funktsiya berilgan bo‘lsa, uning nabla vektor bilan skalyar ko‘paytmasini qarasak,

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = di\vartheta\vec{F}$$

Bo‘lishiga ishonch hosil qilish qiyin emas. Bu holda vektor funktsiyaning (maydonning) divergentsiyasiga ega bo‘lishimizni ko‘ramiz.

3) Endi $\nabla$ simvolik vektor bilan (9.1.1) vektor ko‘paytmasini hisoblasak,

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = rot\vec{F}$$

bo‘lishiga ishonch hosil qilamiz. Demak, bu holda vektor maydonning rotoriga (uyurmasiga) ega bo‘lamiz.

4) Agar biror $D$ sohada (9.1.1) bilan aniqlangan vektor maydon potensial bo‘lsa, $\vec{F}$ vektor funktsiya maydon potensial funktsiyasining gradiyentidan iborat bo‘lishini ko‘ramiz:

$$\vec{F} = gradU$$

yaoni

$$\vec{F} = \vec{i}\,\frac{\partial u}{\partial x} + \vec{j}\,\frac{\partial u}{\partial y} + \vec{k}\,\frac{\partial u}{\partial z}.$$

Bu vektor uchun (9.3.2) shartning bajarilishini tekshirib ko‘rish oson (mustaqil bajarish) yaoni $rot\vec{F} = 0$ bo‘lishi aniqdir:

$$rot(gradU) = 0 \qquad (9.4.1)$$

(9.4.1) ni $\nabla$ operator yordamida

$$\nabla \times \nabla u = 0$$

ko‘rinishda ham yoki vektor ko‘paytma xossasidan foydalanib,

$$(\nabla \times \nabla)u = 0$$

kabi ham yozish mumkin.

Shunday qilib, biror skalyar maydon gradiyenti vositasida tuzilgan vektor maydon potensial bo‘lishini ko‘ramiz.

5) (9.1.1) vositasida $\vec{F}$ vektor maydon berilgan bo‘lib, $\vec{F}$ ning proeksiyasi mos uzluksiz xususiy hosilaga ega bo‘lsa,

$$di\vartheta\left(rot\vec{F}\right) = 0 \qquad (9.4.2)$$

bo‘lishini ko‘rsataylik. Agar $\nabla$ operatordan va vektorlarning aralash ko‘paytmasi xossasidan foydalansak,

$$\nabla \cdot (\nabla \times F) = \nabla\nabla F = 0$$

bo‘lishini ko‘rish qiyin emas.

Bundan biror $D$ sohada (9.1.1) vositasida aniqlangan vektor maydon rotori (uyurmasi) tuzilgan vektor maydon solenoidan bo‘lishini ko‘ramiz.

6) Aytaylik, biror bog‘liq $D$ sohada aniqlangan $U = U(x; y; z)$ funktsiya vositasida skalyar maydon berilgan bo‘lsin. Gradintlar maydonini aniqlaymiz:

$$gradU = \vec{i}\,\frac{\partial u}{\partial x} + \vec{j}\,\frac{\partial u}{\partial y} + \vec{k}\,\frac{\partial u}{\partial z}\,.$$

Bu gradiyent vositasida olingan vektor maydon divergentsiyasini hisoblaylik:

$$di\vartheta(gradU) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\,.$$

Bu tenglikning o‘ng tomoni $U(x; y; z)$ funktsiyaning Laplas operatori deb ataladi va uning uchun $\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ belgilashni qabul qilingan bo‘lib, uni $U$ funktsiyaga qo‘llash natijasini

$$\nabla U = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

kabi yoziladi. Oxirgilarni hisobga olgan holda Laplas operatorini qo‘llashni

$$\nabla U = di\vartheta(gradU)$$

ko‘rinishda yoki $\nabla$ operator yordamida

$$\nabla u = (\nabla \xi \nabla u) = \nabla^2 u$$

kabi yozish mumkinligini ko‘ramiz.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (9.4.3)$$

Yoki

$$\nabla U = 0$$

ni *Laplas tenglamasi* deb, bu tenglamani qanoatlantiruvchi $U(x; y; z)$ funktsiyani esa *garmonik funktsiya* deb ataymiz.

Ham potensial ham solenoidal bo‘lgan vektor maydonni *garmonik vektor maydon* deb ataladi. Bunday maydonning potensial funktsiyasi garmonik funktsiya bo‘lishini ko‘rsatish qiyin emas (mustaqil bajaring).

## 9-bobga doir mashqlar

1. Gradiyentning

1) $grad(cu) = cgradu, \quad c = const$,

2) $grad\left(\sum_{i=1}^{m} c_i u_i\right) = \sum_{i=1}^{m} c_i gradu_i\,, \quad c_i = const$,

3) $grad(u\vartheta) = ugrad\vartheta + \vartheta gradu$

xossalarini isbotlang.

2. $r=\sqrt{x^2+y^2+z^2}$ funktsiya uchun $gradr$, $gradr^2$, $grad\frac{1}{r}$, $grad\, f(r)$ larni toping, bu yerda $f(r)$ qandaydir differensiallanuvchi funktsiya.

Javob. $\frac{\vec{r}}{r}$; $2r$; $-\frac{\vec{r}}{r^3}$; $f'(r)\frac{\vec{r}}{r}$ , bu yerda $\vec{r}$ $(x;y;z)$ nuqtaning radius vektori.

3. $di\vartheta(\vec{A}+\vec{B})=di\vartheta\vec{A}+di\vartheta\vec{B}$ - divergentsiya xossasini isbotlang.

4. $\vec{r}=\{x;y;z\}$ bo‘lsa, $di\vartheta\vec{r}$ ni hisoblang.

Javob: 3.

5. $rot(c_1\vec{A_1}+c_2\vec{A_2})=c_1rot\vec{A}+c_2rot\vec{A_2}$ xossani isbotlang, bu yerda $c_1$ va $c_2$ lar o‘zgarmaslardir.

## 9-bob bo‘yicha bilimingizni sinab ko‘ring.

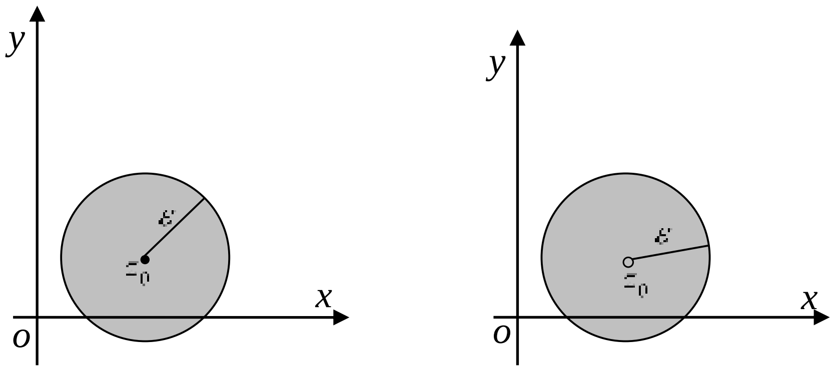
1. Skalyar maydon nimaligini tuShuntiring.
2. Vektor maydon deb nimaga aytiladi?
3. Vektor maydonning vektor chizig‘i deyilganda nimani tuShunasiz?
4. Vektor sirt nima?
5. Vektor quvurni tuShuntiring.
6. Vektor maydonning chiziqli integralini va tsirkulyatsiyasini tuShuntiring.
7. Vektor oqimi nimadan iborat?
8. Yopiq sirt bo‘yicha vektor oqimining maonosini tuShuntiring.
9. Vektor maydon divergentsiyasini yozing va uni tuShuntiring.
10. Solenoidal vektor maydonni bayon qiling.
11. Vektor maydon rotori (uyurmasi) ni yozing va uni tuShuntiring.
12. Stoks formulasining vektor shaklini yozing va uni tuShuntiring.
13. Potensial vektor maydon deyilganda nimani tuShunasiz?
14. Potensial vektor maydonning potensial funktsiyasi nima va u qanday topiladi?
15. Gamilton (nabla) operatorini va uning tatbiqlarini bayon qiling.
16. Laplas operatorini yozing.
17. Laplas tenglamasini yozing.
18. Garmonik vektor maydon nima?
19. Garmonik maydonning garmonik funktsiyasi deyilganda nimani tuShunasiz?
20. Garmonik funktsiya qaysi tenglamani qanoatlantiradi?

# 10-bob. Kompleks o‘zgaruvchili funksiyalar nazariyasining elementlari

## 10.1. Kompleks taxlilning asosiy tushunchalari

Mazkur bandda kompleks tekislikda olib boriladigan mulohazalar jarayonida zarur bo‘ladigan tushunchalar bilan tanishamiz.

**10.1.1-ta’rif.** Kompleks tekislikda berilgan $Z_0$ nuqtaning $\varepsilon\ (\varepsilon > 0)$ atrofi ($\varepsilon$ yaqin atrofi) deb, uning $|z - z_0| < \varepsilon\ \ (0 < |z - z_0| < \varepsilon)$ tengsizlikni qanoatlantiruvchi barcha nuqtalarining to‘plamiga, ya’ni markazi $Z_0$ nuqtada bo‘lgan $\varepsilon$ *radiusli doiraga* (*markazsiz doiraga*) aytamiz (10.1.1-rasm).




$\varepsilon$-atrof. $\varepsilon$- yaqin atrof.

10.1.1-rasm.

**10.1.2-ta’rif.** Agar $Z_0$ nuqtaning ixtiyoriy yaqin atrofida kompleks tekislikka tegishli $D$ to‘plamning elementi (nuqtasi) mavjud bo‘lsa, $Z_0$ ni $D$ to‘plamning *limitik nuqtasi* deymiz.

Bu ta’rifdan kompleks tekislikdagi $D$ to‘plamning limitik nuqtasi uning o‘ziga tegishli bo‘lishi ham tegishli bo‘lmasligi ham mumkin ekanligini ko‘ramiz. Bu o‘rinda to‘plamning limitik nuqtasini ba’zan uning *quyuqlik nuqtasi* deb ham yuritilishini aytamiz.

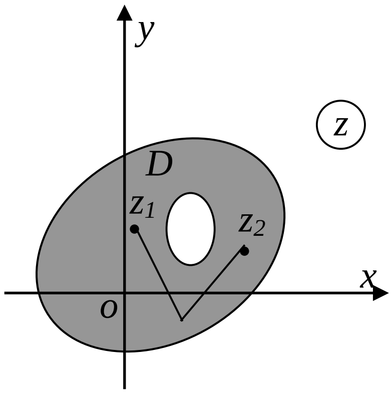
**10.1.3-ta’rif.** Agar kompleks tekislikdagi $D$ to‘plam o‘zining $Z_0$ nuqtasini uning biror $\varepsilon$ atrofi bilan birgalikda o‘z ichiga olsa, u holda $Z_0$ nuqtani $D$ to‘plamning *ichki nuqtasi* deyiladi.

**10.1.4-ta’rif.** Kompleks tekislikda *soha* deb, quyidagi xossalarga ega bo‘lgan $D$ to‘plamga aytiladi:

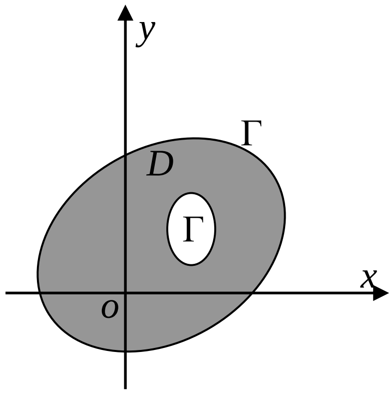
1) $D$ to‘plamning barcha nuqtalari ichki – buni $D$ to‘plamning *ochiqlik xossasi* deyiladi;

2) $D$ to‘plamning har qanday ikkita nuqtasini $D$ to‘plamda to‘lig‘icha yotuvchi qandaydir siniq chiziq bilan tutashtirish mumkin – buni $D$ to‘plamning *bog‘liklik xossasi* deyiladi.

10.1.2-rasmda $D$ sohaning kompleks tekislikdagi geometrik tasviri ifodalangan.




10.1.2-rasm.




10.1.3-rasm.

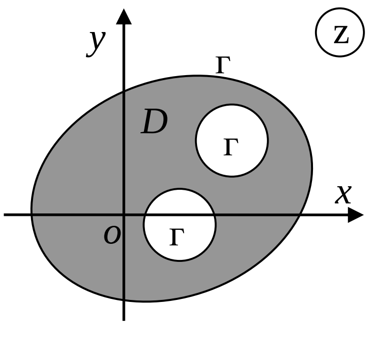
**10.1.5-ta'rif.** Kompleks tekislikdagi $D$ *sohaning chegaraviy nuqtasi* deb, uning o'ziga tegishli bo'lmagan limitik nuqtasiga aytiladi, bunday chegaraviy nuqtalarning to'plamini esa $D$ *sohaning chegarasi* deb ataymiz va uni $\Gamma$ orqali belgilaymiz (10.1.3.-rasm).

**10.1.6-ta'rif.** Kompleks tekislikdagi $D$ sohaga o'zining chegarasidan iborat bo'lgan $\Gamma$ to'plamni birlashtirib olingan to'plam *yopiq soha* deyilib, uni $\overline{D}$ orqali belgilanadi ($\overline{D} = D \cup \Gamma$ ).

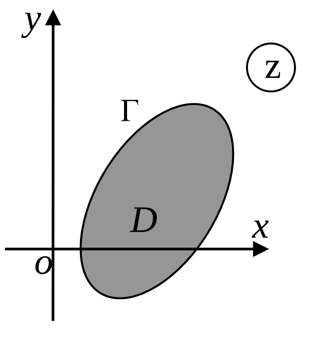
**10.1.7-ta'rif.** Agar kompleks tekislikda $D$ sohani o'z ichiga oluvchi chekli radiusli doira mavjud bo'lsa, $D$ *sohani chegaralangan* deyiladi.

**10.1.8-ta'rif.** Kompleks tekislikdagi $D$ sohaning *bog'lanish tartibi* deb, uning chegarasini tashkil etuvchi konturlar sistemasidagi *konturlar soniga* aytamiz. Faqat bitta kontur bilan chegaralangan sohani *bir bog'lamli* deb ataymiz.

10.1.4-rasmda uch bog'lamli 10.1.5-rasmda esa bir bog'lamli soha tasvirlangan.



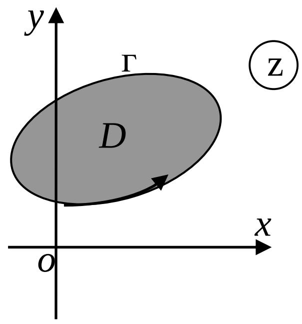

10.1.4-rasm.



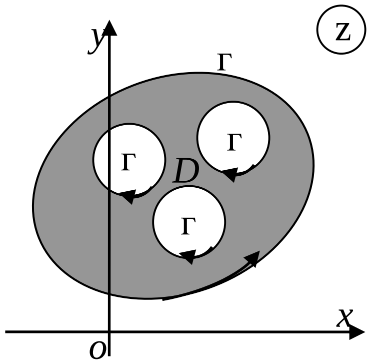

10.1.5-rasm.

**10.1.9-ta'rif.** Kompleks tekislikdagi *soha chegarasining musbat yo'nalishi* deb, uni tashkil etuvchi konturlar sistemasiga tegishli bo'lgan har bir kontur yo'nalishi bo'yicha yurilganda soha chap tomonda joylashgan bo'ladigan yo'nalishga aytiladi.

Masalan, 10.1.6-rasmda bir bog‘lamli, 10.1.7-rasmda esa 4 bog‘lamli soha uchun chegaraning musbat yo‘nalishi ko‘satkich vositasida tasvirlangan.




10.1.6-rasm.




10.1.7-rasm.

## 10.2. Kompleks o‘zgaruvchili funksiya

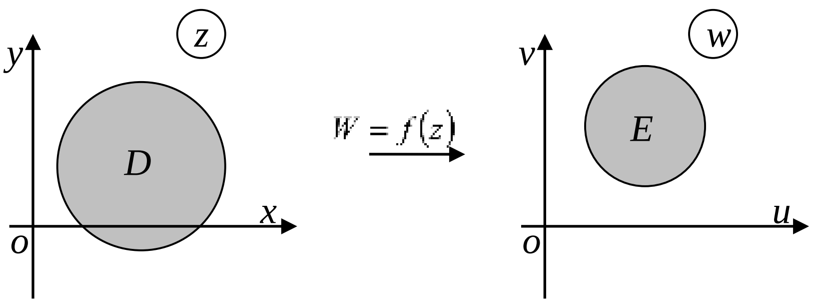
Haqiqiy o‘zgaruvchili funksiyaning tahlilida ko‘rilgan asosiy tushunchalar (masalan: funksiya, uning limiti, uzluksizligi, hosila, integral va boshqalar) kompleks o‘zgaruvchili funksiyaning taxlilida ham deyarlik o‘zgarishsiz qabul qilinadi.

**10.2.1-ta’rif.** Agar $z$ o‘zgaruvchining kompleks tekislikda berilgan $D$ to‘plamga tegishli har bir qiymatiga $W$ o‘zgaruvchining aniq bitta kompleks qiymati mos qo‘yilgan bo‘lsa, u holda $W$ o‘zgaruvchini $z$ *kompleks o‘zgaruvchining funksiyasi* deyiladi va uni $W = f(z)$ kabi belgilaymiz. $D$ to‘plamni funksiyaning *aniqlanish sohasi*, funksiyaning qiymatlari to‘plamini $E$ orqali belgilab, uni funksiyaning *o‘zgarish (qiymatlar) sohasi* deb atalishini hamda ular uchun mos ravishda $D(W)$ va $E(W)$ belgilardan foydalanishimizni aytamiz.

Kompleks sonning ta’rifiga asosan $z = x + iy$ va $W = u + iv$ belgilashlar kiritsak, bu yerda $x, y, u, v \in R$ deb faraz qilinadi, kompleks o‘zgaruvchili funksiyani

$$W = f(x) = f(x + iy) = u(x; y) + iv(x; y) \qquad (10.2.1)$$

ko‘rinishda yoza olamiz. (10.2.1) dan bitta $W = f(z)$ kompleks o‘zgaruvchili funksiyaning berilishi haqiqiy o‘zgaruvchili $u = u(x; y)$ va $v = v(x; y)$ ikkita funksiyalarning berilishiga ekvivalent bo‘lishini ko‘ramiz.




10.2.1-rasm.

(10.2.1) kompleks o‘zgaruvchili funksiyaning geometrik tasviri ⓩ kompleks tekislikda berilgan $D$ sohani $W = f(z)$ funksiya vositasida ⓦ kompleks tekislikdagi qandaydir E sohaga akslantirish natijasi deb qarash mumkinligi 10.2.1-rasmda o‘z aksini topgan.

## 10.3. Kompleks sonli ketma-ketlik va uning limiti

Aytaylik, $\{z_n\}$ ketma-ketlik berilgan bo‘lib, $z_n \in K$ bo‘lsin, u holda $\{z_n\}$ ni *kompleks sonli ketma-ketlik* deb ataymiz va uning limiti tushunchasini haqiqiy sonli ketma-ketlikdagi kabi kiritamiz.

**10.3.1-ta’rif.** Berilgan $\{z_n\}$ kompleks sonli ketma-ketlik uchun Shunday $z_0$ kompleks son mavjud bo‘lib, $\forall \varepsilon > 0$ son uchun $\exists n_0 \in N \Rightarrow \forall n_0 < n \in N$, $|z_n - z_0| < \varepsilon$ bajarilsa, bu holda $z_0$ kompleks sonni $\{z_n\}$ *kompleks sonli ketma-ketlikning limiti* deb ataymiz va $z_n \to z_0$ yoki $\lim z_n = z_0$ kabi yozishni qabul qilamiz.

Bu ta’rifdan ko‘rinadiki, agar $z_n \to z_0$ bo‘lsa, geometrik jihatdan kompleks tekislikda markazi $z_0$ nuqtada bo‘lgan doiraning radiusi $\varepsilon > 0$ har qancha kichik olinmasin, biror nomerdan boshlab $\{z_n\}$ ketma-ketlikning keyinga barcha hadlari Shu doira ichiga joylashadi.

Kompleks sonli ketma-ketlik uchun ham haqiqiy sonli ketma-ketliklarda ko‘rilgan tushunchalar mavjuddir. Masalan, agar $z_n \to 0$ bo‘lsa, $\{z_n\}$ kompleks sonli ketma-ketlik *cheksiz kichik (yoki cheksiz kichik miqdor)* deb ataladi; agar $\{z_n\}$ kompleks sonli ketma-ketlik uchun Shunday $M > 0$ haqiqiy son mavjud bo‘lib, $|z_n| < M$ tengsizlik $\forall n \in N$ bo‘lganda bajarilsa, $\{z_n\}$*kompleks sonli ketma-ketlik chegaralangan* deyiladi; agar $\{z_n\}$ kompleks sonli ketma-ketlik berilgan bo‘lib, $\forall M > 0$ uchun $\exists\, n_0 \in N \Rightarrow \forall n_0 < n \in N$, $|z_n| > M$ bajarilsa, $\{z_n\}$ *cheksiz katta ketma-ketlik (yoki cheksiz katta miqdor)* deb ataladi va bu holda $\lim z_n = \infty$ yoki $z_n \to \infty$ kabi yoziladi. Bu o‘rinda haqiqiy sohada ikkita $+\infty$ va $-\infty$ (cheksiz uzoqlashgan (chetlashgan)) nuqtalarni farqlash qabul qilinganligini eslatgan holda, kompleks tekislikda faqat bitta (cheksiz chetlangan) nuqta mavjud deb qabul qilinganligini va uni $\infty$ (ishorasiz cheksiz) kabi belgilanishini aytamiz.

Kompleks tekislikda $\lim z_n = \infty \Leftrightarrow \lim |z_n| = +\infty$ bo‘lib, haqiqiy sohadagi cheksiz kichik va cheksiz katta miqdorlarning xossalari kompleks tekislikda ham o‘z kuchida qolishini aytamiz. Bu o‘rinda $z_n \to \infty$ bo‘lishi geometrik jihatdan kompleks tekislikdagi markazi koordinata boshida ($z = 0$ nuqtada) bo‘lgan doiraning radiusi M har qancha katta olinmasin biror $n_0$ nomerdan keyingi $\{z_n\}$ kompleks sonli ketma-ketlikning barcha hadlari bu doiradan tashqarida joylashgan ekanligini anglatishini takidlaymiz.

## 10.4. Kompleks o‘zgaruvchili funksiyaning limiti, uzluksizligi va hosilasi

**10.4.1-ta’rif.** Agar $f(z)$ kompleks o‘zgaruvchili funksiya kompleks tekislikning $z_0$ nuqtasining biror yaqin atrofida aniqlangan bo‘lib, $\forall\varepsilon > 0$ uchun $\exists\delta > 0,\ 0 < |z - z_0| < \delta$ bo‘lganda $|f(z) - A|$ bajarilsa, u holda *A kompleks son* $f(z)$ *funksiyaning* $z_0$ *nuqtasidagi (yoki* $z \to z_0$ *dagi) limiti* deyiladi va $\lim\limits_{z\to z_0} f(z) = A$ ko‘rinishda yoziladi.

Bu ta’rifdan va (10.2.1) dan $\lim\limits_{z\to z_0} f(z)$ limitning $z_0 = x_0 + iy_0$ nuqtada mavjud bo‘lishi $u = u(x; y)$ va $v = v(x; y)$ funksiyadarning $(x_0 ; y_0)$ nuqtadagi limitlarining mavjud bo‘lishiga ekvivalent bo‘lib,

$$\lim_{z\to z_0} f(z) = \lim_{\substack{x\to x_0 \\ y\to y_0}} u(x; y) + i \lim_{\substack{x\to x_0 \\ y\to y_0}} v(x; y)$$

o‘rinli bo‘lishini ko‘ramiz.

Bu o‘rinda

$$\lim_{z\to\infty} f(z) = w_0,\quad \lim_{z\to z_0} f(z) = \infty,\quad \lim_{z\to\infty} f(z) = \infty$$

hollar haqida fikrlab ko‘rish va ularning ma’nolarini ochishni o‘quvchining o‘ziga qoldiramiz.

**10.4.2-ta’rif.** Agar $f(z)$ funksiya kompleks tekislikning $z_0$ nuqtasining biror atrofida aniqlangan bo‘lib, $\lim\limits_{z\to z_0} f(z) = f(z_0)$ bajarilsa, u holda $f(z)$*funksiya* $z_0$ *nuqtada uzluksiz* deyiladi.

Bu ta’rifdan va (10.2.1) dan $f(z)$ kompleks o‘zgaruvchili funksiyaning $z_0 = x_0 + iy_0$ nuqtadagi uzluksizligi ikkita $u = u(x; y)$ va $v = v(x; y)$ ikki o‘zgaruvchili funksiyalarning $(x_0 ; y_0)$ nuqtadagi uzluksizligi bilan ekvivalent ekanligini ko‘ramiz.

**10.4.3-ta’rif.** Agar $f(z)$ kompleks o‘zgaruvchili funksiya $D$ sohaning barcha nuqtalarida uzluksiz bo‘lsa, u holda uni $D$ *sohada uzluksiz* deyiladi.

Yuqorida keltirilgan $f(z)$ kompleks o‘zgaruvchili funksiyaning limiti va nuqtadagi uzluksizligi tushunchalari uning aniqlanish sohasiga tegishli bo‘lgan ichki nuqta uchun ta’riflangandir. Agar $f(z)$ yopiq $\overline{D}$ sohada aniqlangan bo‘lib, $z_0$ $\overline{D}$ ning chegaraviy nuqtasi bo‘lsa, bu nuqtadagi funksiya limitini (va uzluksizligini)

$$\lim_{\substack{z\to z_0 \\ z\in \overline{D}}} f(z) \qquad \left( \lim_{\substack{z\to z_0 \\ z\in \overline{D}}} f(z) = f(z_0) \right)$$

ma’nosida tuShunmoq lozimdir.

**10.4.4-ta’rif.** Agar $f(z)$ kompleks o‘zgaruvchili funksiya $\overline{D}$ yopiq sohada aniqlangan bo‘lib, u $D$ sohada va uning chegaraviy nuqtalarining barchasida uzluksiz bo‘lsa, u holda bu funksiya $\overline{D}$ *yopiq sohada uzluksiz* deyiladi.

Bu o‘rinda yopiq sohada uzluksiz bo‘lgan kompleks o‘zgaruvchili funksiya ham yopiq oraliqda (kesmada) uzluksiz bo‘lgan haqiqiy o‘zgaruvchili funksiyaning xossalariga ega bo‘lishini aytamiz.

Endi $w = f(z)$ kompleks o‘zgaruvchili funksiya kompleks tekislikning $z_0$ nuqtasining biror atrofida aniqlangan va $z$ bu atrofga tegishli bo‘lsin deb faraz qilaylik, u holda $z - z_0 = \Delta z$ kabi belgilab, uni $z_0$ *nuqtadagi kompleks argumentning orttirmasi*, $f(z_0 + \Delta z) - f(z_0) = \Delta w$ ni esa *funksiya orttirmasi* deb ataymiz (10.4.1-rasm).

**10.4.5-ta’rif.** Agar $w = f(z)$ funksiya kompleks tekislikning $z_0$ nuqtasining biror atrofida aniqlangan bo‘lib,

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

chekli limit mavjud bo‘lsa, uni $w = f(z)$ funksiyaning $z_0$ *nuqtadagi hosilasi* deb ataymiz va $\frac{dw}{dz}$ yoki $\frac{df(z_0)}{dz}$ yoki $f'(z_0)$ kabi belgilaymiz. Bu holda $f(z)$ ni $z_0$ *nuqtada differensiallanuvchi* deb ataladi.

Agar (10.2.1) ni va kompleks sonlar ustidagi amallarni hisobga olsak,

$$\begin{aligned} &z_0 = x_0 + iy_0, \quad z = x + iy, \\ &\Delta z = z - z_0 = (x - x_0) + i(y - y_0) = \Delta x + i\Delta y, \\ &\Delta w = f(z_0 + \Delta z) - f(z_0) = \Delta u + i\Delta v, \\ &\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta 2 \to 0, \\ \Delta y \to 0}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}. \end{aligned} \qquad (10.4.1)$$

Endi (10.4.1) da $\Delta y = 0$ deb olsak,

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad (10.4.2)$$

ni, $\Delta x = 0$ deb faraz qilib esa

$$\frac{dw}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad (10.4.3)$$

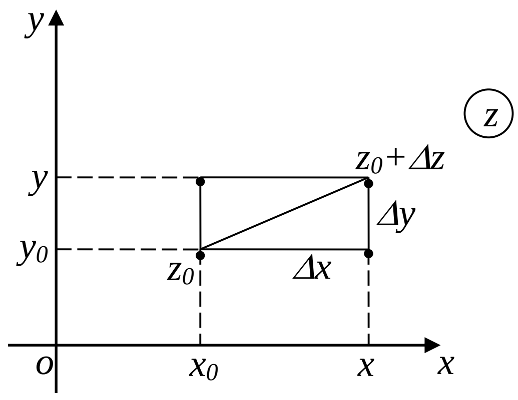
ni olamiz. (10.4.2) va (10.4.3) tengliklardan

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (10.4.4)$$

kelib chiqadi. (10.4.4) *Koshi-Riman shartlari* deb ataladi va ular kompleks o‘zgaruvchili funksiyaning differensiallanuvchi bo‘lishining zaruriy shartidan iboratdir. Ba‘zi bir qo‘shimchalar, masalan, $u(x;y)$ va $\vartheta(x;y)$ funktsiyalar differensialga ega bo‘lishi (10.4.4) ning yetarli bo‘lishini ta’minlaydi

Bu o‘rinda haqiqiy o‘zgaruvchili funksiya uchun o‘rinli bo‘lgan differensiallashning barcha xossalari va qoidalari kompleks o‘zgaruvchili funksiya uchun ham saqlanib qolishini aytamiz.

**10.4.6-ta’rif.** Agar $f(z)$ funksiya kompleks tekislikka tegishli bo‘lgan biror $D$ sohaning ixityoriy nuqtasida differensiallanuvchi bo‘lsa, uni $D$ *sohada analitik (golomorf yoki regulyar yoki monogen) funksiya* deb ataladi.




10.4.1-rasm.

Analitik funksiyaning garmonik funksiya bilan aloqasi bor ekanligi Koshi-Riman shartlaridan bevosita kelib chiqadi.

Ma’lumki, garmonik funksiya deb, biror $D$ sohada ikkinchi tartibli xususiy hosilalari mavjud bo‘lgan

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Laplas tenglamasini qanoatlantiruvchi $u = u(x;y)$ - ikki o‘zgaruvchili funksiyaga aytiladi.

Agar biror sohada (10.4.4) Koshi-Riman shartlari bajarilsa, u holda $w = f(z)$ kompleks o‘zgaruvchili funksiya analitik bo‘lib, uning haqiqiy va mavhum qismlaridan iborat bo‘lgan $u(x;y)$ va $v(x;y)$ funksiyalar ikkinchi tartibli xususiy hosilalarga ega deyilgan faraz asosida, ularning ikkalasi ham Laplas tenglamasini qanoatlantirishini keltirib chiqarish qiyin emas (buni mustaqil bajaring). Bunday holda $u(x;y)$ va $v(x;y)$ lar *o‘zaro qo‘shma garmonik funksiyalar* deb ataladi. Shuningdek, bu o‘zaro qo‘shma garmonik funksiyalar berilgan bo‘lsa, ular vositasida olingan $w = f(z) = u + iv$ kompleks o‘zgaruvchili funksiya analitik ekanligi ravshandir. Shu sababli, biror $D$ sohada o‘zaro qo‘shma garmonik bo‘lgan $u(x;y)$ va $v(x;y)$ funksiyalar $w = f(z) = u + iv$ analitik funksiyaning *tashkil etuvchilari (komponentlari)* deb ham yuritiladi.

Agar analitik funksiyaning tashkil etuvchilaridan biri berilgan bo‘lsa, ikkinchisini Koshi-Riman shartlaridan foydalanib topish mumkin. Masalan, (10.2.1) ko‘rinishda yozilgan analitik funksiyaning haqiqiy qismidan iborat bo‘lgan $u(x;y)$ kompleks tekislikka tegishli biror $D$ sohada berilgan va ikkinchi tartibligacha xususiy hosilalari mavjud funksiya deb faraz qilaylik. U vaqtda (10.4.4) Koshi-Riman shartlarining birinchi tengligi asosida $(x_0\,;y_0) \in D$ deb faraz qilib,

$$v(x;y) = \int_{y_0}^{y} \frac{\partial u(x;\eta)}{\partial x} d\eta + C(x)$$

ni olamiz. Buni $x$ bo‘yicha differensiallab,

$$\frac{\partial v}{\partial x} = \int_{y_0}^{y} \frac{\partial^2 u(x;\eta)}{\partial x^2} d\eta + C'(x)$$

ga ega bo‘lamiz. $u(x;y)$ - garmonik funksiya ekanligidan, u Laplas tenglamasini qanoatlantirishi tufayli va (10.4.4) Koshi-Riman shartlarining ikkinchi tenglamasi asosida

$$-\frac{\partial u(x;y)}{\partial y}=-\int_{y_0}^{y}\frac{\partial^2 u(x;\eta)}{\partial y^2}d\eta+C'(x)\Rightarrow$$

$$-\frac{\partial u(x;y)}{\partial y}=-\frac{\partial u(x;y)}{\partial y}+\frac{\partial u(x;y_0)}{\partial y}+C'(x)\Rightarrow$$

$$C'(x)=-\frac{\partial u(x;y_0)}{\partial y}$$

ni olamiz. Bundan

$$C(x)=-\int_{x_0}^{x}\frac{\partial u(\xi;y_0)}{\partial y}d\xi+C_0$$

ga kelamiz, bu yerda $C_0$ - ixtiyoriy o‘zgarmas. Yuqoridagilarga binoan $v(x;y)$ ni topish uchun

$$v(x;y)=\int_{y_0}^{y}\frac{\partial u(x;\eta)}{\partial x}d\eta-\int_{x_0}^{x}\frac{\partial u(\xi;y_0)}{\partial y}d\xi+C_0 \qquad (10.4.5)$$

ni olamiz.

**Misol.** $u(x;y)=e^x\cos y$ funksiya kompleks tekislikda garmonik ekanligiga ishonch hosil qilish oson. Agar $u(x;y)$ $w=f(z)=u+iv$ analitik funksiyaning haqiqiy qismi deb faraz qilsak, mavhum qismi $v$ ni (10.4.5) formuladan foydalanib topamiz:

$$v(x;y)=\int_{0}^{y}e^x\cos\eta d\eta-\int_{0}^{x}\left(-e^{\xi}\cdot\sin 0\right)d\xi+C_0=e^x\cdot\sin y+C_0$$

Agar $C_0=0$ deb faraz qilsak, $v(x;y)=e^x\sin y$ ga kelamiz. U holda analitik funksiya uchun

$$f(z)=e^x\cos y+ie^x\sin y=e^x(\cos y+i\sin y)=e^{x+iy}=e^z$$

ko‘rsatkichli funksiyaga kelamiz.

## 10.5. Kompleks o‘zgaruvchili funksiyaning integrali

Kompleks tekislikda analitik funksiyaning xossalarini o‘rganish jarayonida kompleks funksiyaning integrali tushunchasi muhim ahamiyatga egadir.

Aytaylik, kompleks tekislikda biror bog‘liq va o‘lchanuvchi $D$ sohada bir qiymatli va uzluksiz bo‘lgan

$$w=f(z)=f(x+iy)=u(x;y)+iv(x;y)$$

funksiya va bu sohaga tegishli bo‘lakli silliq $C$ egri chiziq o‘zining chetki $A$ va $B$ nuqtalari bilan berilgan bo‘lsin. $C$ egri chiziqni qandaydir usul bilan $A$ chetki nuqtasidan $B$ chetki nuqtasi tomon yo‘nalgan $A=z_0$, $z_1,z_2,...,z_m=B$ kabi belgilangan nuqtalarini olib, $m$ ta bo‘lakka ajratib va

$z_k=x_k+iy_k$, $\Delta x_k=x_k-x_{k-1}$, $\Delta y_k=y_k-y_{k-1}$, $\Delta z_k=z_k-z_{k-1}=\Delta x_k+i\Delta y_k$

belgilashlar kiritib $\left(k=\overline{1;m}\right)$, har bir $k$-bo‘lakdan bitta $\xi_k$ nuqtani olib,

$$S_m=\sum_{k=1}^{m} f(\xi_k)\Delta z_k \qquad (10.5.1)$$

yig‘indini tuzaylik. (10.5.1) ni $f(z)$ funksiyaning $C$ egri chiziqni yuqorida aytilgandek bo‘laklarga ajratish bo‘yicha *integral yig‘indisi* deyiladi.

**10.5.1-ta’rif.** Agar $f(z)$ funksiyaning $C$ egri chiziqni ixtiyoriycha qilib bo‘laklarga ajratish bo‘yicha integral yig‘indisining $\max\limits_{k=\overline{1;m}}|\Delta z_k|\to 0$ dagi chekli limiti mavjud bo‘lib, bu limit $C$ egri chiziqni bo‘laklarga ajratish usuliga va har bir $k$-bo‘lakdan olingan $\xi_k$ nuqtaning o‘rniga bog‘liq bo‘lmasa, uni $f(z)$ *funksiyaning* $C$ *egri chiziq bo‘yicha integrali* deyiladi va $\int\limits_C f(z)dz$ kabi belgilanadi. Bu holda $f(z)$ kompleks o‘zgaruvchili funksiyani $C$ egri chiziq bo‘yicha *integrallanuvchi* deb ataymiz.

Shunday qilib, $f(z)$ kompleks o‘zgaruvchili funksiya $C$ egri chiziq bo‘yicha integrallanuvchi bo‘lsa, yuqoridagi ta’rifga asosan va $\lambda=\max\limits_{k=\overline{1;m}}|\Delta z_k|$ deb belgilasak,

$$\int\limits_C f(z)dz=\lim_{\lambda\to 0}\sum_{k=1}^{m} f(\varsigma_k)\Delta z_k$$

o‘rinli bo‘lib, unda $\Delta z_k=\Delta x_k+i\Delta y_k$, $\varsigma_k=\xi_k+i\eta_k$ va $f(\varsigma_k)=u(\xi_k;\eta_k)+iv(\xi_k;\eta_k)$ larni e’tiborga olgan holda integral yig‘indining haqiqiy va mavhum qismlarini ajratish natijasida

$$\int\limits_C f(z)dz=\lim_{\lambda\to 0}\sum_{k=1}^{m}\left[u(\xi_k;\eta_k)\Delta x_k-v(\xi_k;\eta_k)\Delta y_k\right]+$$

$$+i\lim_{\lambda\to 0}\sum_{k=1}^{m}\left[v(\xi_k;\eta_k)\Delta x_k+u(\xi_k;\eta_k)\Delta y_k\right]$$

tenglikni olamiz. Bundan esa $\int\limits_C f(z)dz$ ning ikkita egri chiziqli integral (ikkinchi jins) vositasida ifodalanishi kelib chiqadi.

$$\int\limits_C f(z)dz=\int\limits_C (udx-vdy)+i\int\limits_C (vdx+udy) \qquad (10.5.2)$$

Olingan (10.5.2) formulada tenglikning o‘ng tomomnidagi integrallar $C$ egri chiziq bo‘yicha integrallardir (ikkinchi jins). Undan foydalangan holda kompleks funksiyaning $C$ egri chiziq bo‘yicha integralining xossalarini keltirib chiqarish mumkin. Masalan,

1) $\int\limits_C [af(z)+b\varphi(z)]dz=a\int\limits_C f(z)dz+b\int\limits_C \varphi(z)dz$, $\quad a=const,\ b=const$

2) $C=C_1\cup C_2$; $\int\limits_C f(z)dz=\int\limits_{C_1\cup C_2} f(z)dz=\int\limits_{C_1} f(z)dz+\int\limits_{C_2} f(z)dz$

3) $\int\limits_C f(z)dz=-\int\limits_{C^-} f(z)dz$, bu yerda $C^-$ orqali $C$ egri chiziq bilan ustma-ust tuShuvchi, ammo u bilan qarama-qarshi yo‘nalishli egri chiziq belgilangan.

4) $\left|\int_C f(z)dz\right| \le \int_{C^-} |f(z)||dz| \le Ml$, bu yerda $M = \sup_{z\in c}|f(z)|$ va $l$ - esa $C$ egri chiziqning o‘lchovi (uzunligi).

Endi $f(z)$ kompleks o‘zgaruvchili funksiya biror $D$ sohada analitik bo‘lgan holni qarasak, bu sohada (10.4.4) Koshi-Riman shartlari bajariladi va (10.5.2) tenglikning o‘ng tomonidagi egri chiziqli integrallarning ikkalasi ham integrallash yo‘liga bog‘liq emasligini ko‘rish mumkin. Haqiqatan ham ulardan birinchisi $\int_C (udx - vdy)$ ni olsak, unda $P = u$, $Q = -v$ deb faraz qilsak,

$$\forall(x;y)\in D,\ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

bo‘lishini (10.4.4) Koshi-Riman shartlaridan ikkinchisi vositasida ko‘ramiz. Xuddi Shunga o‘xshash ishni (10.3.2) tenglikning o‘ng tomonidagi ikkinchi egri chiziqli integral uchun ham bajarish mumkin (mustaqil Shug‘ullaning). Bular asosida biror $D$ sohada analitik bo‘lgan funksiyaning Shu $D$ sohaga tegishli egri chiziq bo‘yicha integrali integrallash yo‘liga bog‘liq bo‘lmasligini ko‘ramiz. Bu quyidagi Koshi teoremasinining isbotidir:

**10.5.1-teorema (Koshi).** Agar $f(z)$ funksiya chegarasi C konturdan iborat bir bog‘lamli $D$ sohada hamda C konturda analitik bo‘lsa, u holda $\int_C f(z)dz = 0$ bo‘ladi.

Koshining bu teoremasi asosida, avvalo, analitik funksiyaning aniqmas integrali tushunchasini kiritish mumkinligini aytamiz. Faraz qilaylik, $f(z)$ funksiya bir bog‘lamli $D$ sohada analitik bo‘lsin. Bu sohaga tegishli bo‘lgan biror A nuqtani va qo‘zg‘aluvchi, ya’ni $D$ sohaga tegishli bo‘lgan ixtiyoriy Z nuqtani olsak, A va Z nuqtalarni tutashtiruvchi egri chiziqlar bo‘yicha integrallarning barchasi bir xil bo‘lishi yuqoridagi Koshi teoremasidan kelib chiqadi. Demak, bunday integralning A va Z nuqtalarning o‘rnigagina bog‘liq bo‘lishi, A tayinlangan ekanligidan esa uning faqat Z ga bog‘liq ekanligi kelib chiqadi. Shu sababli bunday integralni

$$F(z) = \int_A^z f(\varsigma)d\varsigma \qquad (10.5.3)$$

kabi belgilab, uni $f(z)$ analitik funksiyaning *aniqmas integrali* deb ataladi. Bu o‘rinda $F'(z) = f(z)$ bo‘lishini aytamiz (buni keltirib chiqarish bilan mustaqil Shug‘ullaning).

Yuqorida keltirilgan Koshi teoremasini unga ekvivalent bo‘lgan quyidagi shaklda ham ifodalash mumkin.

**10.5.2-teorema (Koshi).** Agar $f(z)$ bir bog‘lamli $D$ sohada analitik bo‘lsa, uning bu sohaga tegishli bo‘lgan ixtiyoriy kontur bo‘yicha integrali nolga teng bo‘ladi.

Aytaylik, $(n+1)$ bog‘lamli $D$ soha berilgan bo‘lib, uning chegarasidan iborat bo‘lgan $C_0, C_1, C_2, ..., C_n$ konturlar sistemasidagi har bir kontur o‘zini o‘zi kesmovchi, $C_1, C_2, ..., C_n$ konturlar $C_0$ kontur ichiga joylashgan holda ulardan har biri boshqasidan tashqarida joylashgan, undan tashqari, $C_0$ musbat, $C_1, C_2, ..., C_n$ lar esa manfiy yo‘nalishli konturlar bo‘lsin (10.5.1-rasmga qarang). $f(z)$ funksiya bu $(n+1)$ bog‘lamli $D$ sohada analitik bo‘lsin. 10.5.1-rasmda ko‘rsatilganidek, $\gamma_1, \gamma_2, ..., \gamma_n$ kesmalar («ko‘prik»lar) vositasida ko‘p bog‘lamli sohani bir bog‘lamliga keltiramiz va uning $C$ chegarasi bo‘yicha $f(z)$ analitik funksiyaning integrali yuqorida keltirilgan Koshi teoremasiga ko‘ra nolga teng bo‘lishini va

$$C = C_0 \cup \left( \bigcup_{k=1}^{n} C_k^- \right) \cup \left( \bigcup_{k=1}^{n} \left( \gamma_k \cup \gamma_k^- \right) \right)$$

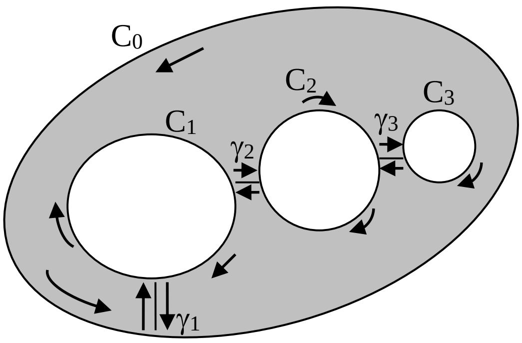
ekanligidan

$$0 = \int_C f(z)dz = \int_{C_0} f(z)dz + \sum_{k=1}^{n} \int_{C_k^-} f(z)dz + \sum_{k=1}^{n} \left( \int_{\gamma_k} f(z)dz + \int_{\lambda_k^-} f(z)dz \right) =$$

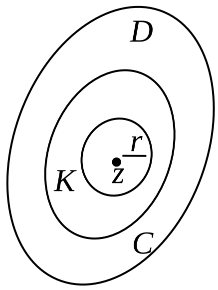
$$= \int_{C_0} f(z)dz - \sum_{k=1}^{n} \int_{C_k} f(z)dz + \sum_{k=1}^{n} \left( \int_{\gamma_k} f(z)dz - \int_{\lambda_k} f(z)dz \right) = \int_{C_0} f(z)dz - \sum_{k=1}^{n} \int_{C_k} f(z)dz$$

ni olamiz. Bundan esa $(n+1)$ bog‘lamli soha uchun Koshi teoremasiga ega bo‘lamiz:

$$\int_{C_0} f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz . \qquad (10.5.4)$$




10.5.1-rasm.




10.5.2-rasm.

Bu o‘rinda, $C_k$ $\left(k = \overline{1;n}\right)$ konturlarning ichida joylashgan sohalar $f(z)$ funksiyaning analitiklik sohasiga tegishli bo‘lishi shart emasdir, ya’ni bu aytilgan sohalarda funksiya analitik bo‘lishi ham bo‘lmasligi ham mumkin ekanligini ta’kidlaymiz.

Koshi teoremasiga asoslanib, kompleks o‘zgaruvchili funksiyalar nazariyasida fundamental ahamiyatga ega va undagi tekshirishning asosiy vositalaridan biri bo‘lgan, biror kontur ichida analitik bo‘lgan funksiyaning kontur ichiga joylashgan nuqtadagi qiymatini konturdagi nuqtalardagi qiymati orqali bog‘lovchi ifodadan iborat *Koshining integral formulasini* keltirib chiqarish mumkin:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} d\xi \qquad (10.5.5)$$

Aytaylik, $C$ - o‘zini o‘zi kesmovchi va $f(z)$ funksiya analitik bo‘lgan bir bog‘lamli $D$ sohada to‘lig‘icha yotuvchi kontur, $z$ esa $C$ kontur ichida yotuvchi tayinlangan nuqta bo‘lsin. (10.5.5) dagi integral osti funksiyasi $D$ sohaning $\xi = z$ nuqtalaridan boshqa barcha nuqtalarida analitik bo‘lishi ravshandir. Bu nuqtani markazi $z$ da va radiusi $r$ bo‘lgan doira bilan chetlantiramiz. Bunda doiraning aylanasini $K$ orqali belgilab uni $C$ kontur ichida yotishiga $r$ ni tanlash bilan erishamiz. U holda $C$ va $K$ konturlar vositasida hosil bo‘lgan halqada $f(z)$ funksiya analitik bo‘lib, Koshi teoremasi asosida

$$\int_C \frac{f(\xi)}{\xi - z} d\xi = \int_K \frac{f(\varsigma)}{\varsigma - z} d\varsigma = \int_K \frac{f(z)}{\varsigma - z} d\varsigma - \int_K \frac{f(z) - f(\varsigma)}{\varsigma - z} d\varsigma$$

$K$ - kontur bo‘yicha integralda $\varsigma$ - nuqta $|\varsigma - z| = r$ aylanaga tegishli ekanligidan $\varsigma = z + re^{i\varphi}, \quad d\varsigma = ire^{i\varphi} d\varphi, \quad 0 \le \varphi \le 2\pi$ bo‘lishi aniqdir (10.5.2-rasm). Shu sababli

$$\int_K \frac{f(z)}{\varsigma - z} d\varsigma = if(z)\int_0^{2\pi} d\varphi = 2\pi i f(z).$$

Kompleks o‘zgaruvchili funksiya integralining 4) xossasi asosida

$$\left|\int_k \frac{f(\varsigma)}{\varsigma - z} d\varsigma - \int_k \frac{f(z)}{\varsigma - z} d\varsigma\right| = \left|\int_k \frac{f(\varsigma) - f(z)}{\varsigma - z} d\varsigma\right| < \frac{1}{r} \cdot 2\pi r \cdot \max|f(\varsigma) - f(z)| = 2\pi \max|f(z) - f(\varsigma)|$$

Bunda $r \to 0$ da $\max|f(z) - f(\varsigma)| \to 0$ bo‘lishini hisobga olsak,

$$\int_C \frac{f(\xi) d\xi}{\xi - z} = 2\pi i f(z),$$

ya’ni (10.5.5) ni olamiz. (10.5.5) ning o‘ng tomonidagi ifodani *Koshi integrali* deb ataladi.

Endi (10.5.5) formuladan foydalanib, analitik funksiyaning hosilasi uchun quyidagicha formulani keltirib chiqaramiz:

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i \Delta z} \int_C \left[\frac{f(\xi)}{\xi - z - \Delta z} - \frac{f(\xi)}{\xi - z}\right] d\xi = \frac{1}{2\pi i} \int_C \left[\frac{f(\xi)}{(\xi - z)(\xi - z - \Delta z)}\right].$$

Bunda $\Delta z \to 0$ dagi limitga o‘tib, analitik funksiya hosilasi uchun,

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^2} d\xi$$

formulani olamiz. Bu jarayonni davom ettirib, analitik funksiyani $n$ marta differensiallashning

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi; \quad n = 0, 1, 2, ... \qquad (10.5.6)$$

umumiy integral formulasiga ega bo‘lamiz. Bundan analitik funksiyaning asosiy xossalaridan biri bo‘lgan u ixtiyoriy marta differensiallanuvchi bo‘lishi hamda uning o‘zi va ixtiyoriy tartibli hosilalari (10.5.6) formula vositasida uning analitik bo‘lgan soha chegarasidagi qiymatlari orqali ifodalanishining kelib chiqishini ko‘ramiz.

Aytaylik, (10.5.6) da $z$ radiusi $\rho$ bo‘lgan $C$ aylananing markazidan iborat bo‘lib, bu aylana va uning ichida joylashgan nuqtalar $D$ sohaga tegishli bo‘lsin. $M(\rho)$ - orqali funksiya modulining $C$ dagi eng katta qiymatini belgilaylik. U holda funksiya va uning $n$-tartibli hosilasining $z$-nuqtadagi qiymatining moduli uchun

$$\left|f^{(n)}(z)\right| \le \frac{n! M(\rho)}{\rho^n}; \quad n = 0, 1, 2, ... \qquad (10.5.7)$$

ni olamiz. (10.5.7) *Koshi tengsizligi* deb ataladi.

Agar C konturda va u bilan chegaralangan D sohada $f(z)$ analitik funksiya bo‘lsa,

$$\frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}d\xi \qquad (10.5.8)$$

ko‘rinishdagi integralni yuqorida Koshi integrali deb atashga kelishgan edik. Bu holda (10.5.8) bilan aniqlangan funksiya D sohaning ichida $f(z)$ bilan utma-ust tushishini (10.5.5) Koshining integral formulasi yordamida va bu sohadan tashqarida ham $f(z)$ funksiya analitik bo‘lishini talab qilsak, konturdan tashqarida joylashgan $\forall Z$ uchun bu integral nolga teng bo‘lishini ko‘ramiz. Endi C – chiziq ixtiyoriy bo‘lakli silliq yoy (yoki Shunday yoylarning birlashmasi) deb, $f(\xi)$ funksiya esa faqat C qiziqda berilgan va uzluksiz deb faraz qilsak, u holda (10.5.8) ni *Koshi tipidagi integral* deb atash qabul qilingan. Koshi integrali Koshi tipidagi integralning xususiy holi bo‘lishini ko‘rish osondir.

Koshi tipidagi integral vositasida $z \notin C$ faraz asosida

$$F(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)d\xi}{\xi - z}$$

formula bilan aniqlangan funksiya $\frac{f(\xi)}{\xi - z}$ funksiya C chiziqda uzluksiz ekanligi sababli mavjuddir. (10.5.6) formulani keltirib chiqarishda qilingan jarayonni Koshi tipidagi integral uchun takrorlab, $z \notin C$ bo‘lgan nuqtalar uchun

$$F'(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)d\xi}{(\xi - z)^2}$$

ni keltirib chiqarish va Shu bilan Koshi tipidagi integral C chiziqda (yoyda) yotmagan ixtiyoriy nuqtada analitik funksiya bo‘lishiga ishonch hosil qilish mumkin. Koshi tipidagi integralning ixtiyoriy n-tartibli hosilasi uchun (10.5.6) formulaga o‘xshash

$$F^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\xi)d\xi}{(\xi - z)^{n+1}}$$

formula o‘rinli bo‘lishini ko‘rsatish mumkinligini ham aytamiz.

## 10.6. Teylor qatori. Loran qatori. Maxsus nuqtalar.

Kompleks o‘zgaruvchili funksiyalarning taxlilida ham haqiqiy o‘zgaruvchili funksiyalarning matematik tahlilidagi kabi sonli va funksional qatorlar tushunchalari qaraladi. Bunda haqiqiy sohadagi qatorlar nazariyasining barcha asosiy tushunchalari kompleks sohaga ham ko‘chiriladi. Masalan, kompleks sohadagi qatorning absolyut yaqinlashishi, tekis yaqinlashishi, darajali qator va Shu kabi boshqa tushunchalar haqiqiy sohadagidek bo‘lishini aytamiz. Bu o‘rinda, aytilgan fikrimizning tasdig‘ini yaxshiroq anglash maqsadida kompleks o‘zgaruvchili funksional qator va uning tekis yaqinlashishi tushunchalarini keltiramiz.

Aytaylik,

$$\sum_{k=1}^{\infty} f_n(z) \qquad (10.6.1)$$

qatorning har bir hadi kompleks tekislikdagi nuqtalarning biror E sohasida aniqlangan kompleks o‘zgaruvchili funksiyadan iborat bo‘lsin. U holda

$$S_n(z) = \sum_{k=1}^{n} f_k(z) \qquad (10.6.2)$$

ni (10.6.1) funksional qatorning $n$*-qismiy yig‘indisi* deyilishi ma’lumdir.

Agar $\forall\ \varepsilon > 0$ uchun $\exists\ N(\varepsilon) \in N$ bo‘lib, $\forall\ N(\varepsilon) < n \in N$, $m \in N$, $\forall\ z \in E$ uchun

$$|\, S_{n+m}(z) - S_n(z)| < \varepsilon$$

tengsizlik bajarilsa, (10.6.1) funktsional qator E sohada *tekis yaqinlashuvchi* deyiladi. Bu holda $f(z) = \lim S_n(z)$ limit mavjud bo‘lib, u E sohada aniqlangan funksiya bo‘lishi ravshandir.

Endi (10.6.1) funksional qatorning tekis yaqinlashishi ta’rifini $\forall\ \varepsilon > 0$ uchun $\exists\ N(\varepsilon) \in N$ bo‘lib, $N(\varepsilon) < n \in N$, $\forall\ z \in E$ uchun $|\, f(z) - S_n(z)| < \varepsilon$ tengsizlikning bajarilishini talab qilish bilan almashtirish ham mumkin ekanligini aytamiz.

Agar $D$ sohaga tegishli bo‘lgan har qanday $\overline{D}*$ yopiq sohada (10.6.1) funksional qator tekis yaqinlashuvchi bo‘lsa, uni $D$ *sohaning ichida tekis yaqinlashuvchi* deymiz.

Biror $D$ sohada tekis yaqinlashuvchi bo‘lgan qatorning $D$ sohaning ichida tekis yaqinlashuvchi bo‘lishi aniqdir. Ammo, buning teskarisi hamma vaqt ham o‘rinli bo‘lavermasligini aytamiz.

### 10.6.1. Teylor qatori

Haqiqiy sohadagi kabi kompleks sohada ham darajali qator deb,

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (10.6.3)$$

ga aytamiz. Bu yerda $z_0, a_k$ $(k = 0,1,2,...)$ berilgan kompleks sonlar bo‘lib, $z$ kompleks o‘zgaruvchidir. (10.6.3) uchun quyidagi Abel teoremasi o‘rinlidir.

**10.6.1-teorema (Abel).** Agar (10.6.3) darajali qator kompleks tekislikning $z_0$ dan farqli biror $z_1$ nuqtasida yaqinlashuvchibo‘lsa, u $|z - z_0| < |z_1 - z_0|$ bo‘ladigan kompleks tekislikning barcha $z$ nuqtalarida yaqinlashuvchi, biror $z_2$ nuqtasida uzoqlashuvchi bo‘lsa, u kompleks tekislikning $|z - z_0| > |z_2 - z_0|$ bo‘ladigan barcha $z$ nuqtalarida uzoqlashuvchi bo‘ladi.

Bu teoremaning isboti 2.5.2-bandda haqiqiy sohada bajarilgan Abel teoremasining isboti kabi amalga oshirilishini hamda kompleks tekislikda $z_0$ dan farqli bo‘lgan (10.6.3) darajali qator yaqinlashuvchibo‘ladigan $z_1$ va uzoqlashuvchi bo‘ladigan $z_2$ nuqta mavjud bo‘lsa, Shunday $R > 0$ son mavjud bo‘lib, kompleks tekislikdagi $|z - z_0| = R$ aylananing barcha ichki nuqtalarida bu qator yaqinlashuvchi, tashqarisida esa uzoqlashuvchi bo‘lishini aytamiz. Yuqorida aytilgan R son (10.6.3) darajali qatorning *yaqinlashish radiusi*, $|z-z_0|<R$ esa uning *yaqinlashish doirasi* deyiladi. Bu holda (10.6.3) darajali qator $|z-z_0|=R$ aylananing nuqtasida yaqinlashuvchiyoki uzoqlashuvchi bo‘lishi mumkinligini aytamiz. Agar (10.6.3) darajali qator kompleks tekislikning barcha chekli nuqtalarida yaqinlashuvchibo‘lsa $R=\infty$, agar $z_0$ dan boshqa barcha nuqtalarida uzoqlashuvchi bo‘lsa $R=0$ deb qabul qilamiz.

Koshi-Adamar teoremasiga asosan (10.6.1) darajali qatorning yaqinlashish radiusi uchun

$$R=\frac{1}{\overline{\lim}\sqrt[n]{|a_n|}}$$

o‘rinli bo‘lib, bu yerda $\overline{\lim}\sqrt[n]{|a_n|}-\left\{\sqrt[n]{|a_n|}\right\}$ ketma-ketlikning yuqori limitidir.

Bu o‘rinda quyidagi Veyershtrass teoremasini isbotsiz keltiramiz.

**10.6.2-teorema (Veyershtrass).** Agar $D$ soha ichida tekis yaqinlashuvchibo‘lgan (10.6.1) funksional qatorning har bir hadi bu sohada analitik bo‘lsa, (10.6.1) qatorning yig‘indisidan iborat $f(z)$ funksiya ham $D$ sohada analitik bo‘ladi. undan tashqari, (10.6.1) ni k marta hadlab differensiallash natijasida olingan

$$\sum_{n=k}^{\infty} f_n^{(k)}(z) \qquad (10.6.4)$$

qator $(k\in N)$ ham $D$ soha ichida tekis yaqinlashuvchibo‘lib, $f^{(k)}(z)$ ga teng yig‘indisiga ega bo‘ladi. bu teoremaga asosan quyidagi tasdiqning o‘rinli ekanligiga ishonch hosil qilish oson.

**10.6.3-teorema.** Biror $|z-z_0|<R$ doirada analitik bo‘lgan har qanday $f(z)$ funksiyani bu doiraning ichida quyidagicha *Teylor qatoriga* yoyilmasini yagona ko‘rinishida ifodalash mumkin

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n\,, \qquad (10.6.5)$$

bu yerda

$$a_n=\frac{1}{n!}f^{(n)}(z_0)=\frac{1}{2\pi i}\int_{C_0}\frac{f(\xi)}{(\xi-z_0)^{n+1}}d\xi\,, \qquad (10.6.6)$$

markazi $z_0$ nuqtada va $\xi$ dan kichik biror $R_0$ radiusli aylana konturi.

Darhaqiqat $\forall\ z\neq z_0$ nuqta uchun Koshining integral formulasiga asosan

$$f(z)=\frac{1}{2\pi i}\int_{C_0}\frac{f(\xi)}{\xi-z}d\xi \qquad (10.6.7)$$

bunda $z$ kontur ichida yotuvchi nuqta bo‘lib, $C_0-|\xi-z_0|=R_0$ aylanadir. Bu holda $|z-z_0|<R$ bo‘lib,

$$\left|\frac{z-z_0}{\xi-z_0}\right|=\leq q<1$$

tengsizlik bajarilib,

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0} \sum_{n=1}^{\infty} \left( \frac{z - z_0}{\xi - z_0} \right)^n \qquad (10.6.8)$$

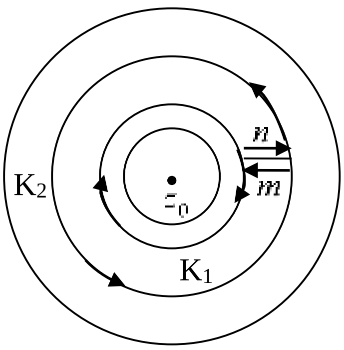
yoyilmadagi qatorning $z_0$ da yotuvchi $\xi$ ga nisbatan tekis yaqinlashuvchibo‘lishi kelib chiqadi.

Endi (10.6.8) ning har ikki tomonini $\frac{1}{2\pi i} f(\xi)$ ga ko‘paytirib, kontur bo‘yicha integrallab, (10.6.7) flrmulani hisobga olgan holda (10.6.5) va (10.6.6) larni olamiz.

Agar $z = z_0$ nuqtada $f(z_0) = 0$ bo‘lsa, u $f(z)$ funksiyaning *noli* deb atalishini eslatamiz. Aytaylik, $f(z)$ analitik funksiya o‘zining noli bo‘lgan $z_0$ nuqtaning atrofida aynan nolga teng bo‘lmasin. U holda $f(z)$ funksiyaning $z_0$ nuqta atrofidagi Teylor qatoriga yoyilmasidagi noldan farqli bo‘lgan koeffisientlarning eng kichik nomeri $z_0$ nolning *tartibi (karraliligi)* deyiladi. Teylor qatori kompleks tekislikning barcha nuqtalarida (barcha chekli $z$ larda) yaqinlashuvchibo‘lgan funksiyani *butun* deb ataladi.

## 10.7. Loran qatorlari va maxsus nuqtalar

Mazkur bandda kompleks tekislikning ba’zi nuqtalarida Koshi-Riman shartlarini qanoatlantirmovchi bir qiymatli funksiyalarni o‘rganish bilan Shug‘ullanamiz. Funksiyaning Koshi-Riman shartlarini qanoatlantirmovchi nuqtalarini uning *maxsus nuqtalari* deb ataladi. Markazi $z_0$ nuqtada radiuslari $r$ va $R$ $(0 < r < R)$ bo‘lgan konsentrik aylanalar bilan chegaralangan $D$ sohada (halqada) analitik bo‘lgan birqiymatli $f(z)$ funksiya berilgan deb faraz qilaylik. Markazi $z_0$ nuqtada va radiuslari mos ravishda $r_1$ va $r_2$ ga teng hamda $0 < r < r_1 < r_2 < R$ o‘rinli bo‘lgan ikkita $K_1$ va $K_2$ konsentrik aylanalarni olsak, ular bilan chegaralangan ikki bog‘lamli $D_1$ sohada (halqada) va bu aylanalar ustidagi nuqtalarda $f(z)$ funksiyaning analitik bo‘lishi aniqdir (10.7.1-rasmga qarang).




10.7.1-rasm.

$D_1$ ikki bog‘lamli sohani $mn$ kesma («ko‘prik») vositasida bir bog‘lamli sohaga keltirib, bu sohada Koshining integral formulasini qo‘llab, kesmaga tegishli bo‘lmagan $D_1$ ning har bir $z$ nuqtasi uchun

$$f(z)=\frac{1}{2\pi i}\int_{K_2}\frac{f(\xi)}{\xi-z}d\xi-\frac{1}{2\pi i}\int_{K_1}\frac{f(\xi)}{\xi-z}d\xi \qquad (10.7.1)$$

ga ega bo‘lamiz. $mn$ kesma ixtiyoriy tanlanganligi sababli (10.7.1) formula $\forall\ z\in D_1$ uchun o‘rinlidir. (10.7.1) ning o‘ng tomondagi $K_2$ bo‘yicha integral uchun

$$\frac{1}{\xi-z}=\frac{1}{\xi-z_0}\cdot\frac{1}{1-\dfrac{z-z_0}{\xi-z_0}}=\frac{1}{\xi-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^n \qquad (10.7.2)$$

yoyilmadan foydalanamiz. Bu qator $\xi\in K_2$ ekanligidan $\left|\dfrac{z-z_0}{\xi-z_0}\right|=q_2<1$ bo‘lib, uning tekis yaqinlashuvchibo‘lishini ko‘ramiz. Xuddi Shunga o‘xshash $K_1$ bo‘yicha integral uchun

$$\frac{1}{\xi-z}=\frac{1}{z-z_0}\frac{1}{1-\dfrac{\xi-z_0}{z-z_0}}=-\frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{\xi-z_0}{z-z_0}\right)^n$$

(10.7.3)

yoyilmadan foydalanamiz. Bu qator $\xi\in K_1$ ekanligidan

$$\left|\frac{\xi-z_0}{z-z_0}\right|=q_1<1$$

bo‘lib, uning ham tekis yaqinlashuvchiligini ko‘ramiz.

(10.7.2) va (10.7.3) larni (10.7.1) ning o‘ng tomonidagi birinchi va ikkinchi integrallarga mos ravishda qo‘yish bilan

$$f(z)=\sum_{n=1}^{\infty}a_{-n}(z-z_0)^{-n}+\sum_{n=0}^{\infty}a_n(z-z_0)^n \qquad (10.7.4)$$

ni olamiz. Bu yerda

$$a_{-n}=\frac{1}{2\pi i}\int_{K_2}\frac{f(\xi)}{(\xi-z_0)^{-n+1}}d\xi\quad (n=1,2,...),$$
$$a_n=\frac{1}{2\pi i}\int_{K_1}\frac{f(\xi)}{(\xi-z_0)^{n+1}}d\xi\quad (n=1,2,...) \qquad (10.7.5)$$

$f(z)$ funksiyani koeffisientlari (10.7.5) formulalar vositasida aniqlanuvchi (10.7.4) ko‘rinishda qatorga yoyilmasi, uning *Loran qatori* deb ataladi.

Agar $f(z)$ funksiya $K_1$ aylana ichida analitik bo‘lsa, Koshining integral teoremasiga asosan $a_{-n}=0\ (n\in N)$ bo‘lib, (10.7.4) Loran qatori Teylor qatoridan iborat bo‘lib qoladi.

$f(z)$ funksiya $D$ sohaning ichida analitik ekanligidan Koshi teoremasiga asosan $K_1$ va $K_2$ aylanalardan iborat integrallash konturlarini $D$ sohada yotuvchi musbat yo‘nalishli qandaydir K kontur ko‘pincha $K$-markazi $z_0$ da $K\subset D$ bo‘lgan aylana deb olamiz bilan almashtirish mumkinligini va bu holda quyidagi teorema o‘rinli bo‘lishini ko‘ramiz.

**10.7.1-teorema (Loran).** $D=\{z: r<|z-z_0|<R\}$ sohada analitik bo‘lgan har qanday $f(z)$ funksiyani bu sohada yaqinlashuvchi bo‘lgan

Qatorga qo‘yish mumkin $f(z)=\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ (10.7.6)

Shunday qilib, Loran qatorining yaqinlashish sohasi, $f(z)$ funksiya analitik bo‘lgan eng katta $D$ halqadan iborat bo‘lishini va bu halqaning chegarasi bo‘lgan aylanada funksiyaning kamida bitta maxsus nuqtasi yotishini ko‘ramiz.

Bu o‘rinda $z_0$ nuqta $f(z)$ funksiyaning $K$ kontur ichidagi yagona maxsus nuqtasi bo‘lgan hol muhimligini va bu holda (10.7.6) Loran qatori $0<|z-z_0|<R$ sohada yaqinlashuvchibo‘lishini hamda $z_0$ ni *yakkalangan maxsus nuqta* deyilishini aytamiz.

(10.7.4) ning

$$\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

dan iborat bo‘lgan qismini *Loran qatorining bosh qismi*,

$$\sum_{n=1}^{\infty} a_n(z-z_0)^n$$

ni esa *to‘g‘ri (regulyar) qismi* deb yuritiladi.

Agar Loran qatorining bosh qismida chekli sondagi hadlar bo‘lib, $a_{-m}$ - so‘nggi noldan farqli koeffisient (ya’ni (10.7.5) bilan aniqlanuvchi koeffisientlar uchun $a_{-m}\neq 0,\ a_{-n}=0,\ n=m+1, m+2,...$) bo‘lsa, $z_0$ ni $m$ *tartibli qutb* deb ataladi va bu holda Loran qatori

$$f(z)=\sum_{n=-m}^{\infty} a_n(z-z_0)^n$$

ko‘rinishda bo‘lishi ravshandir.

Loran qatorining bosh qismi cheksiz ko‘p hadlarni o‘z ichiga olgan holda $z_0$ ni *qat’iy maxsus nuqta (qat’iy qutb)* deyiladi.

Agar Loran qatorining bosh qismi aynan nolga teng bo‘lsa, $z_0$ ni *chetlantiriladigan maxsus nuqta* deyiladi va bu holda Loran qatori

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

ko‘rinishda bo‘lib, funksiyaning $z_0$ nuqtadagi qiymatini $f(z_0)=a_0$ kabi tanlash bilan maxsus nuqta chetlantiriladi.

Shu paytga qadar $f(z)$ kompleks o‘zgaruvchili funksiyani chegaralangan yoki cheksiz chetlangan nuqtasi chegaraviy bo‘lgan sohada aniqlangan deb faraz qilindi. Hosila tushunchasini ham $z$ argumentning chekli qiymati uchun ta’riflandi. Agar biror $D'$ soha cheksiz chetlangan nuqtani ham o‘ziga ichki nuqta sifatida olgan hol qaralsa, bu holda analitik, hosila, qutb, qat’iy maxsus nuqta va chetlantiriladigan maxsus nuqta tushunchalarini $z=\frac{1}{\xi}$ almashtirish vositasida $z$ ga nisbatan cheksiz chetlangan nuqtani $\xi$ ga nisbatan koordinata boshiga keltirish usuli bilan kiritish qabul qilinadi.

## Ko'mpleks o'zgaruvchili asosiy elementar funksiyalar

Mazkur bandda kompleks tekislikda asosiy elementar funksiyalarning ta'riflarini va ularning xossalarini bayon qilamiz.

### 10.8.1. Kompleks o'zgaruvchili ko'rsatkichli, trigonometrik va giperbolik funksiyalar

Matematik analizdan ma'lum bo'lgan $x$ ning haqiqiy qiymatlari uchun $e^x$, $\sin x$, $\cos x$ funksiyalarning darajali qatorga yoyilmasini e'tiborga olgan holda $z$ ning kompleks qiymatlari uchun ta'rif bo'yicha

$$e^z = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots\ , \qquad (10.8.1)$$

$$\sin z = z - \frac{z^3}{3!} + \ldots + (-1)^m \frac{z^{2m+1}}{(2m+1)!} + \ldots, \qquad (10.8.2)$$

$$\cos z = 1 - \frac{z^2}{2!} + \ldots + (-1)^m \frac{z^{2m}}{(2m)!} + \ldots \qquad (10.8.3)$$

larni qabul qilamiz. Bunday ta'rif bilan aniqlangan kompleks o'zgaruvchili funksiyalar butun kompleks tekislikda yaqinlashuvchibo'lgan darajali qatorlar yordamida berilganligini ko'rish osondir. Bu fikrimizga (10.8.1)-(10.8.3) formulalarning o'ng tomonlaridagi qatorlarni tekshirish bilan ishonch hosil qilishingizni tavsiya qilamiz.

Bu kompleks o'zgaruvchili funksiyalarni o'zaro bog'lovchi quyidagi Eyler formulasi o'rinlidir:

$$e^{iz} = \cos z + i \sin z \qquad (10.8.4)$$

Haqiqatdan ham, agar (10.8.1) formulada $z$ ni $iz$ bilan almashtirsak,

$$e^{iz} = 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{4!} - \frac{z^6}{6!} - i\frac{z^7}{7!} + \ldots$$

ga ega bo'lamiz. Endi (10.8.2) ning har ikki tomoniga $i$ ni hadlab ko'paytirib, (10.8.3) bilan hadlab qo'shsak, natijaning o'ng tomoni oxirgi tenglikning o'ng tomoni bilan bir xil bo'lishiga ishonch hosil qilamiz. Bu yuqoridagi (10.8.4) Eyler formulasining o'rinli bo'lishining tasdig'idir.

(10.8.4) da $z$ ni $-z$ bilan almashtirib, $\sin z$ ning toqlik, $\cos z$ ning esa juftlik xossalaridan foydalanib (bu xossalarga mustaqil ishonch hosil qiling),

$$e^{-iz} = \cos z - i \sin z \qquad (10.8.5)$$

ni olamiz.

(10.8.4) va (10.8.5) larga asosan

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad (10.8.6)$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad (10.8.7)$$

larga ega bo‘lamiz.

Bu o‘rinda ko‘rsatkichli $e^x$ funksiyaning haqiqiy sohada o‘rinli bo‘lgan ba’zi xossalari kompleks o‘zgaruvchili ko‘rsatkichli $e^z$ funksiya uchun ham saqlanib qolishini ko‘rsatish mumkinligini aytamiz. Masalan,

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$$

xossani bevosita $e^z$ ning ta’rifi bo‘lgan (10.8.1) ga asoslanib ko‘rsatish mumkin (mustaqil bajaring). Xususan, $z = x + iy$ bo‘lganda, bu yerda $x; y \in R$,

$$e^z = e^{x+iy} = e^x(\cos y + i \sin y) \qquad (10.8.8)$$

o‘rinli bo‘lishini va bundan $\left|e^z\right| = e^x$, $\arg e^z = y$ ekanligini ko‘ramiz. (10.8.8) formula $e^z$ funksiyani $z$ ning ixtiyoriy kompleks qiymatida hisoblash imkoniyatini beradi. Masalan, $k \in z$ bo‘lganda

$$e^{2k\pi i} = 1, \; e^{(2k+1)\pi i} = -1, \; e^{\left(k + \frac{1}{2}\right)\pi i} = (-1)^{1c} i,$$

$$e^{z + 2k\pi i} = e^z \cdot e^{2k\pi i} = e^z \cdot 1 = e^z.$$

Oxirgi tengnlikdan ko‘rinadiki, kompleks o‘zgaruvchili ko‘rsatkichli $e^z$ funksiya davriylik xossasiga ega bo‘lib, $y$ $2\pi i$ davrga ega ekanligini ko‘ramiz. Haqiqiy sohada bu xossa o‘rinli emasligini eslatamiz.

Xuddi Shunga o‘xshash

$$e^{i(z + 2k\pi)} = e^{iz + 2k\pi i} = e^{iz},$$

$$e^{-i(z + 2k\pi)} = e^{-iz - 2k\pi i} = e^{-iz},$$

bo‘lishini hisobga olib, (10.8.6) va (10.8.7) lardan kompleks o‘zgaruvchili $\cos z$ va $\sin z$ funksiyalarning $2\pi$ davrli davriy funksiyalar ekanligiga ishonch hosil qilamiz.

Endi trigonometrik tangens va kotangens funksiyalarni kompleks tekislikda quyidagicha ta’riflaymiz:

$$tgz = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}, \qquad (10.8.9)$$

$$ctgz = \frac{\cos z}{\sin z} = \frac{i\left(e^{iz} + e^{-iz}\right)}{e^{iz} - e^{-iz}} \qquad (10.8.10)$$

Giperbolik funksiyalarni esa quyidagicha aniqlaymiz:

$$shz = \frac{e^z - e^{-z}}{2}, \quad chz = \frac{e^z - e^{-z}}{2},$$

$$thz = \frac{shz}{chz}, \quad cthz = \frac{chz}{shz}.$$

Giperbolik va trigonometrik funksiyalar orasida quyidagi bog‘lanishlar mavjudligiga ishonch hosil qilish oson.

$$shz = -i \sin iz, \quad chz = \cos iz,$$
$$thz = -itgiz, \quad ctgz = ictgiz. \qquad (10.8.11)$$

Yuqoridagicha aniqlangan trigonometrik tangens va kotangenslar $\pi$ davrli, giperbolik sinus va kosinuslar $2\pi i$, giperbolik tangens va kotangenslar esa $\pi i$ davrli funksiyalar ekanligiga ishonch hosil qilish osondir (mustaqil bajaring).

Sinus va kosinus funksiyalarning haqiqiy sohadagi chegaralanganlik xossasi kompleks sohada o‘rinli emas. Haqiqatdan ham (10.8.11) bog‘lanishlarning birinchisidan $\sin iz = ishz$ ekanligini olamiz. Bunda $z = x \in R$ deb faraz qilsak, $|\sin ix| = |shx|$ ga ega bo‘lamiz. Endi $|x| \to +\infty$ bo‘lgan holni qarasak, $|shx| \to +\infty$ ekanligidan $|\sin ix| \to +\infty$ bo‘lishi kelib chiqadi. Xuddi Shunga o‘xshash kosinusning ham kompleks sohada chegaralanmaganligi ko‘rsatiladi.

### 10.8.2. Kompleks o‘zgaruvchili logarifmik funksiya

**10.8.1-ta’rif.** Agar $z \neq 0$ bo‘lib, $e^w = z$ o‘rinli bo‘lsa, $w$ ni $z$ kompleks sonning *logarifmi* ($e$ asosli) deyiladi va

$$w = Lnz$$

kabi belgilanadi.

Agar $w = u + i\vartheta, \quad u;\ \vartheta \in R$ deb faraz qilsak, (10.8.8) ga asosan

$$\left|e^w\right| = e^u, \qquad Arge^w = \vartheta + 2k\pi \quad (k \in Z)$$

bo‘lishini ko‘ramiz. $e^w = z$ ekanligidan $e^u = |z|$ yoki bundan $u = \ln|z|$ bo‘lishi kelib chiqadi (bu yerda $|z|$ musbat haqiqiy son ekanligidan uning haqiqiy sohadagi natural logarifmi mavjudligi aniqdir). Shuningdek, $\vartheta = \arg z$ bo‘lishini ham ko‘rish qiyin emas (bu yerda $agrz$ kompleks $z$ sonining bosh argumenti). Shunday qilib,

$$Lnz = \ln|z| + iArgz = \ln|z| + i\arg z + 2k\pi i, \quad (k \in Z). \qquad (10.8.12)$$

(10.8.12) kompleks $z$ sonining natural logarifmini hisoblash formulasidir. Bu formulaning o‘ng tomonidagi $Argz = \arg z + 2k\pi i$ qo‘shiluvchi ko‘p qiymatli (sanoqli qiymatli) bo‘lganligi sababli, $Lnz$ natural logarifmning ham ko‘p qiymatli funksiya ekanligi kelib chiqadi.

*Logarifmning bosh qiymati* deb, $z$ ning argumentining bosh qiymatiga mos keluvchi qiymatga (ya’ni (10.8.12) da $k = 0$ deb olingan qiymatga) aytamiz. Bu holda

$$\ln z = \ln|z| + i\arg z$$

kabi belgilashdan foydalanamiz.

**1-misol.** $\ln(-1)$ va $Ln(-1)$ larni toping.

**Yechish.** $agr(-1) = \pi$ bo‘lganligi uchun

$$\ln(-1) = \ln 1 + \pi i = \pi i,$$
$$Ln(-1) = \ln 1 + i\arg z + 2k\pi i = \pi i + 2k\pi i = (2k+1)\pi i,$$
$$k \in Z.$$

bo‘ladi. $k = 0$ desak, $\ln(-1) = \pi i$ ga ega bo‘lamiz.

**2-misol.** $\ln(3+4i)$ va $Ln(3+4i)$ larni toping.

**Yechish.** $|3+4i|=\sqrt{3^2+4^2}=\sqrt{25}=5$,

$$\arg(3+4i)=arctg\frac{4}{3}.$$

$$\ln(3+4i)=\ln 5+iarctg\frac{4}{3},$$

$$Ln(3+4i)=\ln 5+i\left(arctg\frac{4}{3}+2\pi k\right),\quad k\in Z.$$

Ma’lumki,

$$\arg(z_1\cdot z_2)=\arg z_1+\arg z_2,$$

$$\arg\frac{z_1}{z_2}=\arg z_1-\arg z_2,$$

$$\arg(z^n)=n\arg z,\qquad n\in Z,$$

$$\arg\sqrt[n]{z}=\frac{1}{n}\arg z,\qquad n\in Z.$$

U holda (10.8.12) formulaga asosan

$$Ln(z_1\cdot z_2)=Lnz_1+Lnz_2,$$

$$Ln\frac{z_1}{z_2}=Lnz_1-Lnz_2,$$

$$Ln(z^n)=nLnz+2k\pi,\ k\in Z,$$

$$Ln\sqrt[n]{z}=\frac{1}{n}Lnz.$$

xossalarni olamiz.

(10.8.4) formula $e$ ning ixtiyoriy kompleks ko‘rsatkichli darajasini hisoblash imkonini beradi. Ixtiyoriy kompleks sonni kompleks ko‘rsatkichli darajaga ko‘tarish amalini aniqlash uchun, avvalo logarifmik funksiyaning ta’rifidan noldan farqli ixtiyoriy $a$ kompleks son berilganda $e^{Lna}=a$ o‘rinli bo‘lishi kelib chiqishini aytamiz.

Haqiqiy sohada $a>0$ bo‘lganda $a^z=e^{z\ln a}$ ayniyat o‘rinlidir. Endi ixtiyoriy kompleks $a$ va $z$ lar uchun $a^z$ ni

$$a^z=e^{zLna}\tag{10.8.13}$$

formula bilan aniqlaymiz.

Yuqorida aytilgan logarifmik funksiyaning ko‘p qiymatlilik xossasiga asosan $a^z$ ifoda ham ko‘p qiymatlidir. Uning bosh qiymati deb, (10.8.13) ning o‘ng tomonidagi ifodada $Lna$ o‘rniga $\ln a$ ni qo‘yish natijasida olingan ifoda qiymatiga aytamiz.

**3-misol.** $i^i$ ni toping.

**Yechish.** $i^i=e^{iLni}=e^{i\left(\frac{\pi}{2}i+2k\pi i\right)}=e^{-\frac{\pi}{2}-2k\pi}$, $(k\in Z)$. $i^i$ ni bosh qiymati $(k=0)$ $e^{-\frac{\pi}{2}}$ ga tengligini ko‘ramiz.

### 10.8.3. Teskari trigonometrik funksiyalar

(10.8.1) bandda kompleks tekislikda aniqlangan trigonometrik funksiyalarga teskari bo‘lgan kompleks o‘zgaruvchili funksiyalarni *teskari trigonometrik funksiyalar* deb ataladi. Bu funksiyalarning ta’riflarini keltiramiz.

Agar $z=\sin w$ o‘rinli bo‘lsa $w$ ni $z$ kompleks sonning *arksinusi* deyiladi va

$$w=Arc\sin z$$

kabi yoziladi. Xuddi Shunga o‘xshash, agar $z=\cos w$ o‘rinli bo‘lsa, $w$ ni $z$ ning *arkkosinusi* deyiladi va

$$w=Arc\cos z$$

bilan belgilanadi. Agar $z=tgw$ bo‘lsa, $w$ ni $z$ ning *arktangensi* deb,

$$w=Arctgz$$

kabi belgilanadi va nihoyat, $z=ctgw$ o‘rinli bo‘lganda, $w$ ni $z$ ning *arkkotangensi* deb atalib,

$$w=Arcctgz$$

kabi yoziladi.

Endi, $z$ berilganda, yuqorida ta’riflangan teskari trigonometrik funksiyalarning qiymatlarini topish masalasini qaraylik.

$z=\sin w$ o‘rinli bo‘lsin deb faraz qilaylik. U holda, (10.8.1) ga asosan

$$z=\frac{e^{iw}-e^{-iw}}{2i}$$

ni olamiz. Bundan

$$e^{iw}-2iz-e^{-iw}=0$$

yoki oxirgi tenglamaning har ikki tomoniga $e^{iw}$ ni hadlab ko‘paytirib,

$$\left(e^{iw}\right)^2-2ize^{iw}-1=0$$

ga, ya’ni $e^{iw}$ ga nisbatan kvadrat tenglamaga kelamiz va uni echib,

$$e^{iw}=iz+\sqrt{1-z^2}$$

ni yoki

$$iw=Ln\left(iz+\sqrt{1-z^2}\right)$$

ni olamiz. Oxirgidan

$$w=Arc\sin z=-iLn\left(iz+\sqrt{1-z^2}\right) \qquad (10.8.14)$$

formulaga kelamiz. (10.8.14) formulada kvadrat ildiz chiqarish amali kompleks sohada ikki qiymatli bo‘lishini eslatamiz.

Endi $z=tgw$ deb faraz qilaylik. U holda

$$z=\frac{e^{iw}-e^{-iw}}{i\left(e^{iw}+e^{-iw}\right)}$$

bo‘lib, bundan

$$e^{2iw}=\frac{1+iz}{1-iz}$$

ni yoki

$$2iw=Ln\frac{1+iz}{1-iz}$$

va nihoyat

$$w = Arctgz = -\frac{i}{2} Ln \frac{1+iz}{1-iz} \qquad (10.8.15)$$

formulani olamiz. Xuddi yuqoridagiga o‘xshash ish jarayonini takrorlab,

$$Arc\cos z = -iLn\left(z + \sqrt{-1+z^2}\right), \qquad (10.8.16)$$

$$Arcctgz = \frac{i}{2} Ln \frac{z-i}{z+i} \qquad (10.8.17)$$

formulalarni keltirib chiqarish osondir.

**4-misol.** $Arc\sin\sqrt{2}$ ni toping.

**Yechish.**

$$z = \sqrt{2},$$

$$Arc\sin\sqrt{2} = -iLn\left(i\sqrt{2} + \sqrt{1-2}\right) = -iLn\left(i\left(\sqrt{2} \pm 1\right)\right) =$$

$$= -i\left(\ln\left(\sqrt{2} \pm 1\right) + \left(\frac{\pi}{2} + 2k\pi\right)i\right) = \frac{\pi}{2} + 2k\pi - i\ln\left(\sqrt{2} \pm 1\right), \quad k \in Z.$$

**Tekshirish.**

$$\sin Arc\sin\sqrt{2} = \sin\left(\frac{\pi}{2} + 2k\pi - i\ln\left(\sqrt{2} \pm 1\right)\right) = \cos\left(i\ln\left(\sqrt{2} \pm 1\right)\right) = \frac{ei^{i\ln\left(\sqrt{2}\pm 1\right)} + e^{-i\cdot i\ln\left(\sqrt{2}\pm 1\right)}}{2} =$$

$$= \frac{e^{-\ln\left(\sqrt{2}\pm 1\right)} + e^{\ln\left(\sqrt{2}\pm 1\right)}}{2} = \frac{\frac{1}{\sqrt{2} \pm 1} + \left(\sqrt{2} \pm 1\right)}{2} = \frac{\sqrt{2} \mp 1 + \sqrt{2} \pm 1}{2} = \sqrt{2}.$$

**5-misol.** $Arctg2i$ ni toping.

**Yechish.**

$$Arctg2i = -\frac{i}{2} Ln \frac{1+i\cdot 2i}{1-i\cdot 2i} = -\frac{i}{2}\cdot Ln\left(-\frac{1}{3}\right) =$$

$$= -\frac{i}{2}\left(\ln\frac{1}{3} + \pi i + 2k\pi i\right) = \frac{\pi}{2} + k\pi + \frac{\ln 3}{2} i.$$

Giperbolik $shz$, $chz$, $thz$, $cthz$ funksiyalarga teskari bo‘lgan funksiyalarni *teskari giperbolik funksiyalar* deyiladi va ularni mos ravishda $Arshz$, $Archz$, $Arthz$, $Arcthz$ kabi belgilanadi.

Agar $z = shw$ bo‘lsa, $w = Arshz$ bo‘lib,

$$z = \frac{e^w - e^{-w}}{2},$$

bundan

$$e^{2w} - 2ze^w - 1 = 0$$

ga ega bo‘lamiz. Bu tenglamani $e^w$ ga nisbatan echib

$$e^w = z + \sqrt{z^2 + 1}$$

ni olamiz yoki

$$w = Arshz = Ln\left(z + \sqrt{z^2 + 1}\right)$$

(10.8.18)

Xuddi Shunday ish jarayonini takrorlab,

$$Arcchz = Ln\left(z + \sqrt{z^2 - 1}\right),$$
$$Arthz = \frac{1}{2} Ln \frac{1+z}{1-z}, \qquad (10.8.19)$$
$$Arcthz = \frac{1}{2} Ln \frac{1+z}{z-1}.$$

formulalarni ham keltirib chiqish mumkin.

(10.8.14)-(10.8.19) formulalarga asoslanib teskari trigonometrik va giperbolik funksiyalar ko‘p qiymatli funksiyalar ekanligi haqida xulosa chiqarish osondir.

## 10.9. Chegirmalar nazariyasi

Agar $f(z)$ funksiya kompleks tekislikning biror $a$ nuqtasida golomorf bo‘lsa, u holda Koshi teoremasiga ko‘ra

$$\int_c f(z)dz = 0$$

bo‘ladi, bu yerda C $a$ nuqtani o‘z ichiga oluvchi Shunday bo‘lakli silliq kontur bo‘lib, u bilan chegaralangan sohada va uning nuqtalarida $f(z)$ funksiya golomorfdir. Agar $a$ nuqta $f(z)$ funksiyaning maxsus nuqtasi bo‘lib, C kontur ustidagi va u bilan chegaralangan sohaning $a$ nuqtasidan boshqa barcha nuqtalarida golomorf bo‘lsa, $a$ ni $f(z)$ funksiyaning yakkalangan maxsus nuqtasi deyilib, bu holda yuqorida yozilgan integral noldan farqli bo‘lishi mumkin hamda uning qiymati C konturning shakliga bog‘liq bo‘lmasligi aniqdir. Bunday qaralayotgan holda $f(z)$ funksiyani $a$ nuqtaning qandaydir yaqin atrofida $(0 < |z-a| < r)$ Loran qatoriga yoyish mumkin bo‘ladi:

$$f(z) = \sum_{n=-\infty}^{+\infty} C_n (z-a)^n \qquad (10.9.1)$$

(10.9.1) Loran qatori yuqorida aytilgan C konturning nuqtalarida tekis yaqinlashuvchibo‘lishi aniqdir, chunki C kontur $a$ nuqtaning $f(z)$ golomorf bo‘lgan qandaydir yaqin atrofiga tegishlidir. (10.9.1) qatorni S kontur bo‘yicha hadlab integrallab

$$\int_C f(z)dz = 2\pi i C_{-1} \qquad (10.9.2)$$

ga ega bo‘lamiz. Haqiqatdan ham

$$\int_C (z-a)^m dz = 0, \qquad (m \in Z_0).$$
$$\int_C \frac{dz}{z-a} = 2\pi i,$$
$$\int_C \frac{dz}{(z-a)^n} = 0, \qquad (n = 2;3;...).$$

### 10.9.1. Yakkalangan maxsus nuqtaga nisbatan funksiyaning chegirmasi

Aytaylik, kompleks tekislikning $a$ nuqtasi $f(z)$ funksiyaning yakkalangan maxsus nuqtasi bo‘lsin.

**10.9.1-ta’rif.** $f(z)$ funksiyaning $a$ yakkalangan maxsus nuqtasiga nisbatan chegirmasi deb, $\frac{1}{2\pi i}\int\limits_C f(z)dz$ ga aytamiz va $\operatorname{Re}s[f(z);a]$ kabi belgilaymiz.

Bu ta’rif va (10.9.2) dan funksiya chegirmasi Loran qatorining $C_{-1}$ koeffisientidan iborat ekanligini ko‘ramiz, ya’ni

$$\operatorname{Re}s[f(z);a]=C_{-1}=\frac{1}{2\pi i}\int\limits_C f(z)dz. \qquad (10.9.3)$$

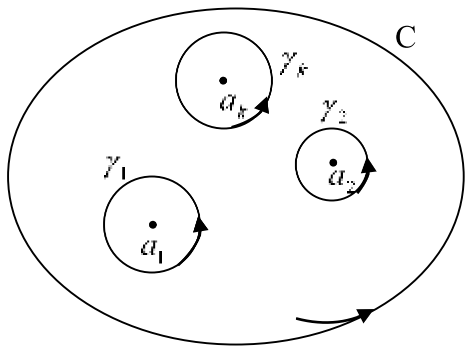
Endi, $f(z)$ funksiya D sohada uning chekli sondagi $a_1,a_2,...,a_k$ maxsus nuqtalaridan boshqa barcha nuqtalarida golomorf bo‘lsin, C esa barcha $a_1,a_2,...,a_k$ maxsus nuqtalarni o‘z ichiga oluvchi va D sohada yotuvchi bo‘lakli silliq konturdan iborat deb faraz qilaylik.

**10.9.1-terema (chegirmalar haqidagi asosiy teorema).** Yuqoridagi shart bajarilganda $\frac{1}{2\pi i}\int\limits_C f(z)dz$ $f(z)$ funksiyaning barcha maxsus nuqtalariga nisbatan chegirmalarining yig‘indisiga tengdir.

**Isbot.** Maxsus $a_1,a_2,...a_k$ nuqtalarni markazi Shu nuqtalarda bo‘lgan $\gamma_1,\gamma_2,...,\gamma_k$ aylana konturlari bilan o‘raymiz va ularning radiuslarini Shunday tanlaymizki, ularga hech qanday juftligi umumiy nuqtaga ega bo‘lmasin va har biri D sohada yotuvchi S kontur ichiga joylashsin (10.9.1-rasmga qarang).

$f(z)$ funksiya chegarasi $K=C\cup\left(\bigcup\limits_{n=1}^{k}\gamma_n^-\right)$ murakkab kontur bilan chegaralangan $k+1$ bog‘lamli yopiq sohaning har bir nuqtasida golomorf bo‘lishi aniq. U holda Koshi teoremasiga ko‘ra

$$\frac{1}{2\pi i}\int\limits_C f(z)dz=\sum_{n=1}^{k}\frac{1}{2\pi i}\int\limits_{\gamma_n} f(z)dz=\sum_{n=1}^{k}\operatorname{Re}s[f(z);a_n]. \qquad (10.9.4)$$




10.9.1-rasm.

### 10.9.2. Funksiyaning qutbga nisbatan chegirmasini hisoblash

Bu bandda kompleks tekislikning biror $a$ nuqtasi $f(z)$ funksiyaning yakkalangan qutb nuqtasi bo‘lgan holda chegirmani hisoblash usulini keltiramiz. Aytaylik, $a$ nuqta $f(z)$ funksiyaning oddiy (1-tartibli) qutb nuqtasi bo‘lsin. U holda $f(z)$ ning $a$ nuqta atrofidagi Loran qatoriga yoyilmasi

$$f(z)=\frac{C_{-1}}{z-a}+C_0+C_1(z-a)+...+C_n(z-a)^n+... \qquad (10.9.5)$$

bo‘ladi. endi (10.9.5) ning har ikki tomonini $(z-a)$ ga hadlab ko‘paytirsak,

$$(z-a)f(z)=C_{-1}+\sum_{k=0}^{\infty}C_k(z-a)^{k+1} \qquad (10.9.6)$$

ni olamiz. Olingan darajali qator yig‘indisi $a$ nuqtada uzluksiz funksiya bo‘lishi aniqdir. (10.9.6) da $z\to a$ dagi limitga o‘tib,

$$\lim_{z\to a}(z-a)f(z)=C_{-1}$$

ni olamiz. Bundan chegirma uchun

$$\operatorname{Re}s[f(z);a]=\lim_{z\to a}(z-a)f(z) \qquad (10.9.7)$$

formulani olamiz. (10.9.7) $a$ oddiy qutbga nisbatan $f(z)$ funksiyaning chegirmasini hisoblash formulasidir. Agar $f(z)$ funksiya

$$f(z)=\frac{\varphi(z)}{\psi(z)}$$

ko‘rinishda ikkita analitik funksiyalarning nisbati kabi ifodalangan va $\varphi(a)\neq 0$, $\psi(a)=0$, $\psi'(a)\neq 0$ bo‘lsa, $a$ nuqta $f(z)$ uchun oddiy qutb bo‘lib, (10.9.7) formula o‘rinli va

$$\operatorname{Re}s[f(z);a]=\lim_{z\to a}(z-a)\frac{\varphi(z)}{\psi(z)}=\lim_{z\to a}\frac{\varphi(z)}{\dfrac{\psi(z)-\psi(a)}{z-a}}=\frac{\lim\limits_{z\to a}\varphi(z)}{\lim\limits_{z\to a}\dfrac{\psi(z)-\psi(a)}{z-a}}=\frac{\varphi(a)}{\psi'(a)},$$

ya’ni $f(z)$ ning $a$ oddiy qutbiga nisbatan chegirmasi uchun $\operatorname{Re}s\left[\frac{\varphi(z)}{\psi(z)};a\right]=\frac{\varphi(a)}{\psi'(a)}$ formulaga ega bo‘lamiz.

Endi, $a$ nuqta $f(z)$ ning $n$-tartibli qutb nuqtasi bo‘lsin deb faraz qilsak, $a$ nuqta atrofidagi $f(z)$ ning Loran qatoriga yoyilmasi uchun

$$f(z)=\sum_{k=-n}^{+\infty}C_k(z-a)^k$$

ni olamiz. Bu tenglikning har ikki tomonini $(z-a)^n$ ga hadlab ko‘paytirib,

$$(z-a)^n f(z)=\sum_{k=0}^{+\infty}C_{k-n}(z-a)^{k-n}$$

ga ega bo‘lamiz va uni $n\geq 2$ faraz asosida $(n-1)$ marta hadlab differensiallash natijasida

$$\frac{d^{n-1}\left[(z-a)^n f(z)\right]}{dz^{n-1}}=(n-1)^1-C_{-1}+\sum_{k=0}^{+\infty}(k-n)!C_{k-n}(z-a)^{k-n}$$

ga kelamiz. Bunda $z\to a$ da limitga o‘tib,

$$(n-1)!C_{-1} = \lim_{z\to a} \frac{d^{n-1}\left[(z-a)^n f(z)\right]}{dz^{n-1}}$$

ni, ya'ni $n$-tartibli qutbga nisbatan chegirma uchun

$$\operatorname{Re}s\left[f(z);a\right] = \frac{1}{(n-1)!}\lim_{z\to a} \frac{d^{n-1}\left[(z-a)^n f(z)\right]}{dz^{n-1}}$$

formulani olamiz.

### 10.9.3. Chegirmalar yordamida integrallarni hisoblash

Chegirmalar yordamida ba'zi bir xosmas integrallarni hisoblash mumkin bo‘ladi.

Avvalo, cheksiz uzoqlashgan nuqta $f(z)$ funksiyaning ikkinchi yoki undan ham yuqoriroq tartibli nolidan iborat bo‘lishini talab qilamiz. U holda $z=\infty$ atrofidagi bu funksiyaning Loran qatoriga yoyilmasi

$$f(z) = \sum_{k=2}^{+\infty} \frac{C_{-k}}{z^k} \qquad (10.8.20)$$

ko‘rinishga ega bo‘ladi. Shu bilan birga $f(z)$ funksiya haqiqiy o‘qda analitik bo‘lishini, yuqori yarim tekislikda $(\operatorname{Im} z > 0)$ faqat chekli sondagi $a_1, a_2, ..., a_n$ maxsus nuqtalarga ega deb faraz qilamiz. U holda yuqori yarim tekislikdagi $f(z)$ funksiyaning barcha maxsus nuqtalarini radiusi yetarlicha katta bo‘lgan markazi koordinata boshiga joylashgan R radiusli yuqori yarim tekislikdagi yarim doira ichiga joylashtirish mumkin bo‘ladi. bu yarim doira chegarasidan iborat C kontur bo‘yicha $f(z)$ funksiyadan olingan integral uchun chegirmalar haqidagi asosiy teoremaga asosan

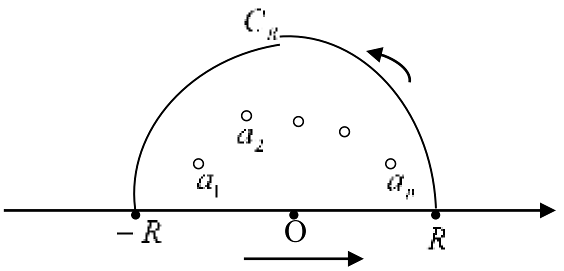
$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Re}s\left[f(z);a_k\right]$$

o‘rinli bo‘ladi. Shuningdek, yarim doira radiusi R ni bundan keyingi kattalashtirish natijasida bu tenglik saqlanib qolishini aytamiz.

Aytaylik, $C_R$-yarim doira chegarasining tarkibiga kiruvchi yarim aylana bo‘lsin (10.8.2-rasmga qarang), u holda

$$\int_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx$$

bo‘lishi ravshandir.




10.8.2-rasm.

(10.8.20) ga asosan $f(z)=\frac{1}{z^2}\varphi(z)$ deb yozaolamiz, bu yerda

$$\varphi(z)=C_{-2}+\frac{C_{-3}}{z}+\frac{C_{-4}}{z^2}+\ldots.$$

Bundan cheksiz uzoqlashgan nuqta $\varphi(z)$ uchun to‘g‘ri nuqta ekanligini ko‘ramiz hamda $\lim\limits_{z\to\infty}\varphi(z)=C_{-2}$ bo‘lib, $\varphi(z)$ ning cheksiz uzoqlashgan nuqta atrofida xususan $C_R$ ning barcha nuqtalarida chegaralanganligi kelib chiqadi, ya’ni $\exists M>0$

$$|\varphi(z)|\le M\ .$$

Bular asosida

$$\left|\int\limits_{C_R} f(z)dz\right|=\left|\int\limits_{C_R}\frac{\varphi(z)dz}{z^2}\right|\le\frac{M}{R^2}\pi R=\frac{M\pi}{R}.$$

Oxirgi munosabat asosida

$$\lim_{R\to\infty}\int\limits_{C_R} f(z)dz=0$$

bo‘lishini olamiz. Nihoyat

$$\lim_{R\to\infty}\int\limits_{C} f(z)dz=0+\lim_{R\to\infty}\int\limits_{-R}^{R} f(x)dx.$$

Bundan

$$\int\limits_{-\infty}^{\infty} f(x)dx=2\pi i\sum_{k=1}^{n}\mathrm{Re}\,c[f(z);a_k] \qquad (10.8.21)$$

ekanligini olamiz.

Masalan, $\int\limits_{-\infty}^{+\infty}\frac{dx}{(x^2+1)^2}$ xosmas integralni hisoblaylik. $f(z)=\frac{1}{(z^2+1)^2}$ funksiya uchun kompleks tekislikning cheksiz uzoqlashgan nuqtasi 4-tartibli noldan iborat va uning maxsus nuqtalari $z=\pm i$ bo‘lib, ulardan faqat bittasi $z=i$ ikkinchi tartibli qutb yuqori yarim tekislikda yotadi.

Demak, (10.8.21) ga asosan

$$\int\limits_{-\infty}^{+\infty}\frac{dx}{(x^2+1)^2}=2\pi i\,\mathrm{Re}\,s[f(z);i]$$

$$\mathrm{Re}\,s\left[\frac{1}{(z^2+1)^2};i\right]=\mathrm{Re}\,s[f(z);i]=\lim_{z\to i}\frac{d}{dz}\left[\frac{(z-i)^2}{(z^2+1)^2}\right]=\lim_{z\to i}\frac{d}{dz}\left[\frac{1}{(z+i)^2}\right]=$$

$$=\lim_{z\to i}\frac{-2}{(z+i)^3}=-\frac{2}{8i^3}=-\frac{1}{4}i.$$

Demak, $\int\limits_{-\infty}^{+\infty}\frac{dx}{(x^2+1)^2}=2\pi i\cdot\left(-\frac{i}{4}\right)=\frac{\pi}{2}.$

## 10-bobga doir mashqlar

1. Tenglama vositasida qanday chiziq berilgan?

a) $z=(1+i)t \quad (t\in R)$. Javob: $y=x$ to‘g‘ri chiziq.

b) $z=a\cos t+ib\sin t \quad (a>0,\ b>0,\ 0\le t\le 2\pi)$. Javob: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ - ellips.

c) $z=t+\frac{i}{t} \quad (t\in R,\ t\ne 0)$. Javob: $y=\frac{1}{x}$ - giperbola.

g) $z=t^2+\frac{i}{t^2} \quad (t\in R,\ |t|>0)$. Javob: $y=\frac{1}{x}$ - giperbolaning I chorakdagi qismi.

2. Ifoda qiymatini toping.

a) $Ln(1+i)$. Javob: $\frac{1}{2}\ln 2+i\left(\frac{\pi}{4}+2k\pi\right) \quad (k\in Z)$.

b) $Ln(-i)$. Javob: $\left(-\frac{\pi}{4}+2k\pi\right)i \quad (k\in Z)$.

c) $Ln(-3+4i)$. Javob: $\ln 5+i\left(-arctg\frac{4}{3}+(2k+1)\pi\right) \quad (k\in Z)$.

g) $\sin i$. Javob: $i\,sh1$.

d) $\cos(1+i)$. Javob: $\cos 1\cdot ch1-i\sin 1 sh1$.

e) $ch\,i$. Javob: $\cos 1$.

j) $sh(-2+i)$. Javob: $-\cos 1\cdot sh2+i\ \sin 1ch2$.

z) $Arc\sin 3$. Javob: $\left(2k+\frac{1}{2}\right)\pi-i\ln\left(3\pm 2\sqrt{2}\right) \quad (k\in Z)$.

i) $Arc\,tg\frac{i}{3}$. Javob: $k\pi+\frac{i}{2}\ln 2$.

3. Berilgan funksiya uchun Koshi-Riman shartini tekshiring.

$a)\ w=\sin z$; $\quad b)\ w=\cos z$; $\quad v)\ w=Lnz$; $\quad g)\ w=Arc\sin z$

4. Haqiqiy qismi $x^3-3xy^2$ dan iborat bo‘lgan $z$ $(z=x+iy)$ o‘zgaruvchili analitik funksiyani toping.

Javob: $z^3+C_i$.

5. Mavhum qismi $2xy+3x$ dan iborat bo‘lgan $z$ kompleks o‘zgaruvchining analitik funksiyasini toping.

Javob: $z^2+3iz+C$.

6. Agar S-kontur markazi koordinata boshida radiusi 3 ga teng aylanadan iborat bo‘lsa, $\int_C \frac{z^2dz}{z-2i}$ integralni hisoblang.

Javob: $-8\pi i$.

7. Agar S-kontur markazi $2i$ nuqtada radiusi 2 ga teng aylanadan iborat bo‘lsa, $\int_C \frac{dz}{z^2+9}$ ni hisoblang.

Javob: $\frac{\pi}{54}$.

8. $\frac{1}{(z-a)(z-b)}$ funksiyani $|a|<|z|<|b|$ halqa ichida Loran qatoriga yoying.

Javob: $e\left(1+z+\frac{3}{2}z^2+\frac{13}{6}z^3+\ldots\right)$, yaqinlashish radiusi $r=1$.

9. $z^2 e^{\frac{1}{z}}$ funksiyani $z=0$ nuqta atrofida Loran qatoriga yoying.

Javob: $\frac{1}{2}+z+z^2+\frac{1}{3!\,z}+\frac{1}{4!\,z^2}+\ldots+\frac{1}{(n+2)!\,z^n}+\ldots$.

10. Agar S-kontur musbat yo‘nalish bo‘yicha $x^2+y^2=2x$ aylanani bir marta o‘rovchi chiziq bo‘lsa, $\int_C \frac{dz}{z^4+1}$ ni hisoblang.

Javob: $-\frac{\pi i}{\sqrt{2}}$.

11. $\int_{-\infty}^{+\infty} \frac{x^2+1}{x^4+1}dz$ xosmas intagralni chegirmalar yordamida hisoblang.

Javob: $\pi\sqrt{2}$.

## 10-bob bo‘yicha bilimingizni sinab ko‘ring

1. Kompleks tekislikda berilgan nuqtaning $\varepsilon$ atrofi va $\varepsilon$ qisqa atrofi tushunchalarini bayon qiling.
2. Kompleks tekislikda sohaning limitik nuqtasini ta’riflang.
3. Kompleks tekislikda soha, uning ichki nuqtasi, chegaraviy nuqtasi va sohaning bog‘liqligi tushunchalarini keltiring.
4. Soha chegarasi va yopiq soha tushunchalarini bayon qiling.
5. Soha chegarasidan iborat konturlar sistemasini musbat yo‘nalishi va ko‘p bog‘lamli soha tushunchalarini keltiring.
6. Kompleks o‘zgaruvchili funksiyaga ta’rif berish.
7. Kompleks sonli ketma-ketlik va uning limitini ta’riflang.
8. Kompleks tekislikning cheksiz uzoqlashgan nuqtasi nimadan iborat va u qanday ta’riflanadi?
9. Kompleks o‘zgaruvchili funksiyaning limiti, uzluksizligi va hosilasi tushunchalarini ta’riflang.
10. Koshi-Riman shartlarini keltiring.
11. Kompleks o‘zgaruvchili funksiyaning integrali tushunchasini ta’riflang va xossalarini keltiring.
12. Koshi teoremasini ayting.
13. Koshi teoremasini yozing. Koshi integralini bayon qiling.
14. Teylor va Loran qatorlarini bayon qiling.
15. Funksiya maxsus nuqtalari va ularning tasvirlarini keltiring.

16. Kompleks o‘zgaruvchili asosiy elementar funksiyalarni yozing va xossalarini bayon qiling.
17. Kompleks tekislikda funksiya chegirmasini ta’riflang.
18. Yakkalangan maxsus nuqtaga nisbatan funksiya chegirmasini hisoblash usullarini keltiring.
19. Chegirmalar yordamida integrallarni hisoblash usulini bayon qiling.

# 11-bob. Operatsion hisob

Bu bobda Laplas almashtirishning ta’rifini va uning xossalarini hamda unga asoslanib operatsion hisob usuli nazariyasini qurishni keltiramiz. Buning uchun bizga kompleks o‘zgaruvchili funksiyalar nazariyasini mukammal bilish lozimligini aytamiz. Bu o‘rinda operatsion hisob usuli matematikaning ba’zi muhim ahamiyatga ega bo‘lgan bo‘limlarida, masalan, differensial tenglamalarni yechishda, elektrotexnikaning nazariy asoslariga tegishli bo‘lgan bir qator masalalarni hal qilishda, elektr zanjirlaridagi o‘tish jarayonlarini o‘rganishda va boshqa ko‘p masalalarni hal qilishda salmoqli tatbiqqa egaligini aytamiz.

## 11.1.Laplas almashtirishning ta’rifi.

Aytaylik, $f(t)$ haqiqiy o‘zgaruvchili funksiya uchun quyidagi shartlar bajarilsin:
1) $t \geq 0$ bo‘lganda aniqlangan va bo‘lakli uzluksiz yani (0;+∞)oraliqqa tegishli har qanday chekli chekli sondagi intervalda birinchi jins uzilish nuqtalari mavjud bo‘lishi mumkin);
2) $t \leq 0$ bo‘lganda $f(t) = 0$;
3) $t$ ning o‘sishi bilan $f(t)$ funksiyaning moduli uchun

$$|f(t)| \leq Me^{\tau}$$

o‘rinli bo‘lib, bu yerda *m>o* va b qandaydir o‘zgarmaslar.
Bu shartlar bajariladi deb faraz qilgan holda Laplas almashtirishi va integrali tushunchalarini ta’riflaymiz.
<u>1-ta’rif.</u> Haqiqiy o‘zgaruvchili $f(t)$ funksiyaning Laplas almashtirishi deb,

$$F(p) = \int_0^{+\infty} f(t)e^{-pt}dt \qquad (11.1.1)$$

formula vositasida aniqlangan r kompleks o‘zgaruvchili *F(r)* funksiyaga aytamiz, (1.1) ning o‘ng tomonidagi ifodani esa Laplas integrali deb ataymiz.
Yuqorida bayon qilingan 1)-3) shartlarni qanoatlantiruvchi ixtiyoriy haqiqiy o‘zgaruvchili $f(t)$ funksiyani original $F(p)$ funksiyani esa, $f(t)$ ning tasviri (ba’zan

Laplas bo‘yicha tasviri) deb ataymiz. $f(t)$- original va *f(r)*- tasvir orasidagi moslikni

$F(t) \leftrightarrow F(p)$ yoki $F(p) \leftrightarrow F(t)$ kabi yozamiz. Bundan tashqari,

$$f(t) \leftarrow F(p),\ F(p) = L[f(t)]$$

ko‘rinishdagi belgilashlar ham qo‘llanishini aytamiz.

**1-misol.** Hevisaydning birlik funksiyasining Laplas tasviri topilsin.

**Yechish.** Bu funksiyani quyidagicha

$$\sigma_0(t) = \begin{cases} 0, & агар\ \ t < 0; \\ 1, & агар\ \ t \geq 0. \end{cases}$$

yozish mumkin. Bundan ko‘rinadiki, uning uchun 1)-3) shartlar bajarilib, *M=1* va $\delta = 0$ bo‘lishi aniqdir. U holda (1.1) formula yordamida bu funksiyaning Laplas tasviri uchun

$$F(p) = \int_0^{+\infty} e^{-pt} dt$$

ni olamiz. Bunda $r \neq 0$ deb faraz qilib,

$$F(p) = \lim_{T \to +\infty} \int_0^T e^{-pt} dt = \frac{1}{p}(1 - \lim_{T \to +\infty} e^{-pT})$$

ga ega bo‘lamiz. Bundan va oxirgi chekli limit mavjud bo‘lsa,u nolga teng bo‘lib,

$$F(p) = \frac{1}{p}$$

ekanligi kelib chiqadi. Haqiqatdan ham $p = \alpha + i\beta$ deb faraz qilsak,

$$e^{-pT} = e^{-\alpha T}(\sin \beta T - iSin\beta T)$$

bo‘lib, oxirgi ifodaning $T \to +oo$ da chekli limiti mavjud bo‘lishi uchun $\propto \geq 0$ bo‘lishi zarur va yetarlidir va bu limit 0 ga tengdir. Demak,

$$1 \leftrightarrow \frac{1}{p} \quad (\operatorname{Re} p > 0). \qquad (11.1.2)$$

Laplas almashtirishi ta’rifidagi $f(t)$ haqiqiy o‘zgaruvchili kompleks funksiya bo‘lishi ham mumkinligini va bu holda uning xaqiqiy va mavhum qismlari bo‘lgan haqiqiy o‘zgaruvchili haqiqiy funksiyalarning ikkalasi ham yuqoridagi 1)-3) shartlarni qanoatlantirishini talab qilish lozimligini aytamiz.

**2-misol.** $e^{at}$ funksiyaning Laplas tasviri topilsin, bu yerda $t \in (0;+\infty), a \in K$ (*a*- kompleks o‘zgarmas).

**Yechish.** $a = \alpha_0 + \beta_0 i\ (\alpha_{0,}\beta_0 \in R)$ deylik.

U holda

$$e^{at} = e^{\alpha_0 - t}(\cos\beta_0 t + iSin\beta_0 t) = e^{\alpha_0 t} Cos\beta_0 t + ie^{\alpha_0 t}\sin\beta_0 t$$

bo‘lib, bu funksiyaning haqiqiy qismi $\varphi(t) = e^{\alpha_0 t}\cos\beta_o t$ va mavhum qismi $\psi(t) = e^{\alpha_o t} Sin\beta_0 t$ larning ikkalasi ham 1)-3) shartlarni qanoatlantirishini tekshirish qiyin emas. Agar *f(t)* ni

$$f(t) = \begin{cases} 0, & t < 0; \\ e^{at}, & t \geq 0 \end{cases}$$

Kabi aniqlasak, uning uchun 1)-3) shartlarning bajarilishini ko‘ramiz. Endi *f(t)* ning Laplas tasvirini topaylik:

$$F(p) = \int_0^{+\infty} e^{at}.e^{-pt} = \int_0^{+\infty} e^{-(p-a)t} dt = \frac{1}{p-a,} \quad (\mathrm{Re}\, p > \mathrm{Re}\, a). \qquad (11.1.3)$$

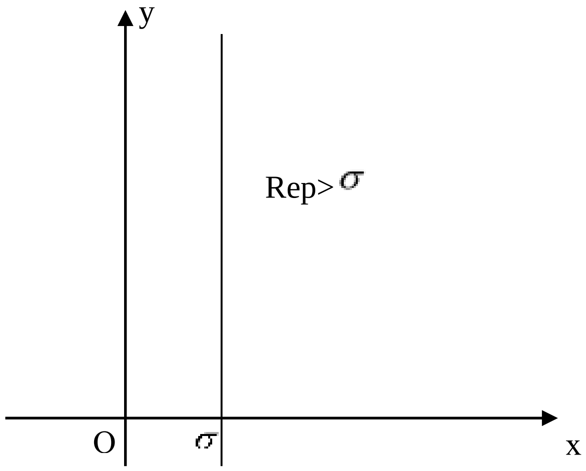
ni olamiz.

Endi har bir *F(t)* originalga ma’lum bitta *F(p)* tasvir mos kelishini va Laplas almashtirishining umumiy xossalarini keltirib chiqarishga o‘tamiz.

**1.1-teorema.** Agar haqiqiy o‘zgaruvchili *f(t)* funksiya originaldan iborat bo‘lsa (yani 1)-3) shartlarni qanoatlantirsa), u holda

$$F(p) = \int_0^{+\infty} f(t) e^{-pt} dt \qquad (11.1.4)$$

Laplas integrali kompleks *p* o‘zgaruvchining $\mathrm{Re}\, p > \sigma$ shartni qanoatlantiruvchi barcha qiymatlarida absolyut yaqinlaShuvi bo‘ladi (bu yerda $\sigma$ 3) shartda qatnashgan o‘zgarmasdir) va bu integral vositasida aniqlangan *F(p)* tasvir $\mathrm{Re}\, p > \sigma$ yarim tekislikda analitik funksiyadan iborat bo‘ladi.




11.1.1-rasm

**Isbot.** Laplas integralining absolyut yaqinlashuvchi ekanligini isbotlash uchun 3)shartdan foydalanamiz. Unga ko‘ra $|f(t)| \le Me^{\sigma t}$ dir.Agar $p = \propto + \beta t$ deb faraz qilsak,

$$\left|e^{-pt}\right| = e^{-\propto t} \text{ bo‘lib, } \left|f(t)e^{-pt}\right| \le Me^{(g\sigma\tau - a)t} \qquad (11.1.4)$$

Endi $$\varphi(T) = \int_0^T \left|f(t)e^{-pt}\right| dt$$

yordamchi funksiyani kiritamiz, bunda $T \in [0;+\infty)$ bo‘lgan tayin qiymat deb qarasak, $\varphi(T)$ $[0;+\infty)$ oraliqda aniqlangan va kamaymovchi funksiya bo‘lishi 1) va 2) shartlardan kelib chiqadi. Endi (11.1.4)ni hisobga olib,

$$\varphi(T) \le M \int_0^T e^{-(\alpha - \sigma)t} dt$$

ni olamiz va nihoyat, $\alpha = \mathrm{Re}\, p > \sigma$ ekanligidan

$$\varphi(T) \le \frac{M}{\alpha-\sigma}(1-e^{-(\alpha-\sigma)t}) < \frac{M}{\alpha-\sigma}$$

ga ega bo‘lamiz. Bu $[o;+\infty)$ oraliqda aniqlangan va kamaymovchi $\varphi(T)$ funksiyaning $T \to +\infty$ bo‘lganda yuqoridan chegaralangan ekanligini ko‘rsatadi. Demak, $T \to +\infty$ bo‘lganda $\varphi(T)$ funksiyaning chekli limiti mavjuddir,yani Laplas integrali absolyut yaqinlashuvchidir.

Endi Laplas almashtirishi natijasida olingan *F(p)* funksiyaning $\operatorname{Re} p > \sigma$ bo‘lganda analitik ekanligini ko‘rsatish maqsadida (11.1.4) ning har ikki tomonini differentsiyallab hamda uning o‘ng tomonidagi ifodada integral ostida *P* parametr bo‘yicha differensiallash mumkinligini faraz qilgan holda quyidagi formal formulaga ega bo‘lamiz:

$$\frac{dF(p)}{dp} = -\int_0^{+\infty} tf(t)e^{-pt}dt. \qquad (11.1.5)$$

Bu formula haqiqatdan ham o‘rinli bo‘lishi uchun uning o‘ng tomonidagi hosmas integralning tekis yaqinlashuvchi ekanligini ko‘rsatish kifoya.

(11.1.5) ning o‘ng tomonidagi integral osti funksiyasi uchun $\operatorname{Re} p \ge \sigma_0 > \sigma$ bo‘lganda $\left|tf(t)e^{-pt}\right| \le Mte^{-(\sigma_0-\sigma)t}$ baho o‘rinlidir.U holda

$$\int_0^{+\infty}\left|tf(t)e^{-pt}\right|dt \le M\int_0^{+\infty} te^{-(\sigma_0-\sigma)t}dt$$

bo‘lib, oxirgi xosmas integralni bo‘laklab integrallash usulini qo‘llab hisoblash natijasida

$$\left|\int_0^{+\infty} tf(t)e^{-pt}dt\right| \le \frac{M}{(\sigma_0-\sigma)^2}$$

dan iborat *p* parametrga nisbatan tekis bo‘lga n bahoni olamiz. Bu (11.1.5) ning o‘ng tomonidagi xosmas integralning qaraliyotgan holda tekis yaqinlashishini ko‘rsatadi.Bundan *F(p)* kompleks o‘zgaruvchili funksiyaning analitik ekanligi kelib chiqadi.Bu o‘rinda tekis yaqinlashuvchi parametrga bog‘liq integrallar haqidagi teoremalarda integral osti funksiyasinining integral o‘zgaruvchisi hamda parametr bo‘yicha uzluksiz bo‘lishini talab qilinishini eslatgan holda bu teoremalar integral osti funktsisining integral o‘zgaruvchisi bo‘yicha bo‘lakli uzluksiz parametr bo‘yicha uzluksiz bo‘lgan holda ham o‘rinli bo‘lishini aytamiz.

Endi (1.4) ning har ikki tomonini p martda formal tarzda integral ostida differensiallab olingan

$$\frac{d^nF(p)}{dp^n} = \int_0^{+\infty}(-t)^n f(t)e^{-pt}dt \qquad (11.1.6)$$

ni yozib, o‘ng tomonidagi xosmas integral $\operatorname{Re} p \ge \sigma_0 > \sigma$ bo‘lganda absolyut va tekis yaqinlashuvchi bo‘lishini xuddi yuqoridagidek, *n* marta ketma-ket bo‘laklab integrallash yordamida ko‘rsatish mumkin bo‘lib va buning natijasida (1.6) ning o‘rinli ekanligi kelib chiqishini aytamiz.

Demak,( 1.4) Laplas almashtirishi vositasida aniqlangan *F(p)* funksiya $\operatorname{Re} P \geq \sigma_0 > \sigma$ yarim tekislikda analitik ekanligini ko‘ramiz va $\sigma_0 < \sigma$ ixtiyoriy son bo‘lgani sababli tasvirning *Rep*> $\sigma$ yarim tekislikda analitik bo‘lishini olamiz. Teorema isbotlandi.

## 11.2.Laplas almashtirishining xossalari.

Bu bandda originallarni $f(t), g(t),...$ kabi ularga mos tasvirlarni esa $F(p), \varsigma(p),..$ kabi belgilashga kelishib olamiz, yani

$$f(t) \leftrightarrow F(p), g(t) \leftrightarrow C(p),...$$

$1^0$ **.Chiziqlilik xossasi.** Agar A va B lar ixtiyoriy kompleks o‘zgarmaslar bo‘lsa,

$$af(t) + bg(t) \leftrightarrow aF(p) + aG(p) \qquad (11.2.1)$$

o‘rinlidir.

Bu xossaning isboti integralning chiziqlilik xossasidan kelib chiqadi.

$2^0$**. O’xshashlik teoremasi (xossasi).** Ixtiyoriy $\lambda > 0$ o‘zgarmas uchun

$$f(\lambda t) \leftrightarrow \frac{1}{\lambda} F(\frac{p}{\lambda}). \qquad (11.2.2)$$

**Isbot.** Haqiqatdan ham Laplas almashtirishining ta’rifiga ko‘ra

$$f(\lambda t) \leftrightarrow \int_0^{+\infty} f(\lambda t) e^{-pt} dt.$$

Ohirgi integralda $\lambda t = t_1$ almashtirish qilib,

$$f(\lambda t) \leftrightarrow \frac{1}{\lambda} \int_0^{+\infty} f(t_1) e^{-\frac{p}{\lambda} t_1} dt_1 = \frac{1}{\lambda} F\left(\frac{p}{\lambda}\right)$$

**3-misol.** *sint* va *cost* $(t \geq 0)$ larning tasviri topilsin.

**Yechish.** $\sin t = \frac{1}{2i}(e^{it} - e^{-it}), \cos t = \frac{1}{2}(e^{it} + e^{-it})$ ekanligidan (11.2.1) va (11.1.3) lar asosida

$$\sin t \leftrightarrow \frac{1}{2i}(\frac{1}{p-i} - \frac{1}{p+i}) = \frac{1}{p^2+1},$$

$$cost \leftrightarrow \frac{1}{2}(\frac{1}{p-t} + \frac{1}{p-t}) = \frac{p}{p^2+1}$$

yani

$$\sin t \leftrightarrow \frac{1}{p^2+1}$$

$$cost \leftrightarrow \frac{p}{p^2+1} \qquad (11.2.3)$$

larni olamiz.

**4-misol.** $\sin\omega t$ va $\cos\omega t$ $(t \geq 0)$ larning tasviri topilsin.
**Yechish.** (2.2)-(.2.3) larga asosan

$$sin\omega t \leftrightarrow \frac{1}{\omega}\frac{1}{(\frac{p}{\omega})^2+1} = \frac{\omega}{p^2+\omega^2}, \qquad (11.2.4)$$

$$cos\omega t \leftrightarrow \frac{1}{\omega}\frac{\frac{p}{\omega}}{(\frac{p}{\omega})^2+1} = \frac{p}{p^2+\omega^2}$$

larni olamiz.

**5-misol.** $sh\omega t$ va $ch\omega t$ $(t \geq o)$ larning tasviri topilsin.
**Yechish.** $sht = \frac{1}{2}(e^t - e^{-t}), cht = \frac{1}{2}(e^t + e^{-t})$ lar uchun (2.1) va (1.3) larni qo‘llab,

$$sht \leftrightarrow \frac{1}{p^2-1}, cht = \frac{p}{p^2-1}$$

larni olamiz.
Endi (11.2.2) va yuqoridagilarga asosan

$$sh\omega t \leftrightarrow \frac{1}{\omega}(\frac{1}{(\frac{p}{\omega})^2-1}) = \frac{\omega}{p^2-\omega^2}$$

$$chs\omega t \leftrightarrow \frac{1}{\omega}\left(\frac{\frac{p}{\omega}}{\left(\frac{p}{\omega}\right)^2-1}\right) = \frac{p}{p^2-\omega^2} \qquad (11.2.5)$$

larni olamiz. Ya’ni

$$sh\omega t \leftrightarrow \frac{\omega}{p^2-\omega^2}, ch\omega t \leftrightarrow \frac{p}{p^2-\omega^2}$$

$3^0$. **Siljish (so‘nish) teoremasi.** Ixtiyoriy $a$-kompleks son uchun $f(t) \leftrightarrow F(p)$ bo‘lganda

$$e^{\omega t} f(t) \leftrightarrow F(p-a) \qquad (11.2.6)$$

o‘rinlidir, yani originalni $e^{\omega t}$ ga ko‘paytirish tasvirning argumenti p ni $a$ ga siljishiga olib keladi.
**Isbot.** Laplas almashtirishiga asosan

$$e^{at} f(t) \leftrightarrow \int_0^{+\infty} e^{at} f(t) e^{-pt} dt = \int_0^{+\infty} f(t) e^{-(p-a)t} dt$$

Bundan (2.6) ni olamiz.
**6-misol.** $e^{at}\sin\omega t, e^{at}Cos\omega t$ $(t \geq o)$ larning tasviri topilsin.

**Yechish.** (11.2.6), (11.2.4) larga asosan

$$e^{at}sin\omega t \leftrightarrow \frac{\omega}{(p-a)^2+\omega^2}, e^{at}\cos\omega t \leftrightarrow \frac{(p-a)}{(p-a)^2+\omega^2} \qquad (11.2.7)$$

**7-misol.** $e^{\omega t}sh\omega t$ va $e^{\omega t}ch\omega t$ $(t \geq o)$ larning tasvirini toping.

**Yechish.** (2,6), (2.5) larga asosan

$$e^{at}sh\omega t \leftrightarrow \frac{\omega}{(p-a)^2-\omega^2}, e^{at}ch\omega t \leftrightarrow \frac{p-a}{(p-a)^2-\omega^2} \qquad (11.2.8)$$

$4^0$**. Kechikish teoremasi.** Ixtiyoriy τ>*0* o'zgarmas uchun

$$f(t-\tau) \leftrightarrow e^{-p\tau}\tau(p) \qquad (11.2.9)$$

o'rinlidir, bunda $F(p) \leftrightarrow f(t)$; yani original argumentining τ ga kechikishi tasvirni $e^{-\tau}$ ga ko'paytirishga olib keladi.

**Isbot.** τ>*o* va *t*<*o* bo'lganda *f(t)=o* bo'lganidan *f(t-*τ*)* funksiyaning *t*<τ (jumladan [0; τ) oraliqda) nolga teng bo'lishi kelib chiqadi. Buni e'tiborga olib, *f(t-*τ*)* ning tasviri uchun

$$f(t-\tau) \leftrightarrow \int_0^{+\infty} f(t-\tau)e^{-pt}dt = \int_\tau^{+\infty} f(t-\tau)e^{-pt}dt$$

ni olamiz va so'ngi integralda $t-\tau = t_1$ almashtirish qilib,

$$f(t-\tau) \leftrightarrow \int_0^{+\infty} f(t_1)e^{-p(t_1+\tau)}dt_1 = e^{-p\tau}F(p)$$

ni, yani (11.2.9) ni olamiz,bunda $F(p \leftrightarrow f(t)$ dir.

Demak,

$$f(t-\tau) \leftrightarrow e^{-p\tau}F(p).$$

(11.2.10)

Teorema (xossa) isbotlandi.

**8-misol.** $\tau$ -vaqt oralig'ida $(\tau > 0)$ ta'sir etuvchi (11.2.1-rasmga qarang) $\varphi(t)$ birlik impulsning Laplas tasviri topilsin.

**Yechish.** Aytilgan $(0,\tau)$ oraliqda ta'sir etuvchi birlik impuls funksiyani Xevisaydning birlik funksiyasi orqali quyidagicha ifodalash mumkin.

$$\varphi(t) = \sigma_0(t) - \sigma_0(t-\tau)$$

Endi bu funksiya tasvirini topish uchun $1^0 -$ va $4^0 -$xossalarni qo'llab,

$$\varphi(t) \leftrightarrow \frac{1}{p} - e^{-p\tau}\frac{1}{p} = \frac{1-e^{-p\tau}}{p}$$

ni olamiz. Demak, $\tau -$ vaqt oralig‘ida ta’sir etuvchi birlik impuls funksiyasining tasviri

$$\varphi(t) \leftrightarrow \frac{1-e^{p\tau}}{p}$$

dan iborat ekan.
Xuddi Shunga o‘xshash, agar birlik impuls $(a; a+\tau)\ (a \geq o)$ oraliqda ta’sir etsa, (11.2.2-rasmga qarang) bu birlik impuls funktsisini

$$\varphi(t) \leftrightarrow \sigma_0(t-a) - \sigma_0(t-a-\tau)$$

ko‘rinishda Xevisaydning birlik funksiyasi orqali ifodalash mumkin bo‘lib, uning tasviri uchun

$$\varphi(t) \leftrightarrow \frac{1-e^{-\tau p}}{p} e^{-ap}$$

ni olamiz.

**9-misol.** $\tau\ (\tau > o)$ vaqt oralig‘ida boshlang‘ich holatdan boshlab *(t=0)* dan *T* davr bilan $(T \geq \tau)$ davriy ravishda ta’sir etuvchi birlik impuls funksiyasining (11.2.3-rasmga qarang) Laplas tasviri topilsin.
**Yechish.** Bunday impuls funksiyasi uchun

$$\varphi(t) = \sum_{k=0}^{+\infty}\left[\sigma_0(t-kT) - \sigma_0(t-kT-\tau)\right]$$

ifodani yozish mumkin bo‘lib, unga $1^0 -$ va $4^0 -$ xossalarni qo‘llab,

$$\varphi(t) \leftrightarrow \sum_{k=0}^{+\infty} \frac{1-e^{-\tau p}}{p} e^{-kT} = \frac{1-e^{-\tau p}}{p} .(1+e^{-Tp}+e^{-2Tp}+...+e^{-kTp}+...) = \frac{1-e^{-\tau p}}{p} \cdot \frac{1}{1-e^{-Tp}}$$

tasvirni olamiz. Demak, qaralayotgan holda

$$\varphi(t) \leftrightarrow \frac{1}{p} \cdot \frac{1-e^{-\tau p}}{1-e^{-TP}}$$

ga ega bo‘lamiz.Endi $T = 2\tau$ bo‘lgan xususiy holni qarasak,bu formula yanada soddaroq ko‘rinishni oladi:

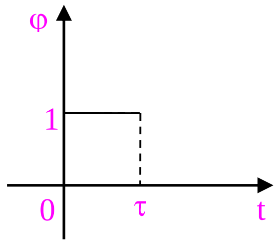
$$\varphi(t) \leftrightarrow \frac{1}{p(1+e^{-\tau p})} \cdot$$

Agar boshlang‘ich impuls $t$=$o$ boshlang‘ich paytdan emas, balki, $t=\tau$ paytdan ta’sir qiladi (2.4-rasmga qarang) deb faraz qilsak, $4^0$ – xossaga ko‘ra oxirgi tasvirni yana $e^{-P\tau}$ ga ko‘paytirish kerak bo‘ladi,yani bu holda

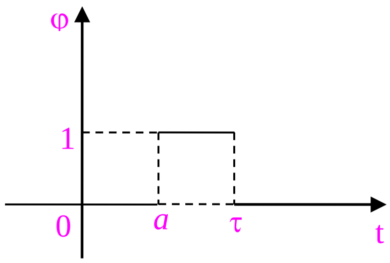
$$\varphi(t) \leftrightarrow \frac{e^{-p\tau}}{p(1+e^{-p\tau})} = \frac{1}{p(e^{p\tau}+1)}$$

ga ega bo‘lamiz.

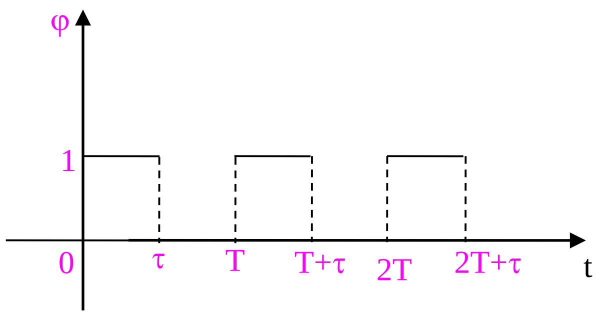
Kechikish teoremasiga o‘xshash ilgarilash teoremasi (xossasi) ham mavjudligini aytamiz va bu holda *f(t)* original berilgan bo‘lib, $f(t) \leftrightarrow F(p)$ deb faraz qilsak, $f(t+\tau)$ $(\tau > o)$ funksiya argumenti $\tau$ ga ilgarilanganligi aniqdir.



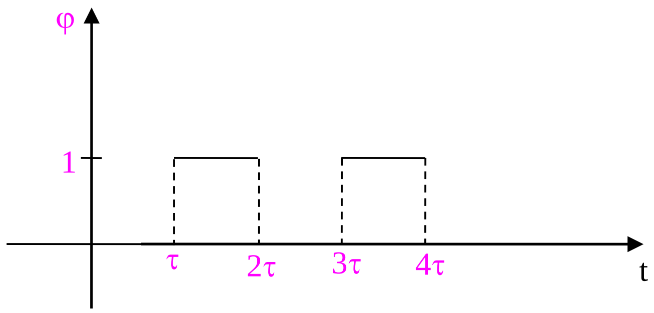
11.2.1-расм



11.2.2-расм



11.2.3-расм



11.2.4-rasm

**Ilgarilash teoremasi(xossasi):**

$$f(t+\tau) \leftrightarrow e^{p\tau}\left[F(p) - \int_0^{\tau} f(t)e^{-pt}dt\right]$$

o‘rinlidir.
Haqiqatdan ham,

$$f(t+\tau) \leftrightarrow \int_0^{+\infty} f(t+\tau)e^{-pt}dt$$

integralda $t+\tau = t_1$ almashtirish qilib,

$$f(t+\tau) \leftrightarrow e^{p\tau}\int_{\tau}^{+\infty} f(t_1)e^{-pt_1}dt = e^{p\tau}\left[\int_0^{\infty} f(t_1)e^{-pt_1}dt_1 - \int_0^{\tau} f(t_1)e^{-pt_1}dt_1\right]$$

Bundan teorema isboti kelib chiqadi.

$5^0$. Parametr bo‘yicha differensiallash haqidagi teorema (xossa). Agar x ning ixtiyoriy qiymatida *F(t;x)* originalga *f(p;x)* tasvir mos kelsa, hamda *f(t;x) x* parametr bo‘yicha xususiy hosilaga ega va $\frac{\partial f(t;x)}{\partial x}$ ham originaldan iborat bo‘lsa,

$$\frac{\partial f(t;x)}{\partial x} \leftrightarrow \frac{\partial F(p;x)}{\partial x} \qquad (11.2.11)$$

bo‘ladi.

Bu teoremani isbotsiz qabul qilgan holda uning ba’zi tatbiqlarini keltirish bilan kifoyalanamiz. Masalan,

$$e^{at} \leftrightarrow \frac{1}{p-a}$$

ekanligini bilamiz. Bu yerda *a* ni parametr deb qarab,oxirgi munosabatga (11.2.11) ni qo‘llaylik:

$$\frac{\partial e^{at}}{\partial a} \leftrightarrow \frac{\partial}{\partial a}\left(\frac{1}{p-a}\right)$$

Bundan

$$te^{at} \leftrightarrow \frac{1}{(p-a)^2}$$

bo‘lishini olamiz. Differensiallashni ketma-ket davom ettirib,

$$t^2e^{at} \leftrightarrow \frac{2}{(p-a)^3}, t^3e^{at} \leftrightarrow \frac{3!}{(p-a)^4}, ...,$$

$$t^n e^{at} \leftrightarrow \frac{n!}{(p-a)^{n+1}}$$

bo‘lishini ko‘ramiz. $a = o$ desak, oxirgidan

$$t^n \leftrightarrow \frac{n!}{p^{n+1}}, n \in Z_0$$

munosabat kelib chiqadi. Xuddi Shuningdek,

$$sin\omega t \leftrightarrow \frac{\omega}{p^2+\omega^2}, \cos\omega t \leftrightarrow \frac{P}{p^2+\omega^2}$$

munosabatlarni $\omega$ parametr bo‘yicha differensiallab,

$$tcos\omega t \leftrightarrow \frac{p^2-\omega^2}{(p^2+\omega^2)^2}, tSin\omega t \leftrightarrow \frac{2p\omega}{(p^2+\omega^2)^2}$$

ni olamiz.

$6^0$. **Originalni differensiallash teoremasi.** Agar $f(t) \leftrightarrow F(p)$ bo‘lsa, u holda

$$f'(t) \leftrightarrow pF(p) - f(o), \qquad (11.2.12)$$

yani originalni differensiallash tasvirni P ga ko‘paytirib, undan $f(o)$ ni ayirishga olib keladi; bu yerda $f(0) = f(+0)$ deb tuShunmoq kerak.

**Isbot.** $f'(t) \leftrightarrow \int\limits_0^{+\infty} f'(t)e^{-pt}dt$ ekanligidan bu ni bo‘laklab integrallab

$$\int\limits_0^{+\infty} f'(t)e^{-pt}dt = f(t)e^{-pt}\Big|_0^{+\infty} \int\limits_0^{+\infty} + p\int\limits_0^{+\infty} f(t)e^{-pt}dt$$

Original bo‘lish shartlaridan 3)-ga asosan

$$|f(t)| \le Me^{\tau t}$$

ekanligidan $\operatorname{Re} p > \sigma$ bo‘lganda

$$\left|f(t)e^{-pt}\right| < Me^{(\sigma - \operatorname{Re} p)t}$$

bo‘lib, bu munosabatning o‘ng tomonidagi ifoda $\sigma - \operatorname{Re} p < o$ bo‘lgani sababli $t \to oo$ da nolga intiladi.Demak,

$$\int\limits_0^{+\infty} f'(t)e^{-pt}dt = o - f(o) + pF(p) = pF(p) - f(0).$$

Bu teoremani ketma-ket qo‘llab,

$$f^{(n)}(t) \leftrightarrow p^nF(p) - p^{n-1}f'(0) - p^{n-2}f''(0) - ... - f^{(n-1)}(0) \qquad (11.2.13)$$

ni olamiz.

$7^0$. **Originalni integrallash teoremasi.** Agar $f(t) \leftrightarrow F(p)$ va $g(t) = \int\limits_0^t f(t_1)dt_1$ bo‘lsa, u holda

$$g(t) \leftrightarrow \frac{F(p)}{p}, \qquad (11.2.14)$$

yani originalni (0;t) oraliq bo‘yicha integrallash tasvirni r ga bo‘lishga olib keladi.

**Isbot.** Avval, $g'(t) = f(t)$ va $g(o) = o$,bo‘lishini eslatamiz. U holda $g(t) \leftrightarrow G(p)$ deb belgilab, (2.12) ga asosan

$$g'(t) = f(t) \leftrightarrow pG(p) - g(0) = pG(p)$$

Bundan $pG(p) = F(p)$ ni olamiz va (11.2.14) ga kelamiz.

$8^0$. **Tasvirni differensiallash teoremasi.** Agar $f(t) \leftrightarrow F(p)$ bo‘lsa, u holda

$$-tf(t) \leftrightarrow F'(p) \qquad (11.2.15)$$

yani tasvirni differensiallash originalni –*t* ga ko‘paytirishga olib keladi.

**Isbot.** Buning isboti *F(p)* tasvirning analitik ekanligidan kelib chiqadi, yani Laplas integralini *P* bo‘yicha differensiallash uchun integral osti funktsiasini r bo‘yicha differensiallab so‘ngra integrallash amalini bajarish kifoyadir, yani

$$F'(p) = \int_0^{+\infty} f(t)\cdot(-t)\cdot e^{-pt}dt$$

Bu teoremani ketma-ket tatbiq qilib, F(p) tasvirning yuqori tartibli hosilalarining originallarini topamiz.

$$t^2 f(t) \leftrightarrow F''(p),$$

$-t^3 f(t) \leftrightarrow F'''(p)$

va umumiy holda

$$(-t)^n f(t) \leftrightarrow F^{(n)}(p) \qquad (11.2.16)$$

**10-misol.** $t^n \ (n \in Z_0)$ ning tasvirini toping.

**Yechish.** $n=0$ bo‘lganda $1 \leftrightarrow \frac{1}{p}$ bo‘lishi ma’lum. $n \ge 1$ bo‘lsin, u holda

$$n=1,\ (-t)\cdot 1 \leftrightarrow (\frac{1}{p})' \Rightarrow -t \leftrightarrow -\frac{1}{p^2};$$

$$n=2,\ (-t)\cdot(-t) \leftrightarrow (-\frac{1}{p^2})' \Rightarrow t^2 \leftrightarrow \frac{2}{p^3};$$

$$n=3,\ (-t)\cdot t^2 \leftrightarrow (\frac{2}{p^3})' \Rightarrow -t^3 = -\frac{3!}{p^4}$$

va hakozo $n \in N$ uchun

$$t^n \leftrightarrow \frac{n!}{p^{n+1}}$$

munosabatga kelamiz. Olingan bu munosabat $5^0$ bandda olingan bilan bir xildir.

$9^0$. **Tasvirni integrallash teoremasi.** Agar $f(t) \leftrightarrow F(p)$ bo‘lib, $\int_0^{+\infty} F(x)dx$ xosmas integral yaqinlashuvchi bo‘lsa, u holda

$$\frac{f(t)}{t} \leftrightarrow \int_p^{+\infty} F(z)dz \qquad (11.2.17)$$

munosabat o‘rinlidir, yani tasvirni $(p, +\infty)$ oraliq bo‘yicha integrallash originalni t ga bo‘lishga mos keladi.

**Isbot.** $$\frac{d}{dp}(\int_p^{+\infty} F(z)dz) = -\frac{dp}{dp}\cdot F(p) = -F(p),$$

$$\frac{d}{dp}(\int_0^{+\infty} \frac{f(t)}{t} e^{-pt}dt) = \int_0^{+\infty} \frac{f(t)}{t}\cdot(-t)e^{-pt}dt = -F(p)$$

munosabatlardan teoremaning isboti kelib chiqadi.

**10-misol.** $\frac{s\text{int}}{t}$ originalning tasviri topilsin.

**Yechish.**

$$\sin t \leftrightarrow \frac{1}{p^2+1}, \int\limits_p^{+\infty} \frac{dz}{1+z^2} = arctgz\Big|_p^{+\infty} = \frac{\pi}{2} - arctgp = arcctgp,$$

U holda

$$\frac{s\text{int}}{t} \leftrightarrow arcctgp.$$

**11-misol.** $\int\limits_0^t \frac{sin\tau}{\tau} d\tau$ ning tasvirini toping.

**Yechish.** Oldingi misolda $\frac{S\text{int}}{t}$ originalning tasvirini topdik, uni *(o;t)* oraliq bo‘yicha integrallab $7^0$ – xossa asosida

$$\int\limits_0^t \frac{sin\tau}{\tau} d\tau \leftrightarrow \frac{arcctgp}{p}$$

ni olamiz.

**12-misol.** $1 \leftrightarrow \frac{1}{p}$ va $e^{at} \to \frac{1}{p-a}$ bo‘lishi ma’lum. Bulardan $1-e^{at} \leftrightarrow \frac{1}{p} - \frac{1}{p-a}$ ni olamiz.

$$\int\limits_p^{+\infty} \left( \frac{1}{z} - \frac{1}{z-a} \right) dz = \ell n \frac{z}{z-a}\Big|_p^{+\infty} = \ell n 1 - \ell n \frac{p}{p-a} = \ell n \frac{p-a}{p}$$

Bulardan va (11.2.17) dan $\frac{1-e^{at}}{t} \leftrightarrow \ell n \frac{p-a}{p}$ bo‘lishini olamiz.

$10^0$. **Tasvirlarni ko‘paytirish teoremasi.**

Avvalo, berilgan ikkita $f(t)$ va $\varphi(t)$ funksiyalarining uyurmasi (svyortkasi) deb,

$$\int\limits_0^t f(\tau)\varphi(t-\tau)d\tau \qquad (11.2.18)$$

integralga aytilishini va uni, odatda $f * \varphi$ kabi belgilanishini aytamiz.

funksiyalarning uyurmasi amali kommutativlik xossasiga egadir. Haqiqatdan ham,(11.2.18) integralda $t - \tau = \tau_1$ almashtirish qilib,

$$f * \varphi = \int\limits_0^t f(t_1 - \tau_1)\varphi(\tau_1)d\tau_1 = \varphi * f$$

ekanligini ko‘ramiz.

Bu o‘rinda agar $f(t)$ va $\varphi(t)$ funksiyalar originallardan iborat bo‘lsa, ulraning o‘ramasi $f * \varphi$ ham original bo‘lishini ko‘rsatish qiyin emasligini aytamiz hamda tasvirlarni ko‘paytirish teoremasi deb ataluvchi quyidagi tasdiqni keltiramiz.

Agar $f(t) \leftrightarrow F(p)$ va $g(t) \leftrightarrow G(p)$ bo‘lsa, u holda ular $f * g$ uyurmasi tasvirlarining ko‘paytmasiga mos keladi, yani

$$f * g \leftrightarrow F(p) \cdot G(p). \qquad (11.2.19)$$

**Isbot.** $f * g$ o‘ramaning Laplas tasvirini yozamiz.

$f*g \leftrightarrow \int_0^{+\infty}\left[\int_0^t f(\tau)g(t-\tau)d\tau\right]e^{-pt}dt$ va uni 11.2.5-rasmda tasvirlangan D soha bo‘yicha ikkilangan integral sifatida yozish mumkinligini aytamiz.

D sohani $0 \le t < +\infty, 0 \le \tau \le t$ tengsizliklar sistemasi vositasida ifodalash mumkin ekanligini ko‘rish oson. Unda integrallash tartibini o‘zgartirib, D sohani ifodalash uchun $0 \le \tau < +\infty$ $\tau \le t < +\infty$ tengsizliklar sitemasini olamiz va unga mos ikki karrali integralni yozamiz:

$$\int_0^{+\infty} e^{-pt}dt\int_0^t f(\tau)g(t-\tau)d\tau = \int_0^{+\infty} d\tau \int_\tau^{+\infty} f(\tau)g(t-\tau)e^{pt}dt$$

oxirgining ichki integralida $t-\tau = t_1$ almashtirish qilib, ikkilangan integralni

$$\int_0^{+\infty} f(\tau)e^{-p\tau}d\tau \int_0^{+\infty} g(t_1)e^{-pt_1}dt_1$$

ko‘rinishga keltiramiz. Bu takroriy ikki karrali integralda ichki va tashqi integrallar bir-biriga bog‘liq bo‘lmaganligi sababli va ularning har biri f(t) va g(t) originallarning mos tavirlari ekanligidan (11.2.19) kelib chiqadi.

**11-misol.** Agar $f(t)=t$, $g(t)=e^t$ $(t \ge o)$ bo‘lsa, $f*d = t*e^t$ ning tasvirini toping.

**Yechish.**

$$t \leftrightarrow \frac{1!}{p^2} = \frac{1}{p^2}, e^t \leftrightarrow \frac{1}{p-1}, \operatorname{Re} p > o$$

ekanligi ma’lum.Demak,

$$t*e^t \leftrightarrow \frac{1}{p^2}\cdot\frac{1}{p-1} = \frac{1}{p^2(p-1)}$$

ni olamiz.Endi bu natijani uyurmani hisoblab, so‘ngra tasvirni topish bilan tekshiraylik.

$$t*e^t = \int_0^t (t-\tau)\cdot e^\tau d\tau$$

Buni bo‘laklab integrallaymiz:

$$u = t-\tau, d\vee = e^\tau d\tau \Rightarrow du = -d\tau, \vee = e^\tau$$

$$t*e^t = (t-\tau)e^\tau\Big|_0^t + \int_0^t e^\tau d\tau = -t + e^t - 1$$

$$-t + e^t - 1 \leftrightarrow -\frac{1}{p^2} + \frac{1}{p-1} - \frac{1}{p} = \frac{-p+1+p^2-p^2+p}{p^2(p-1)}.$$

Bundan ko‘rinadiki, yana oldingi natijani oldik.

Endi $pF(p)G(p)$ ko‘paytmaga mos originallarni qidiraylik. Ma’lumki,originalni differentsiyalash teoremasiga ko‘ra

$$pF(p) - f(0) \leftrightarrow \int_0^{+\infty} f'(t)e^{-pt}dt$$

U holda

$$pF(p)G(p)=\left[pF(p)-f(0)\right]G(p)+f(0)G(p)\leftrightarrow f'(t)g(t)+f(0)g(t)=f(0)g(t)+$$

$$+\int_0^t f'(\tau)g(t-\tau)d\tau=f(0)g(t)+\int_0^t g(\tau)f'(t-\tau)d\tau,$$

yani
$$pF(p)G(p)\leftrightarrow f(0)g(t)+\int_0^t g(\tau)f'(t-\tau)d\tau.$$

Oxirgi munosabatni Dyuamel integrali deb ataladi.

$11^0$. **Orginallarni ko‘paytirish teoremasi.** Avvalo kompleks tekislikda tasvirlarning o‘ramasi tushunchasini kiritamiz.faraz qilaylik, $\operatorname{Re} p>\sigma$ bo‘lganda berilgan *f(t)* va g*(t)* originallarning tasvirini aniqlovchi Laplas integrallari absolyut yaqinlashuvchi bo‘lsin. U holda

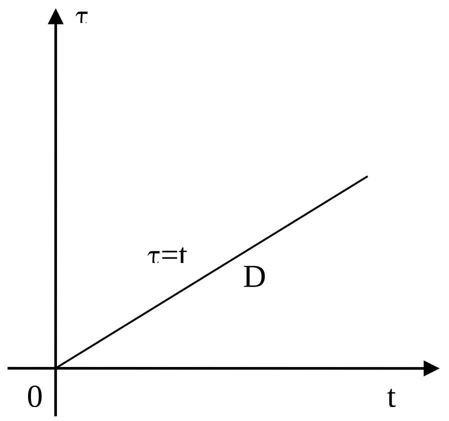
$$f(t)\leftrightarrow F(p), g(t)\leftrightarrow G(p)$$

tasvirlar mavjud bo‘lib, ular $\operatorname{Re} P>\sigma$ yarim tekislikda analitikdir.

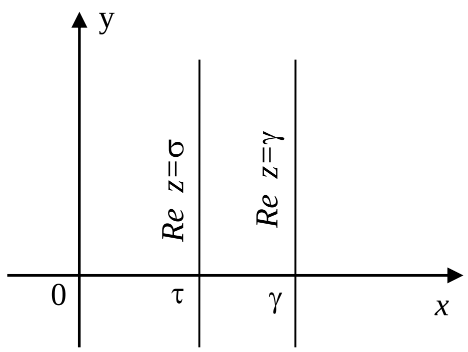
Bu ikki tasvirlarning **kompleks sohadagi uyurmasi** deb

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}F(z)G(p-z)dz \qquad (11.2.20)$$

integralga aytiladi, bunda integralash yo‘li (chizig‘i) $\operatorname{Re} z=\gamma>\sigma$ to‘g‘ri chiziqdan iborat bo‘lib, uning yo‘nalishi quyidan yuqoriga tomondir (11.2.6-rasmga qarang).




11.2.5-расм




11.2.6-расм

Endi originallarni ko‘paytmasiga doir quyidagi teoremani keltiramiz.

**Originallarni ko‘paytirish teoremasi.** Agar $f(t)\leftrightarrow F(p)$ *ва* $g(t)\leftrightarrow G(p)$ bo‘lsa, u holda

$$f(t)g(t)\leftrightarrow\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma-i\infty}F(z)G(p-z)dz \qquad (11.2.21)$$

o‘rinlidir, yani originallar ko‘paytmasiga tasvirlarning uyurmasi mos keladi.

Agar (11.2.21) da g*(t)=f(t)* deb faraz qilsak,

$$f^2(t)\leftrightarrow\frac{1}{2ni}\int_{\gamma-i\infty}^{\gamma+i\infty}F(z)F(p-z)dz \qquad (11.2.22)$$

ni olamiz.
Bu o‘rinda (11.2.21) va  (11.2.22) lardagi integrallarni bosh qiymat ma’nosida tuShunish lozimligini eslatamiz, yani

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(z)G(p-z)dz = \lim_{\omega\to+\infty}\frac{1}{2\pi i}\int_{\gamma-i\omega}^{\gamma+i\omega} F(z)G(p-z)dz. \qquad (11.2.23)$$

$12^0$. **O’girish teoremasi.** Bu yerda berilgan *F(p)* tasvir bo‘yicha unga mos originalni topish masalasini qaraymiz. Buning uchun bizga quyidagi teorema qo‘l keladi.
**O’girish teoremasi.** Agar *f(t)* funksiya original *F(p)* esa uning tasviri bo‘lsa, u holda original uzluksiz bo‘lgan ixtiyoriy t nuqtada

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(p)e^{pt}dp \qquad (11.2.24)$$

formula  o‘rinli bo‘ladi, bu yerda integrallash Laplas integrali absolyut yaqinlashuvchi bo‘ladigan kompleks tekislikning yarim tekisligidagi $\operatorname{Re} p = \gamma$ cheksiz to‘g‘ri chiziq bo‘yichadir.

Bu teoremani isbotlash jarayonida ushbu kitobda ko‘rilmagan ba’zi tushunchalardan foydalanishga to‘g‘ri kelishi sababli uni isbotsiz qabul qilamiz.
(2.24) integralni ixtiyoriy analitik funksiya uchun hisoblash qiynchiliklar keltirib chiqarishi sababli tatbiqda muhim ahamiyat kasb qiluvchi ba’zi bir xususiy hollar bilan cheklanamiz.
*F(p)* funksiyani chekli sondagi maxsus nuqtalaridan boshqa kompleks tekislikning barcha nuqtalarida analitik va

$$\lim_{p\to\infty} F(p) = 0, \qquad (11.2.25)$$

undan tashqari cheksiz uzoqlashgan $(z=\infty)$ nuqtada ham funksiya analitik deb faraz qilamiz.Agar *F(p)* funksiya cheksiz uzoqlashgan nuqtada analitik bo‘lsa,(11.2.25) shartning bajarilishi kelib chiqishini aytamiz, yani u funksiya cheksiz uzoqlashgan nuqtada analitik bo‘lishining zaruriy shartidir. Endi $\operatorname{Re} p = \gamma$ to‘g‘ri chiziqni Shunday tanlaymizki, (11.2.24) integraldagi F(p) funksiyaning barcha maxsus nuqtalari bu to‘g‘ri chiziqning chap tomonida joylashsin. U holda bu to‘g‘ri chiziqda va uning o‘ng tomonida, yani $\operatorname{Re} p \geq \gamma$ bo‘lganda F(p) funksiya maxsus nuqtaga ega bo‘lmaydi. Bu shartlar bo‘yicha (11.2.24) integralni Jordan lemmasiga asosan hisoblash mumkin bo‘ladi.

Bu lemmani qaralayotgan hol uchun moslashtirgan holda isbotsiz keltiramiz.
**Jordan lemmasi.** Agar yuqorida aytilgan shartlar bajarilsa, $t>0$ bo‘lganda

$$\int_{C_R} F(p)e^{pt}dp$$

integral $R\to+\infty$ da nolga intiladi, bu yerda $C_R$, $|p| = R$ aylananing Shunday yoyiki unda $\operatorname{Re} p < \gamma$ bajariladi (2.7-rasm); yani

$$\lim_{\substack{R\to+\infty \\ t>0}} \int_{C_R} F(p)e^{pt}dp = 0$$

Agar $t < 0$ bo‘lsa, $|p| = R$ aylananing $C_R$ yoyini aylanaga to‘ldiruvchisi bo‘lgan, $C_R'$ yoyi bo‘yicha integral $R \to +\infty$ bo‘lganda nolga intiladi, yani bu holda $\operatorname{Re} p > \gamma$ bo‘lib,

$$\lim_{\substack{R\to+\infty \\ t<0}} \int_{C_R'} F(p)e^{pt}\alpha p = 0$$

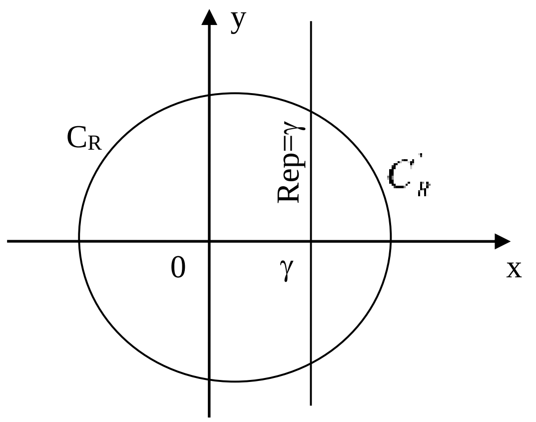
Bu lemmani tatbiq qilish natijasida *t>o* bo‘lsa,

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(p)e^{pt}dp = \sum_k \operatorname{Re} s\left[F(p)e^{pt}; a_k\right], \qquad (11.2.26)$$

bu yerda $a_k - F(p)$ funksiyaning barcha mahsus nuqtalari ; $t < o$ bo‘lganda esa

$$\int_{\gamma-ia}^{\gamma+i\infty} F(p)e^{pt}dp = 0$$

bo‘lishini olamiz.




11.2.7-расм

**13-misol.** $F(p) = \dfrac{1}{p^2+1}$ tasvir berilgan. Unga mos originalni toping.

**Yechish.** (11.2.26) ni tatbiq qilib, *t>o* bo‘lganda

$$f(t) = \operatorname{Re} s\left[\frac{e^{pt}}{p^2+1}; i\right] + \operatorname{Re} s\left[\frac{e^{pt}}{p^2+1}; -i\right] = \lim_{p\to i}\left(\frac{e^{pt}}{(p^2+1)}\right) + \lim_{p\to -i}\left(\frac{e^{pt}}{(p^2+1)}\right) = \lim_{p\to i}\left(\frac{e^{pt}}{2p}\right) + \lim_{p\to i}\left(\frac{e^{pt}}{2p}\right) =$$

$$= \frac{e^{it}}{2i} + \frac{e^{-it}}{-2i} = s\operatorname{int}.$$

$13^0$. F (p) tasvir ratsional funksiya bo‘lgan holda (11.2.26) formulani qo‘llaylik. Bu yerda

$$F(p) = \frac{A(p)}{B(p)} \qquad (2.27)$$

bo‘lib, *A(p)* va B*(p)* lar ko‘phadlar hamda *A(p)* ning darajasi B*(p)* ning darajasidan kichik deb faraz qilamiz. Shu bilan birga

$$\lim_{p\to\infty} F(p) = \lim_{p\to\infty} \frac{A(p)}{B(p)} = 0,$$

yani (11.2.25) shart bajariladi; *F(P)* funksiya chekli sondagi maxsus nuqtalari mavjud bo‘lib, *A(P)* va *B(P)* lar umumiy ildizga ega emas deb faraz qilamiz. Bu holda *F(P)* ning maxsus nuqtalari *B(P)* ning nollaridan iborat bo‘lib, ular *F(P)* uchun qutblar, (11.2.27) ning o‘ng tomonidagi kasr qisqarmaydigan bo‘lishi kelib chiqadi.

Endi $F(P)$ ning $a_k$ qutbi $n_k$- tartibli deb faraz qilib, (11.2.26) formulani qo‘llab, qutb maxsus nuqtaga nisbatan funksiyaning chegirmasini hisoblash uchun formulani eslab, quyidagi

$$\frac{A(p)}{B(p)} \leftrightarrow \sum_k \frac{1}{(n_k-1)!} \lim_{p\to\infty} \left[ (p-a_k)^{n_k} \frac{A(p)e^{pt}}{B(p)} \right]_p^{(n_k-1)} \qquad (11.2.28)$$

formulani olamiz. (11.2.28) **yoyish formulasi** (yoki **yoyish teoremasi**) deb atalishini aytamiz.

Agar *F(p)* ning barcha maxsus nuqtalari oddiy qutblardan iborat bo‘lgan holni qarasak, (11.2.28) ni

$$\frac{A(p)}{B(p)} \leftrightarrow \sum_k \frac{A(a_k)}{B'(a_k)} e^{a_k t} = f(t) \qquad (11.2.29)$$

ko‘rinishga keltirish oson. Olingan (11.2.29) formula qo‘llash uchun qulaydir.

Agar $F(p)$ tasvirni aniqlovchi ratsional funksiyaning surati va maxrajidagi $A(p)$ va $B(p)$ ko‘phadlar haqiqiy koeffitsientli bo‘lsa, $B(p)$ ning $a_k$ nollari orasida mavhumi ham mavjud bo‘lsa, u holda bu $a_k$ mavhum ildizga qo‘shma bo‘lgan $\bar{a}_k$ kompleks son ham $B(p)$ ning noli bo‘lishi bizga ma’lum . Bunday holda $a_k$ va $\bar{a}_k$ mavhum ildizlarga mos keluvchi (11.2.29) dagi yig‘indining qo‘shiluvchilari ham qo‘shma mavhum ifodalardan iborat bo‘lib, ularning yig‘indisi haqiqiy qismining ikkilanganidan iborat bo‘lishini ko‘rish oson:

$$\frac{A(a_k)}{B'(a_k)} e^{a_{n}t} + \frac{A(\bar{a}_k)}{B'(a_k)} \cdot e^{\bar{a}_{n}t} = 2\,\mathrm{Re}\,\frac{A_k(a_k)}{B'(a_k)} e^{a_k t}$$

$B(p)$ ko‘phadning barcha ildizlari oddiy bo‘lgan holda (11.2.29) dagi yig‘indini haqiqiy va mavhum ildizlarga mos ravishda ikkita qismga ajratib, yuqoridagi mulohaza asosida quyidagicha yozish mumkin:

$$f(t) = \sum\nolimits_v \frac{A(a_k)}{B'(a_k)} e^{a_k t} + 2\,\mathrm{Re} \sum\nolimits_v \frac{A(a_k)}{B'(a_k)} e^{a_k t} \qquad (11.2.29)$$

Bundagi birinchi yig‘indi $B(p)$ ning haqiqiy oddiy ildizlariga ikkinchisi esa oddiy mavhum ildizlariga mos keladi.

**14-misol.** Berilgan $F(p) = \dfrac{1}{p(p+1)(p+2)}$ tasvir bo‘yicha originalni toping.

**Yechish.** $A(p)=1,\ B(p) = p(p+1)(p+2)$ desak, $F(P)$ ning maxsus nuqtalari *B(p)* ning $p=0,\ p=-1,\ p=-2$ oddiy nollaridan iborat bo‘lib, $F(p) = \dfrac{A(p)}{B(p)}$ to‘g‘ri ratsional kasrning barcha maxsus nuqtalari oddiy qutblar ekanligidan (11.2.29) formuladan foydalanish mumkin.

$$B'(p) = (p+1)(p+2) + p(p+2) + (p(p+1),$$
$$B'(0) = 2,\ B'(-1) = -1,\ B'(-2) = 2,$$
$$A(0) = A(-1) = A(-2) = 1.$$

Demak, original uchun

$$f(t) = \frac{1}{2} + \frac{1}{-1}e^{-t} + \frac{1}{2}e^{-2t} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} = \frac{1}{2}(1-e^{-t})^2$$

ni olamiz.

Tekshirish. $$\frac{(1-e^{-t})^2}{2} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \leftrightarrow \frac{1}{2p} - \frac{1}{p+1}\frac{1}{2(p+2)} = \frac{1}{p(p+1)(p+2)}$$

$14^0$. **Originallar va tasvirlar mosligining jadvali.**

| № | Original | Tasvir | Izoh |
|---|---|---|---|
| 1 | af(t)+bg(t) | aF(p)+bG(p) | Chiziqlilik xossasi, a=cont., b=cont., |
| 2 | $f(\lambda t)$ | $\frac{1}{\lambda}F(\frac{p}{\lambda})$ | O'xshashlik teoremasi. $\lambda > 0$ |
| 3 | $e^{at}f(t)$ | $F(p-a)$ | Siljish teoremasi. *a*=const. |
| 4 | $f(t-\tau)$ | $e^{-p\tau}F(p)$ | Kechikish teoremasi. $\tau > 0$ |
| 5 | $\frac{\partial f(t;x)}{\partial x}$ | $\frac{\partial F(p;x)}{\partial x}$ | Parametr bo'yicha differensiallash teoremasi. |
| 6 | $f'(t)$ | $pF(p) - f(0)$ | Originalni differensiallash teoremasi |
| 7 | $f^{(n)}(t)$ | $p^nF(p) - \sum_{k=1}^{n} p^{(n-k)} f^{(k-1)}$ | Originalni *n* marta differensiallash |
| 8 | $(-t)^n \cdot f(t)$ | $F^{(n)}(p)$ | Tasvirni *n* marta differensiallash teremasi. |
| 9 | $\int_0^t f(\tau)d\tau$ | $\frac{F(p)}{p}$ | Originalni integrallash teoremasi. |
| 10 | $\frac{f(t)}{t}$ | $\int_p^\infty F(p_1)dp_1$ | Tasvirni integrallash teoremasi. |
| 11 | $\int_0^t f(\tau)g(t-\tau)d\tau = f(t) * g(t)$ | $F(p)G(p)$ | Tasvirni ko'paytirish teoremasi. |

| | | | |
|---|---|---|---|
| 12 | $\int_0^t f'(\tau)g(t-\tau)d\tau + f(0)g(t) =$ $= \int_0^\tau g'(\tau)f(t-\tau)d\tau + g(o)f(t)$ ** | $pF(p)G(p)$ | Dyuamel integrali.** |
| 13 | $f(t)g(t)$ | $\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(z)G(p-z)dz$ | Originallarni ko‘paytirish teoremasi. $(\gamma > \sigma)$ |
| 14 | $f(t)$ | $\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(P)e^{pt}dp$ | O’girish teoremasi. $(\gamma > \sigma)$ |
| 15 | 1 | $\frac{1}{p}$ | $\operatorname{Re} p > 0$ |
| 16 | $e^{at}$ | $\frac{1}{p-a}$ | $a \in K$, $\operatorname{Re} p > \operatorname{Re} a$ |
| 17 | $sin\omega t$ | $\frac{\omega}{p^2+\omega^2}$ | $0 \neq \omega \in R$, $\operatorname{Re} p > 0$ |
| 18 | $cos\omega t$ | $\frac{p}{p^2+\omega^2}$ | $0 \neq \omega \in R$, $\operatorname{Re} p > 0$ |
| 19 | $sh\omega t$ | $\frac{\omega}{p^2-\omega^2}$ | $0 \neq \omega \in R$, $\operatorname{Re} p > 0$ |
| 20 | $ch\omega t$ | $\frac{p}{p^2-\omega^2}$ | $0 \neq \omega \in R$, $\operatorname{Re} p > 0$ |
| 21 | $e^{at}sin\omega t$ | $\frac{\omega}{(p-a)^2+\omega^2}$ | $a \in K$, $\omega \in R, \omega \neq 0$, $\operatorname{Re} p > \operatorname{Re} a$ |
| 22 | $e^{at}cos\omega t$ | $\frac{p-a}{(p-a)^2+\omega^2}$ | $a \in K$, $\omega \in R, \omega \neq 0$, $\operatorname{Re} p > \operatorname{Re} a.$ |
| 23 | $t^n$ | $\frac{n!}{p^{n+1}}$ | $n \in Z_0$ $\operatorname{Re} p > 0$ |
| 24 | $t^n e^{at}$ | $\frac{n!}{(p-a)^{n+1}}$ | $\operatorname{Re} p > \operatorname{Re} a$ |
| 25 | $tsin\omega t$ | $\frac{2p\omega}{(p^2+\omega^2)^2}$ | $\operatorname{Re} p > 0$ |
| 26 | $tcos\omega t$ | $\frac{p^2-\omega^2}{(p^2+\omega^2)^2}$ | $\operatorname{Re} p > 0$ |
| 27 | $tsh\omega t$ | $\frac{2p\omega}{(p^2-\omega^2)^2}$ | $\operatorname{Re} p > 0$ |
| 28 | $tch\omega t$ | $\frac{P^2+\omega^2}{(p^2-\omega^2)^2}$ | $\operatorname{Re} p > 0$ |

| 29 | $t^k (-1 < k \in R)$ | $\frac{\Gamma(k+1)}{p^{k+1}}$ | $\Gamma(k+1)$ – gamma funksiya<br>$\mathrm{Re}\, p > 0$ |
|---|---|---|---|

### 11.2.3. Gamma funksiya

Ta’rif bo‘yicha gamma (Eyler) funksiyasi deb,

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt \qquad (11.2.30)$$

xosmas integralga aytiladi.
Bu xosmas integral $x > 0$ bo‘lganda yaqinlashuvchidir (mustaqil Shug‘ullaning).
Endi $x > 1$ deb faraz qilib, (11.2.30) ga bo‘laklab integrallash formulasini qo‘llaylik:
$u = t^{x-1},\ dv = e^{-t} dt;$

$$\int_0^{+\infty} t^{x-1} e^{-1} dt = -t^{x-1} e^{-t} \Big|_0^{+\infty} + (x-1)\int_0^{+\infty} t^{x-2} e^{-t} dt = (x-1)\Gamma(x-1).$$

Oxirgidan

$$F(x) = (x-1)\Gamma(x-1) \qquad (11.2.31)$$

rekkurrent formulani olamiz.
Agar *x=1* desak,

$$\Gamma(1) = \int_0^{+\infty} e^{-t} dt = -e^{-t}\Big|_0^{+\infty} = 1$$

ekanligini olamiz.
Agar $n \in N$ bo‘lsa, (11.2.31) ni ketma-ket qo‘llash yordamida

$$\Gamma(n) = (n-1)\cdot(n-2)\cdot \ldots \cdot 2\cdot 1\cdot \Gamma(1) = (n-1)!\cdot 1 = (n-1)!,$$

yani
$\Gamma(n) = (n-1)!,\ n \in N$
formulani olamiz.

endi $n < x < n+1 (n \in N)$ shartni qanoatlantiruvchi $\forall X \in R$ uchun

$$\Gamma(x) = (x-1)\Gamma(x-1) = \ldots = (x-1)(x-2)\cdot \ldots \cdot (x-n)\Gamma(x-n),\ \Gamma(x) = \prod_{k=1}^{n}(x-k)\cdot \Gamma(x-n)$$

o‘rinli bo‘lib, x ga qo‘yilgan shart asosida $0 < x-n < 1$ bo‘ladi.
Aytaylik, $x = \frac{1}{2}$ bo‘lsin. U holda (11.2.30) dan

$$\Gamma(\frac{1}{2}) = \int_0^{+\infty} t^{\frac{1}{2}-1} e^{-t} dt = \int_0^{+\infty} \frac{e^{-t}}{\sqrt{t}} dt$$

Bunda $t = u^2$ deb almashtirish qilsak,

$$\Gamma(\frac{1}{2}) = \int_0^{+\infty} \frac{e^{-u^2}}{u} \cdot 2udu = 2\int_0^{+\infty} e^{-u^2} du = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi},$$

bu yerda $\int_0^{+\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$ - Puasson integralidan foydalandik.

$F(\frac{1}{2}) = \sqrt{\pi}$ yordamida quyidagilarni hisoblash oson:

$$\Gamma(\frac{3}{2}) = (\frac{3}{2} - 1)\Gamma(\frac{3}{2} - 1) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2},$$

$$\Gamma\left(\frac{5}{2}\right) = \left(\frac{5}{2} - 1\right)\Gamma\left(\frac{5}{2} - 1\right) = \frac{3}{2} \cdot \Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4},$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2} \cdot \frac{3\sqrt{\pi}}{2} = \frac{5 \cdot 3 \cdot 1 \cdot \sqrt{\pi}}{2^3}$$

va hokazo

$$\Gamma\left(\frac{2k+1}{2}\right) = \frac{2k-1}{2}\Gamma\left(k - \frac{1}{2}\right) = \frac{(2k-1)!!\sqrt{\pi}}{2^{k-1}}, k \in N.$$

Bu yerda $(2k-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2k-1)$ belgilangan.

Endi gamma funksiya tushunchasini qo‘llab, $t^k$ - original $(t > 0)$ uchun tasvirni topaylik.

Ta’rif bo‘yicha

$$t^k \leftrightarrow \int_0^{+\infty} t^k e^{-pt} dt$$

Oxirgi integralda $pt = t_1$ almashtirish qilsak, $\operatorname{Re} p > o$ faraz asosida

$$\int_0^{+\infty} t^k e^{-pt} dt = \frac{1}{p^{k+1}} \int_0^{+\infty} t_1^k e^{-t_1} dt_1 = \frac{\Gamma(k+1)}{p^{k+1}},$$

yani

$$t^k \leftrightarrow \frac{\Gamma(k+1)}{p^{k+1}}, \operatorname{Re} p > 0 \qquad (11.2.32)$$

ga ega bo‘lamiz.

Agar (11.2.32) da $k = n \in N$ bo‘lsa, 1-jadvalning 23-satridagi natijani olamiz. $k \geq 0$ bo‘lganda $t^k$ funksiya originallik shartini qanoatlantiradi va uning tasviri (11.2.31) formula bilan aniqlanadi. Agar $-1 < k < 0$ deb faraz qilsak, $t^k$ $t \to +\infty$ da cheksiz katta miqdordir. Ammo (11.2.31) ning o‘ng tomoni bu hol uchun ham mavjuddir. Shu sababli bunday holda $t^k$ ni “maxsus” original deb hisoblaymiz, uning tasvirini esa (11.2.31) formula bilan aniqlaymiz. Masalan, $k = -\frac{1}{2}$ bo‘lgan xususiy holda

$$\frac{1}{\sqrt{t}} \leftrightarrow \frac{\Gamma(\frac{1}{2})}{\sqrt{p}} = \sqrt{\frac{\pi}{p}} \qquad \text{yoki} \quad \frac{1}{\sqrt{\pi t}} \leftrightarrow \frac{1}{\sqrt{p}}$$

Bunday kiritilgan “maxsus” originallar va tasvirlar uchun Laplas almashtirishining barcha xossalari o‘rinli bo‘lishini tekshirib ko‘rish oson (mustaqil Shug‘ullanib ko‘ring.) Masalan, siljish teoremasiga asosan

$$\frac{e^{at}}{\sqrt{\pi t}} \leftrightarrow \frac{1}{\sqrt{p-a}}$$

## 11.3. Laplas almashtirishining tatbiqlari.

Mazkur bandda operatsion hisob usulining asl mohiyati bilan tanishamiz va uning tatbiqiga doir bir qator misollar keltiramiz. Bu usulning mohiyati quyidagichadir:

1) qo‘yilgan masalani biror almashtirish (masalan, Laplas almashtirishi) vositasida nisbatan soddaroq bo‘lgan (hal qilinishi jihatidan) masalaga keltiriladi va uni **tasvir masala** deb ataymiz. Bu o‘rinda boshlang‘ich masala tarkibiga kirgan funksiyalarning almashtirishni qo‘llash mumkin bo‘lishi shartini (masalan, Laplas almashtirishi uchun 1)-3) shartlarni) tekshirib ko‘rish muhimligin eslatamiz;

2) almashtirish natijasida olingan tasvir masalani yechib, noma’lum tasviri topiladi;

3) teskari almashtirish yordamida noma’lumning tasviridan asliga qaytiladi.

Endi operatsion hisob usulini ba’zi masalalarni hal qilishga qo‘llaylik.

### 11.3.1. Differensial tenglamalarni yechish.

Laplas almashtirishi asos bo‘lgan operatsion hisob usulini chiziqli differensial tenglamalarni va ularning sistemalarini yechishga qo‘llash uning muhim tatbiqlaridan biri ekanligini aytamiz.

Aytaylik, p-tartibli o‘zgarmas koeffisientli chiziqli oddiy differensial

$$\sum_{j=0}^{n} в_j y^{(n-j)} = f(t), t \in (0;+\infty) \qquad (11.3.1)$$

tenglamaning

$$y^{(k)}(0) = y_{0k,} k = \overline{0; n-1} \qquad (11.3.2)$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini topish masalasi qo‘yilgan bo‘lsin. Bunda $в_k (k = \overline{o; n})$ berilgan haqiqiy sonlar (koeffitsientlar) bo‘lib, $в_0 \neq 0$ (ko‘pincha $в_0 = 1$) deb hamda $f(t)$ haqiqiy o‘zgaruvchili funksiya $[o;+\infty)$ oraliqda berilgan va uzluksiz deb faraz qilsak, qo‘yilgan (11.3.1) – (11.3.2) masala yagona yechimga ega bo‘lishi aniqdir (14.6.1-bandga qarang).

Endi (11.3.1)-(11.3.2) masalaga Laplas almashtirishiga asoslangan operatsion hisob usulini qo‘llaymiz.

Buning uchun (11.3.1) ning o‘ng tomonidagi $f(t)$ funksiya originaldan iborat bo‘lsin deb talab qilamiz. U holda $y \leftrightarrow У(p)$ deb olsak, originalni differentsiyallash teremasiga ko‘ra

$$y^{(k)} \leftrightarrow p^k У(p) - \sum_{m=0}^{k-1} p^{k-m-1} y_{0,m} \quad K = \overline{1,n}$$

bo‘ladi. Agar $f(t) \leftrightarrow F(p)$ deb faraz qilsak, tasvir tenglama

$$\sum_{j=0}^{n} в_j (p^{n-j} У(p) - \sum_{m=0}^{n-j-1} p^{n-j-m-1} y_{0,m}) = F(p)$$

ko‘rinishda bo‘lib, u $У(p)$ tasvirga nisbatan chiziqli algebraik tenglamadan iboratdir. Uni yechib

$$У(p) = \frac{F(p) + \sum_{j=0}^{n} в_j \sum_{m=0}^{n-j-1} p^{n-j-m-1} y_{0,m}}{\sum_{j=0}^{n} в_j p^{n-j}}$$

ni olamiz. Agar

$$A(p) = F(p) + \sum_{j=0}^{n} в_j \sum_{m=0}^{n-j-1} p^{n-j-m-1} y_{0,m}$$

$$B(p) = \sum_{j=0}^{n} в_j p^{n-j}$$

belgilashni kiritsak, $B(p)-$(11.3.1) ning xarakteristik ko‘phadi, yani $n-$darajali haqiqiy koeffisientli ko‘phaddir. Uning nollari $У(p)$ ning maxsus nuqtalari bo‘lishi aniqdir. Agar $F(p)-$ratsional funksiya bo‘lsa, $F(p)$ ning maxsus nuqtalarini ham hisobga olgan holda $У(p)$ni

$$У(p) = У_1(p) + \frac{A_1(p)}{B_1(p)}$$

ko‘rinishga keltirib, so‘ngra, $\frac{A_1(p)}{B_1(p)}$ qisqarmaydigan kasrga yoyish formulasini qo‘llashga to‘g‘ri keladi, bu yerda $A_1(p)$ va $B_1(p)$ ko‘phadlardan iborat bo‘lib, $A_1(p)$ ning darajasi $B_1(p)$ ning darajasidan kichikdir, $У_1(p)$ esa ko‘phaddan iboratdir.

**15-misol.** $y'' + 3y' + 2y = 1 + t^2; y(0) = 1, y'(0) = 0$

**Yechish.** Tasvirga o‘tamiz: $1 + t^2 \leftrightarrow \frac{1}{p} + \frac{6}{p^3} = \frac{p^2 + 6}{p^3};$

$$y \leftrightarrow У(p); y' = pУ(p) - y(0) = pУ(p) - 1;$$
$$y'' \leftrightarrow p^2 У(p) - py(0) - y'(0) = p^2 У(p) - P\ .$$

Tasvir tenglama:

$$p^2У(p)-p+3(pУ(p)-1)+2У(p)=\frac{p^2+2}{p^3};$$

$$(p^2+3p+2)У(p)=\frac{p^2+2}{p^3}+p+3=\frac{p^2+2+p^4+3p^3}{p^3}$$

Bundan

$$У(p)=\frac{p^4+3p^3+p^2+2}{p^3(p^2+3p+2)}=\frac{p^4+3p^3+p^2+2}{p^3(p+2)(p+1)}$$

ni olamiz. Oxirgining o‘ng tomoni to‘g‘ri ratsional kasr bo‘lib, uning maxsus nuqtalari $a_1=0$ - uchinchi tartibli $a_2=-2$ va $a_3=-1$ lar oddiy qutblardir.

Endi yoyish teoremasini qo‘llab,(11.2.28) formulaga ko‘ra u ni topamiz:

$$y=\sum_{k=1}^{з}\mathrm{Re}\,s\left[\frac{A(p)}{B(p)}e^{pt};a_k\right],$$

bu yerda $A(p)=p^4+3p^3+p^2+2,\ B(p)=p^3(p+2)(p+1)=p^5+3p^4+2p^3,$ $a_1=0$ -uchinchi tartibli qutb, $a_2=-1$ va $a_3=-2$ lar birinchi tartibli qutblardir.Ularni hisobga olib yuqoridagi formuladan foydalanamiz:

$$\mathrm{Re}\,s\left[\frac{A(p)}{B(p)}e^{pt};0\right]=\frac{1}{2!}\lim_{p\to 0}\left[p^3\frac{p^4+3p^3+p^2+2}{p^3(p+1)(p+2)}e^{pt}\right]''_{pp}=\frac{1}{2!}\lim_{p\to 0}\left[\frac{p^4+3p^3p+^2+2}{p^2+3p+2}e^{pt}\right]''_{pp}=$$

$$\frac{1}{2}\lim_{p\to o}\left[\frac{A(p)}{B_0(p)}\cdot e^{pt}\right]''_{pp}=\frac{1}{2}\lim_{p\to o}\left\{\left[\left(\frac{A(p)}{B_0(p)}\right)'_p+\frac{A(p)}{B_0(p)}t\right]e^{pt}\right\}'_p=$$

$$=\frac{1}{2}\lim_{p\to o}\left\{\left[\left(\frac{A(p)}{B_0(p)}\right)''_{pp}+\left(\frac{A(p)}{B_0(p)}\right)'_p t\right]+\left[\left(\frac{A(p)}{B_0(p)}\right)'_p+\frac{A(p)}{B_0(p)}t\right]te^{pt}\right\}=$$

$$=\frac{1}{2}\lim_{p\to 0}\left\{\left[\left(\frac{A(p)}{B_0(p)}\right)''_{pp}+2\left(\frac{A(p)}{B_0(p)}\right)^1_p\cdot t+\frac{A(p)}{B_0(p)}t^2\right]e^{pt}\right\}$$

Bunda

$$B_0(p)=p^2+3p+2$$

$$\left(\frac{A(p)}{B_0(p)}\right)'_p=\frac{A'(p)\cdot B_0(p)-A(p)B_0'(p)}{B_0^2(p)}$$

$$B_0(p)-2B_0'(p)(A'B_0-AB_0)$$

$$\left(\frac{A(p)}{B_0(p)}\right)''_{pp}=\frac{\left(A''(p)\cdot B_0(p)-A(p)\cdot B_0''(p)\right)}{B_0^3(p)}$$

$$p=0;\ A(0)=2,\ B_0(0)=2;\ \frac{A(0)}{B_0(0)}=\frac{2}{2}=1;$$

$$A'(p) = 4p^3 + 9p^2 + 2p;\ A'(0) = 0;$$

$$B'(p) = 2p + 3,\ B'(0) = 3$$

$$A''(p) = 12p^2 + 18p + 2,\ A''(0) = 2;\qquad B_0''(p) = 2.$$

$$\operatorname{Re}s\left[\frac{A(p)}{B_0(p)}e^{pt};0\right] = \frac{1}{2}\left(\frac{9}{2} - 2\cdot\frac{3}{2}t + t^2\right) = \frac{9}{4} - \frac{3}{2}t + \frac{1}{2}t^2$$

$$\operatorname{Re}s\left[\frac{A(p)}{B(p)}e^{pt};-1\right] = \frac{1}{2}\lim_{p\to -1}\left\{\left[(p+1)\frac{A(p)}{p^3(p+1)(p+2)}\right]e^{pt}\right\} = \lim_{p\to -1}\frac{A(p)}{p^3(p+2)}e^{pt} =$$

$$= \left(\frac{A(-1)}{-1\cdot 1}e^{-t}\right) = -e^{-t};$$

$$\operatorname{Re}s\left[\frac{A(p)}{B(p)}e^{pt};-2\right] = \lim_{p\to -2}\left[(p+2)\frac{A(P)}{p^3(p+1)(p+2)}e^{pt}\right] = \frac{A(-2)}{-B(-1)}e^{-2et} = -\frac{1}{4}e^{-2et}$$

Demak,

$$y = \frac{9}{4} - \frac{3}{2}t + \frac{1}{2}t^2 - e^{-t} - \frac{1}{4}e^{-2t}$$

Endi chiziqli differensial tenglamani yechishda tasvirlarni ko‘paytirish teoremasi, xususan, Dyuamel integralining tatbiqlaridan birini bayon qilamiz.

Aytaylik, (11.3.1) chiziqli o‘zgarmas koeffisientli oddiy differensial tenglamaning (11.3.2) boshlang‘ich shartlarni qanoatlantiruvchi yechimini topish talab qilingan bo‘lib,boshlang‘ich shartlar bir jinsli bo‘lsin,yani

$$y^{(k)}(0) = 0,\ k = \overline{0, n-1} \qquad (11.3.3)$$

Agar $L[y] = \sum_{m=0}^{n} b_m \frac{d^{n-m}y}{dt^{n-m}}$ chiziqli differensial operator kiritsak (11.3.4) ni

$$L[y] = f(t) \qquad (11.3.4)$$

ko‘rinishda yozish mumkindir.

(11.3.4) ning (11.3.2) bir jinsli boshlang‘ich shartlarda yechish uchun, avvalo, xususiy holdan iborat bo‘lgan

$$L(y_1) = 1 \qquad (11.3.5)$$

ning (11.3.3) bir jinsli boshlang‘ich shartlardagi yechimini topish masalasini yechishdan boshlaymiz.

Tasvirga o‘tamiz. Buning uchun $y_1 \leftrightarrow Y_1(p)$ desak,

$$y_1^{(k)} \leftrightarrow p^k Y_1(p)$$

bo‘lishi aniqdir. $\frac{1}{p} \leftrightarrow 1$ ekanligidan tasvir uchun

$$Y_1(p)L(p) = \frac{1}{p} \qquad (11.3.6)$$

tenglamani olamiz, bunda $L(p)=\sum_{m=0}^{n}в_ш p^{n-m}$ (11.3.4) ning tasnifiy (xarakteristik) ko‘phadidir. Agar bu masalaning yechimi $y_1(t)$ ma’lum deb faraz qilsak, u holda $y_1(t)\leftrightarrow У_1(p)$ tasvir ham ma’lumdir. (11.3.6) dan

$$L(p)=\frac{1}{PУ_1(p)} \qquad (11.3.7)$$

o‘rinli bo‘lishini olamiz.
Endi (11.3.4) ning (11.3.3) bir jinsli boshlang‘ich shartlarda yechish masalasiga qaytsak va $f(t)\leftrightarrow F(p)$ deb faraz qilsak, bu masalaning tasvir tenglamasi

$$У(p)L(p)=F(p)$$

Bundan (11.3.7) ni hisobga olgan holda

$$У(p)=pУ_1(p)F(p)$$

ni olamiz. Tasvirlarni ko‘paytirish formulasiga asosan

$$y(t)=y_1(0)f(t)+\int_0^t y_1'(\tau)f(t-\tau)dt$$

Dyumel integralini qo‘llab,
(11.3.3) birjinsli boshlang‘ich shartlar asosida $y_1(0)=0$ ekanligidan

$$y(t)=\int_0^t y_1'(\tau)f(t-\tau)d\tau \qquad (11.3.8)$$

yoki

$$y(t)=y_1(t)f(0)+\int_0^t f'(\tau)y_1(t-\tau)dt \qquad (11.3.9)$$

formulani olamiz. Oxirgi (11.3.8) va (11.3.9) formulalar $y(t)$ ni aniqlash uchun teng kuchlidir. Uyurmaning kommutativlik hossasidan foydalanib yana ikkita formulalarni keltirish mumkinligini aytamiz.

**16-misol.** $y'-2y'+y=t^2e^t,\ y(o)=y'(0)=0$ (11.3.10) Koshi masalasini yeching.

**Yechish.** Avval $y_1''-2y_1'+y_1=1,\ y_1(0)=y_1'(0)=0$ (10.3.11) Koshi masalasining yechimini topamiz.

$$L(p)=p^2-2p+1=(p-1)^2 \qquad \text{dan}$$

bo‘lib, (11.3.6)

$$У_1(p)=\frac{1}{p(p-1)^2}$$

ni olamiz. Bundan ko‘rinadiki, $a_1=0$ $У_1(P)$ uchun oddiy, $a_2=1$ esa ikkinchi tartibli qutbdir. (11.2.28) yoyish teoremasini qo‘llab, (11.3.11) ning yechimini olamiz:

$$y_1(t)=\lim_{p\to0}\left[P\frac{e^{pt}}{p(p-1)^2}\right]+\lim_{p\to1}\left[(p-1)^2\frac{e^{pt}}{p(p-1)^2}\right]_p'=$$

$$= \lim_{p\to 0} \frac{e^{pt}}{(p-1)^2} + \lim_{p\to 1}\left(\frac{e^{pt}}{p}\right)_p' = 1 + \lim_{p\to 0}\left(-\frac{1}{p^2} + \frac{1}{p}t\right)e^{pt} = 1 + (-1+t)e^t = 1 + (t-1)e^t.$$

$$y_1(t) = 1 + (t-1)e^t$$

ega bo‘lamiz. Endi (11.3.8) formulani qo‘llab (11.3.10) Koshi masalasining yechimiga ega bo‘lamiz:

$$y(t) = \int_0^t \left[1 + (\tau-1)e^\tau\right]' \left((t-\tau)^2 e^{t-\tau}\right) d\tau = \int_0^t [1 + (\tau - 1]e^\tau (t-\tau)^2 e^{t-\tau} d\tau = \int_0^t \tau (t-\tau)^2 e^t d\tau =$$

$$= e^t \int_0^t \tau (t-\tau)^2 d\tau = \frac{t^4}{12} e^t .$$

demak, $$y(t) = \frac{t^4}{12} e^t .$$

**16-misol.** $y'' + y = f(t)$ differensial tenglamaning $t=0,\ y(0) = y'(0) = 0$ boshlang‘ich shartlarni qanoatlantiruvchi yechimi topilsin.

**Yechish.** Tasvir tenglamaga o‘tamiz. Buning uchun $y(t) \leftrightarrow Y(p)$ deb faraz qilsak,

$$Y(p)(p^2 + 1) = F(p)$$

ni olamiz,bu yerda $$F(p) \leftrightarrow f(t).$$

Oxirgi tenglamadan

$$Y(p) = \frac{1}{1+p^2} F(p)$$

ni olamiz. $\frac{1}{1+p^2} \leftrightarrow S\text{int}$ ekanligi bizga ma’lum (1-jadvalning 17-satrida $\omega = 1$ bo‘lgan hol). U holda, tasvirlarni ko‘paytirish formulasini qo‘llab, izlanyotgan yechim

$$y(t) = \int_0^t f(\tau) sin(t-\tau) d\tau$$

bo‘lishini olamiz.

### 11.3.2. Oddiy differensial tenglamalar sistemasini yechish

Bu bandda o‘zgarmas koeffisientli chiziqli oddiy differensial tenglamalarning

$$y_j' = \sum_{m=1}^{n} a_{jm} y_m + \varphi_j(t),\ j = \overline{1,n} \qquad (11.3.12)$$

normal sistemasini

$$t = 0; y_j(0) = y_{o,j}\ \ j = \overline{1,n} \qquad (11.3.13)$$

boshlang‘ich shartlarni qanoatlantiruvchi yechimini topish masalasini qaraymiz va buning uchun operatsion hisob usulini tatbiq qilamiz.

Agar $y_j \leftrightarrow Y_j(p)$ va $\varphi_j(t) \leftrightarrow F_j(p)$ desak, (11.3.12) – (11.3.13) masala uchun

$$pУ_j(p) - y_{0,j} = \sum_{m=1}^{n} a_{jm}У_m(p) + F_j(p),\ j = \overline{1,n}$$

tasvir tenglamalar sistemasini olamiz.
Bu sistemani

$$\sum_{m=1}^{n}\left(p\delta_{jm} - a_{jm}\right)У_m(p) = y_{0,j} + F_j(p),\ j = \overline{1,n}$$

ko‘rinishga keltiramiz, bu yerda $\delta_{jm}$ –Kronekker belgisi.

Agar bu sistemaning $A = \left[p\delta_{jm} - a\right]$ $nxn$ matritsasini maxsus emas deb talab qilsak, uning yagona yechimi mavjuddir:

$$У_j(p) = F_j(p),\ j = \overline{1,n}.$$

Bu (11.3.12) – (11.3.13) masala yechimining tasviridir. Undan asliga (originalga) qaytib, $y_j(t)(j = \overline{1,n})$ yechimni olamiz.

**18-misol.** $\begin{cases} \dfrac{dx}{dt} = -x + y + e^t \\ \dfrac{dy}{dt} = -3x + 2yx + 2e^t \end{cases}$

differensial tenglamalar sistemasini

$$t = 0; x(0) = y(0) = 1$$

boshlang‘ich shartlarda yeching.

**Yechish.** $x \leftrightarrow X(p), y \leftrightarrow У(p)$ desak, $e^t \leftrightarrow \dfrac{1}{p-1}$ ekanligidan, ko‘rilayotgan masalaning tasvir tenglamasini

$$\begin{cases} pX(p) - 1 = -X(p) + У(p) + \dfrac{1}{p-1}, \\ pУ(p) - 1 = -3X(p) + 2У(p) + \dfrac{2}{p-1} \end{cases}$$

sistema ko‘rinishda bo‘ladi.Uni

$$\begin{cases} (p+1)X(p) - У(p) = \dfrac{p}{p-1}, \\ 3x(p) + (p-2)Y(p) = \dfrac{p+1}{p-1} \end{cases}$$

ko‘rinishga keltirish oson, buni yechib,

$$X(p) = \frac{1}{p-1}, У(p) = \frac{1}{p-1}$$

ni olamiz. Demak yechim:

$x(t) = y(t) = e^t$

bo‘ladi.

**19-misol.** $$\begin{cases} \dfrac{dx}{dt} + y = \varphi_1(t), \\ \dfrac{dy}{dt} + x = \varphi_2(t), \end{cases}$$

bu yerda

$$\varphi_1(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & t \ge 1; \end{cases}$$

$$\varphi_2(t) = \begin{cases} 1, & 0 \le t < 2, \\ 0, & t \ge 2. \end{cases}$$

differensial tenglamalar sistemasini $x(0) = y(0) = 0$ bir jinsli boshlang‘ich shartlarda yeching.

**Yechish.** $\varphi_1(t)$ va $\varphi_2(t)$ funksiyalarni $\varphi_1(t) = \sigma_0(t) - \sigma_0(t-1), \varphi_2(t) = \sigma_0(t) - \sigma_0(t-2)$ kabi ifodalash mumkin, bu yerda $\sigma_0(t) -$ Xevisaydning birlik funksiyasidir.U holda ularning tasvirlari

$$\varphi_1(t) \leftrightarrow \frac{1}{p} - \frac{e^{-p}}{p} = \frac{1-e^{-p}}{p}, \varphi_2(t) \leftrightarrow \frac{1-e^{-2P}}{p}$$

bo‘ladi. $x \leftrightarrow X(p), y \leftrightarrow У(p)$ desak sistema tasviri uchun

$$\begin{cases} pX(p) + У(p) = \dfrac{1-e^{-p}}{p} \\ X(p) + pУ(p) = \dfrac{1-e^{-2p}}{p} \end{cases}$$

sistemani olamiz. Uni yechamiz:

$$\begin{cases} p^2 X(p) + pУ(p) = 1 - e^{-p}, \\ X(p) + pУ(p) = \dfrac{1-e^{-2P}}{p}; \end{cases} \Rightarrow X(p) = \frac{1}{p(p+1)} - \frac{e^{-p}}{p^2-1} + \frac{e^{-2p}}{p(p^2-1)};$$

$$\begin{cases} pX(p) + У(p) = \dfrac{1-e^{-p}}{p}, \\ pX(p) + p^2 У(p) = 1 - e^{-2p}; \end{cases} \Rightarrow У(p) = \frac{1}{p(p+1)} - \frac{e^{-2p}}{p^{2-p}} + \frac{e^{-p}}{p(p^2-1)}$$

Topilgan tasvirlardan asliga (originalga) qaytamiz:

$$X(t) = \begin{cases} 1 - e^{-t}, \ o < t < 1, \\ 1 - e^{-t} - sh\ (t-1), & 1 \le t < 2, \\ -e^{-t} - sh\ (t-1) + ch(t-2), & t \ge 2; \end{cases}$$

$$y(t)=\begin{cases} 1-e^{-t}, 0\le t<1 \\ 1-e^{-t}-sh(t-1), 1\le t<2 \\ -e^{-t}-sh(t-1)+ch(t-2), t\ge 2; \end{cases}$$

sistema yechimini olamiz. Bu yerda “kechikish” teoremasi tatbiq qilinganligini aytamiz.

## 11-bobga doir mashqlar.

1. Differensial tenglamani berilgan boshlang‘ich shartlarda yeching:

1) $\frac{d^2y}{dt^2}-2\frac{dy}{dt}+y=t^2e^t, y(0)=y'(0)=0.$

Javob: $y=\frac{1}{12}t^4e^t.$

2) $\frac{d^3y}{dt^3}+3\frac{d^2y}{dt^2}+3\frac{dy}{dt}+y=6e^{-t}, y(0)=y'(0)=y''(0)=0$

Javob: $y=t^3e^{-t}$ .

3) $\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y=12e^{3t}, y(0)=2, y'(0)=6$

Javob: $y=4e^t-8e^{2t}+6e^{3t}$ .

4) $\frac{d^2y}{dt^2}+4y=3\sin t+10\cos 3t, y(0)=-2, y'(0)=3$

Javob: $y=\sin t+\sin 2t-2\cos 3t$

5) $\frac{d^2y}{dt^2}-y=4\sin t+5\cos 2t, y(0)=-1, y'(0)=3$

Javob: $y=-2\sin t-\cos 2t$

2. Differensial tenglamalar sistemasini berilgan boshlang‘ich shartlarda yeching.

1) $\begin{cases} x'+2x+2y=10e^{2t}, \\ y'-2x+y=7e^{2t}; \end{cases}$ $x(o)=1, y(0)=3$

Javob: $\begin{cases} x=e^{2t}, \\ y=3e^{2t}. \end{cases}$

2) $\begin{cases} x'+y'-y=e^t, \\ 2x'+y'+2y=cost; \end{cases}$ $x(0)=y(0)=0.$

Javob: $$x = -\frac{1}{2} - \frac{11}{34}e^{4t} + e^{t} - \frac{3}{17}\cos t + \frac{5}{17}\sin t$$
$$y = \frac{22}{5}e^{4t} - \frac{2}{3}e^{t} + \frac{4}{17}\cos t - \frac{1}{17}\sin t$$

3) $$\begin{cases} x' = y + z, \\ y' = z + x, \ x(0) = -1, y(0) = 1, z(0) = 0. \\ z' = x + y; \end{cases}$$

Javob: $$\begin{cases} x = -e^{-t}, \\ y = e^{-t}, \\ z = o. \end{cases}$$

4) $$\begin{cases} x'' - 3x - 4y + 3 = 0, \\ y'' + x + y - 5 = 0; \end{cases} \begin{cases} x(0) = x'(0) = 0, \\ y(0) = y'(0) = 0. \end{cases}$$

Javob: $$\begin{cases} x = 17 - 17cht - 7tsht, \\ y = -12 + 12cht - \frac{7}{2}tsht. \end{cases}$$

## 11-bob bo‘yicha bilimingizni sinab ko‘ring.

1. Laplas almashtirishini ta’riflang.
2. Original va tasvir tushunchalarini bayon qiling.
3. Xevisayd birlik funktsisiyaning tasvirini yozing.
4. $e^t$, $S\text{int}$ va $Cost$ funksiyalar tasvirini yozing.
5. Laplas almashtirishining chiziqlilik xossasini bayon qiling.
6. Laplas almashtirishining o‘xshashlik xossasini yozing.
7. Laplas almashtirishining siljish teoremasini bayon qiling.
8. Laplas almashtirishining kechikish teoremasini bayon qiling.
9. Parametr bo‘yicha differensiallash haqidagi teoremani ayting.
10. Originalni differensiallash teoremasini bayon qiling.
11. Originalni integrallash teoremasini bayon qiling.
12. Tasvirni differensiallash teoremasini bayon qiling.
13. Tasvirni integrallash teoremasini bayon qiling.
14. Tasvirni ko‘paytirish teoremasini bayon qiling.
15. Dyuamel integralini yozing.
16. Originallarni ko‘paytirish teoremasini ayting.
17. Gamma funksiya haqida tushuncha bering.
18. O’girish teoremasini bayon qiling.
19. Yoyish formulasini yozing.
20. Operatsion hisob usuli haqida tushuncha bering.

## Oliy matematika fanida Maple dastur tizimidan foydalanish uslubiyoti

Maple dasturi- murakkab hisoblash ishlarini analitik hamda sonli usullarda yechish uchun kuchli hisoblash tizimi hisoblanadi. Maple tizimi tarkibida tasdiqlangan, ishonchli va samarali analitik va sonli hisoblash algoritmi mavjud.

Quyida Maple tizimida oliy matematikaning ayrim masalalarini yechishni ko’rsatamiz:

### 1. Matritsalar.

1.1. $A=\begin{Vmatrix}1 & 2 & 3\\ -3 & -2 & -1\end{Vmatrix}$ matritsaning yozilishi

> $A := matrix([[1,2,3],[-3,-2,-1]]);$

$$A := \begin{bmatrix}1 & 2 & 3\\ -3 & -2 & -1\end{bmatrix}$$

1.2. $A=\begin{Vmatrix}3 & 2\\ 1 & -1\end{Vmatrix}$ matritsani $B=\begin{Vmatrix}-5 & 1\\ 7 & 4\end{Vmatrix}$ matritsaga ko’paytirish

> $A := matrix([[3,2],[1,-1]]);$

$$A := \begin{bmatrix}3 & 2\\ 1 & -1\end{bmatrix}$$

> $B := matrix([[-5,1],[7,4]]);$

$$B := \begin{bmatrix}-5 & 1\\ 7 & 4\end{bmatrix}$$

> $evalm(A\&*B);$

$$\begin{bmatrix}-1 & 11\\ -12 & -3\end{bmatrix}$$

> $A := matrix([[3,2],[1,-1]]);$

$$A := \begin{bmatrix}3 & 2\\ 1 & -1\end{bmatrix}$$

> $B := matrix([[-5,1],[7,4]]);$

$$B := \begin{bmatrix} -5 & 1 \\ 7 & 4 \end{bmatrix}$$

> *evalm*(*A*&**B*);

$$\begin{bmatrix} -1 & 11 \\ -12 & -3 \end{bmatrix}$$

> *v* := *vector*([2, 4]);

$$v := \begin{bmatrix} 2 & 4 \end{bmatrix}$$

> *multiply*(*A*, *v*);

$$\begin{bmatrix} 14 & -2 \end{bmatrix}$$

> *multiply*(*A*, *B*);

$$\begin{bmatrix} -1 & 11 \\ -12 & -3 \end{bmatrix}$$

> *matadd*(*A*, *B*);

$$\begin{bmatrix} -2 & 3 \\ 8 & 3 \end{bmatrix}$$

1.3. Matritsani songa ko'paytirish va qo'shish $C = \begin{Vmatrix} 1 & 1 \\ 2 & 3 \end{Vmatrix}$ berilgan bo'lsa, $2+3C$ ni hisoblash

> C := *matrix*([[1, 1], [2, 3]]);

$$C := \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

> *evalm*(2 + 3 * C);

$$\begin{bmatrix} 5 & 3 \\ 6 & 11 \end{bmatrix}$$

1.4. $A = \begin{Vmatrix} 4 & 0 & 5 \\ 0 & 1 & -6 \\ 3 & 0 & 4 \end{Vmatrix}$ matritsa determinantini topish

> *A* := *matrix*([[4, 0, 5], [0, 1, -6], [3, 0, 4]]);

$$A := \begin{bmatrix} 4 & 0 & 5 \\ 0 & 1 & -6 \\ 3 & 0 & 4 \end{bmatrix}$$

> *det*(*A*);

$$1$$

1.5. Matritsa minorini topish $M_{32}$

> *minor*(*A*, 3, 2);

$$\begin{bmatrix} 4 & 5 \\ 0 & -6 \end{bmatrix}$$

> *trace*(*A*);

1.6. Teskari va transponirlangan matritsani topish

1.7. $A = \begin{Vmatrix} 4 & 0 & 5 \\ 0 & 1 & -6 \\ 3 & 0 & 4 \end{Vmatrix}$ matritsaga teskari matritsani topish

> *inverse*(*A*);

$$\begin{bmatrix} 4 & 0 & -5 \\ -18 & 1 & 24 \\ -3 & 0 & 4 \end{bmatrix}$$

> *multiply*(*A*, %);

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.8. Matritsani transponirlash

> *transpose*(*A*);

$$\begin{bmatrix} 4 & 0 & 3 \\ 0 & 1 & 0 \\ 5 & -6 & 4 \end{bmatrix}$$

1.9. Uchta matritsa kopaytmasi

> *with*(*linalg*) :

> *A* := *matrix*([[4, 3], [7, 5]]);

$$A := \begin{bmatrix} 4 & 3 \\ 7 & 5 \end{bmatrix}$$

> *B* := *matrix*([[−28, 93], [38,−126]]);

$$B := \begin{bmatrix} -28 & 93 \\ 38 & -126 \end{bmatrix}$$

> *C* := *matrix*([[7, 3], [2, 1]]);

$$C := \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$$

> *F* := *evalm*(*A*&**B*&**C*);

$$F := \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

> Det(A) = det(A); Det(B) = det(B); Det(C) = det(C); Det(F) = det(F);

$$Det(A) = -1$$

$$Det(B) = -6$$

$$Det(C) = 1$$

$$Det(F) = 6$$

1.11. $A = \begin{Vmatrix} 8 & -4 & 5 & 5 & 9 \\ 1 & -3 & -5 & 0 & -7 \end{Vmatrix}$ matritsa rangini aniqlash

> *restart*;
> *with*(*linalg*) :
> *A* := *matrix*([[8,-4, 5, 5, 9], [1,-3,-5, 0,-7]]);

$$A := \begin{bmatrix} 8 & -4 & 5 & 5 & 9 \\ 1 & -3 & -5 & 0 & -7 \end{bmatrix}$$

> *r*(*A*) = *rank*(*A*);

$$r(A) = 2$$

1.12. $A = \begin{Vmatrix} 5 & 1 & 4 \\ 3 & 3 & 2 \\ 6 & 2 & 10 \end{Vmatrix}$ matritsa berilgan bo'lsa, $P(A) = A^3 - 18A^2 + 64A$ ni hisoblash

> *A* := *matrix*([[5, 1, 4], [3, 3, 2], [6, 2, 10]]);

$$A := \begin{bmatrix} 5 & 1 & 4 \\ 3 & 3 & 2 \\ 6 & 2 & 10 \end{bmatrix}$$

> *P*(*A*) = *evalm*(*A*^3-18**A*^2 + 64**A*);

$$P(A) = \begin{bmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{bmatrix}$$

## 2. Tenglamalar sistemasini yechish

2.1. $\begin{cases} 2x - 3y + 5z + 7t = 1 \\ 4x - 6y + 2z + 3t = 2 \\ 2x - 3y - 11z - 15t = 1 \end{cases}$

> *eq* := {2**x*-3**y* + 5**z* + 7**t* = 1, 4**x*-6**y* + 2**z* + 3**t* = 2,
2**x*-3**y*-11**z*-15**t* = 1};

$$eq := \{2x - 3y - 11z - 15t = 1, 2x - 3y + 5z + 7t = 1, 4x - 6y + 2z + 3t = 2\}$$

> *s* := *solve*(*eq*, {*x*, *y*, *z*});

$$s := \left\{x = \frac{1}{2} + \frac{3}{2}y - \frac{1}{16}t, y = y, z = -\frac{11}{8}t\right\}$$

$y = 1$; $t = 1$ bo'lsa

> *subs*({*y* = 1, *t* = 1}, *s*);

$$\left\{1 = 1, x = \frac{31}{16}, z = -\frac{11}{8}\right\}$$

2.2. Matritsali tenglamalarni yechish $AX = B$

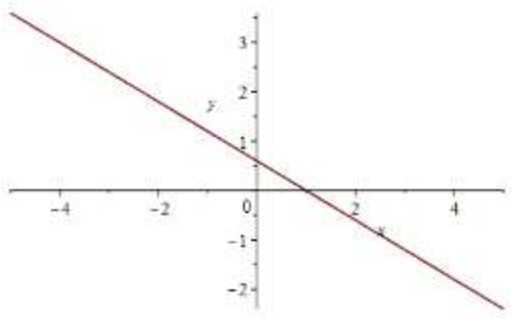
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot X = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}$$

> $A := matrix([[1, 2], [3, 4]]) :$
> $B := matrix([[3, 5], [5, 9]]) :$
> $X := linsolve(A, B);$

$$X := \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}$$

**3. Maple dastur tizimida to'g'ri chiziqni qurish**

Quyidagi to'g'ri chiziqni quring $3x + 5y = 3$
implicitplot(3*x+5*y = 3, x = -5 .. 5, y = -5 .. 5, scaling = constrained)




**4. Ikki to'gri chiziq orasidagi burchakni topish**

$\frac{x-2}{2} = \frac{y+3}{1} = \frac{z-2}{3}$ va $\frac{x+3}{1} = \frac{y+4}{2} = \frac{z-5}{-2}$ to'g'ri chiziqlar orasidagi burchakni toping

> $a := ([2, 1, 3]);$

$$a := [2, 1, 3]$$

> $b := ([1, 2, -2]);$

$$b := [1, 2, -2]$$

> $dotprod(a, b);$

$$-2$$

> $\text{phi} = angle(a, b);$

$$\phi = \pi - \arccos\left(\frac{1}{63}\sqrt{14}\sqrt{9}\right)$$

**5. Nuqtadan to'g'ri chiziqqacha bo'lgan masofa**

$A(1;3;2)$ nuqtadan $3x + y + z - 6 = 0$ tekislikkacha bo'lgan masofa
$with(geom3d);$
> $point(A, 1, 3, 2);$

$$A$$

> $plane(p, 3 \cdot x + y + z - 6 = 0, [x, y, z]);$

$$p$$

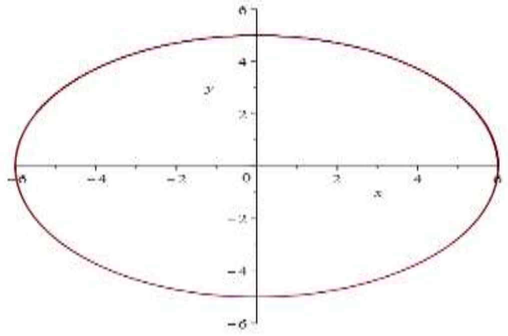
> $distance(A, p);$

$$\frac{2}{11}\sqrt{11}$$

6. **Maple dastur tizimi yordamida ellipsni yasash**.

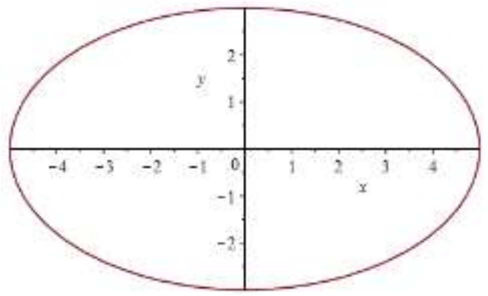
Agar katta yarim o`q a=6; kichik yarim o'q b=5 bo'lsa

a := 6; b := 5; w := 3; plot([a*cos(w*t), b*sin(w*t), t = 0 .. 3], x = -6 .. 6, y = -6 .. 6);




$\frac{x^2}{25} + \frac{y^2}{9} = 1$giperbolani qurish

**implicitplot(x^2/25+y^2/9=1,x=-5..5,y=-5..5,scaling=constrained);**

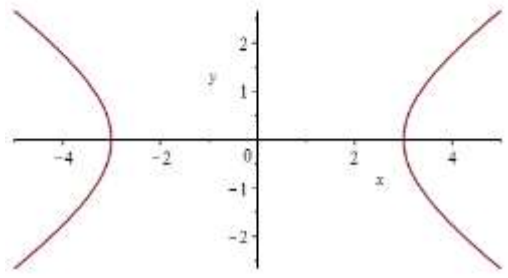



7. **Giperbola grafigini qurish**

$\frac{x^2}{9} - \frac{y^2}{4} = 1$ giperbolani quraylik

with(plots);

implicitplot((1/9)*x^2-(1/4)*y^2 = 1, x = -5 .. 5, y = -5 .. 5, scaling = constrained);




$\frac{x^2}{64}-\frac{y^2}{36}=1$ giperbolaning asimptotasini qurish

```
> with(plots);

> with(geometry) : hyperbola(Hyp_3,x^2/64 - y^2/36 = 1, [x,y]);
```

Hyp_3

```
> detail(Hyp_3);
```

| | |
|---|---|
| name of the object | Hyp_3 |
| form of the object | hyperbola2d |
| center | [0, 0] |
| foci | [[−10, 0], [10, 0]] |
| vertices | [[−8, 0], [8, 0]] |
| the asymptotes | $\left[y+\frac{3x}{4}=0, y-\frac{3x}{4}=0\right]$ |
| equation of the hyperbola | $-1+\frac{x^2}{64}-\frac{y^2}{36}=0$ |

```
> CC := center(Hyp_3);
```

CC := center_Hyp_3

```
> FF := map(coordinates,foci(Hyp_3));
```

FF := [[-10, 0], [10, 0]]

```
> point(F_1, FF[1]);point(F_2, FF[2]);
```

F_1

F_2

```
> VER := map(coordinates, vertices(Hyp_3));
```

VER := [[-8, 0], [8, 0]]

```
> point(VER_1, VER[1]);point(VER_2, VER[2]);
```

VER_1

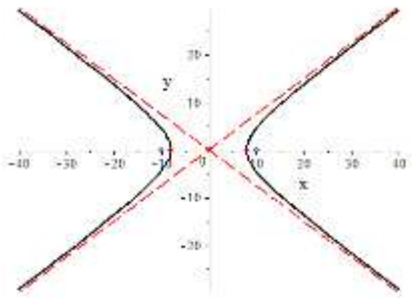
VER_2

```
> AS := map(Equation, asymptotes(Hyp_3));
```

$AS := \left[y+\frac{3}{4}x=0, y-\frac{3}{4}x=0\right]$

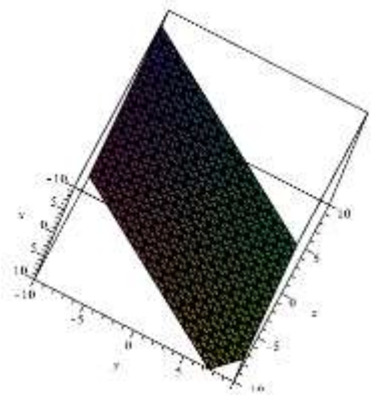
```
> line(AS1, AS[1], [x, y]) : line(AS2, AS[2], [x, y]) :
```

> *draw*([*Hyp_3*(*color* = *black*, *thickness* = 2), *AS1*(*color* = *red*, *linestyle* = *dash*), *AS2*(*color* = *red*, *linestyle* = *dash*), *CC*, *F_1*(*color* = *blue*), *F_2*(*color* = *blue*), *VER_1*, *VER_2*], *axes* = *normal*, *labels* = ["x", "y"], *labelfont* = [*times*, 14]);



## 8.Tekislik

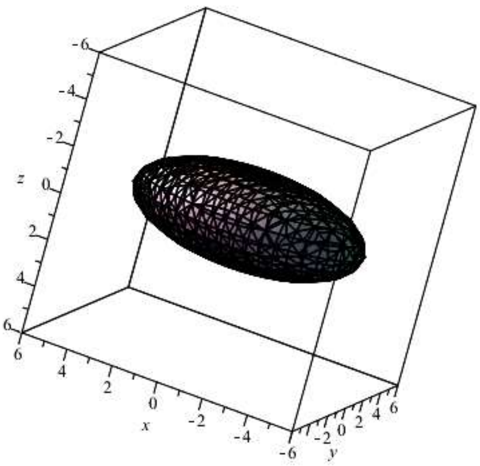
$implicitplot3d(x + 2\cdot y + 3\cdot z + 5 = 0, x = -10..10, y = -10..10, z = -10..10, grid = [23, 23, 23]);$



## 9. Ikkinchi tartibli sirtlar

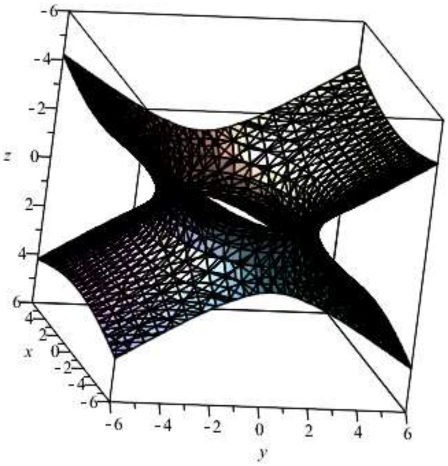
ellipsoid

$$implicitplot3d\left(\frac{x^2}{25} + \frac{y^2}{9} + \frac{z^2}{4} = 1, x = -6..6, y = -6..6, z = -6..6, grid = [23, 23, 23]\right)$$



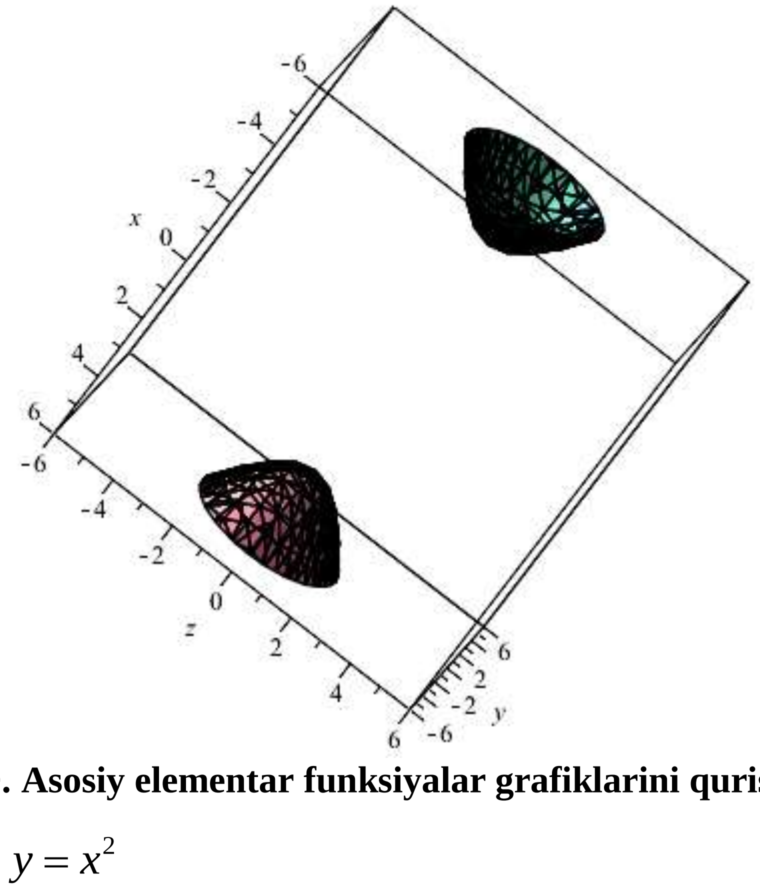
Giperboloid

$$implicitplot3d\left(\frac{x^2}{25}+\frac{y^2}{9}-\frac{z^2}{4}=1, x=-6..6, y=-6..6, z=-6..6, grid=[23, 23, 23]\right)$$




Ikki pallali giperboloid

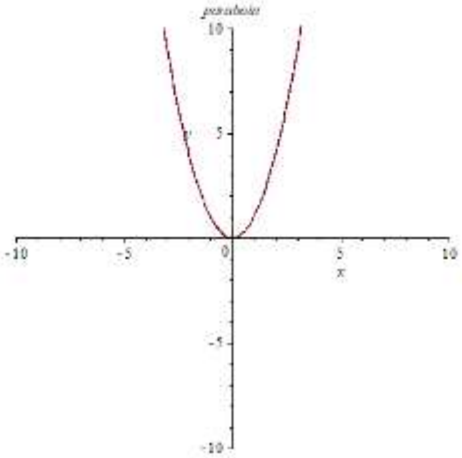
$$implicitplot3d\left(\frac{x^2}{16}-\frac{y^2}{9}-\frac{z^2}{4}=1, x=-6..6, y=-6..6, z=-6..6, grid=[23, 23, 23]\right)$$




**10. Asosiy elementar funksiyalar grafiklarini qurish**

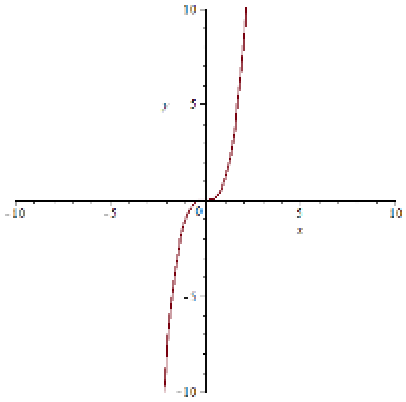
$y=x^2$

$plot((x)\hat{}2, x=-10..10, y=-10..10, title=parabola)$



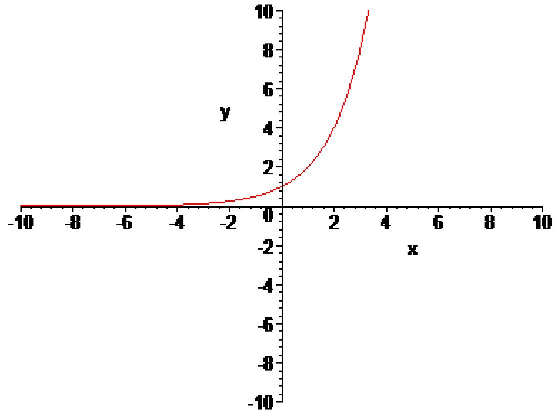

$y = x^3$

$plot([(x)\hat{}3], x = -10..10, y = -10..10)$




$y = a^x, \quad a > 1$

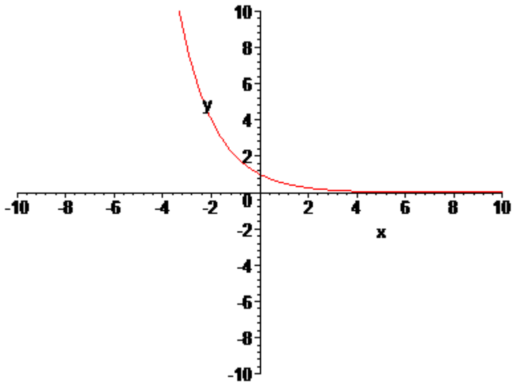
**plot([2^x], x=-10..10,y=-10..10);**




$y = a^x, \quad a < 1$

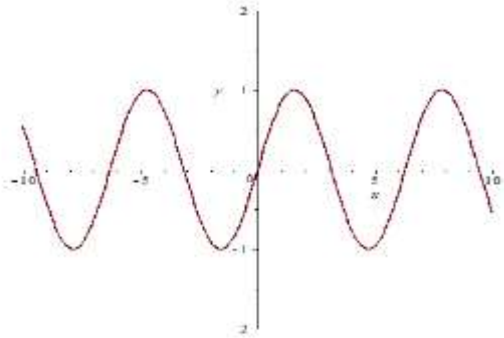
>

> **plot([0.5^x], x=-10..10,y=-10..10);**

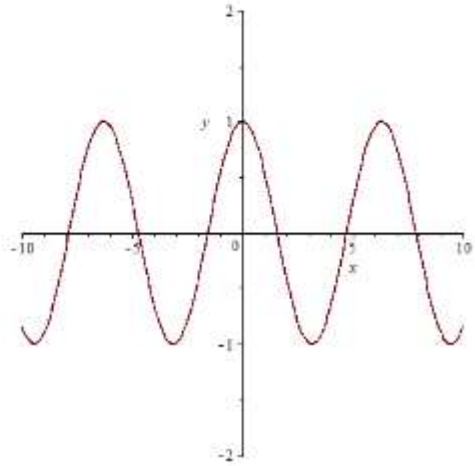



>

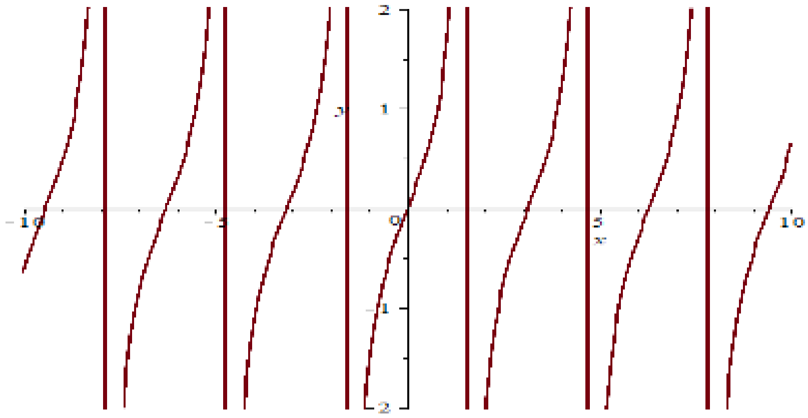
$plot([\sin(x)], x=-10..10, y=-2..2)$



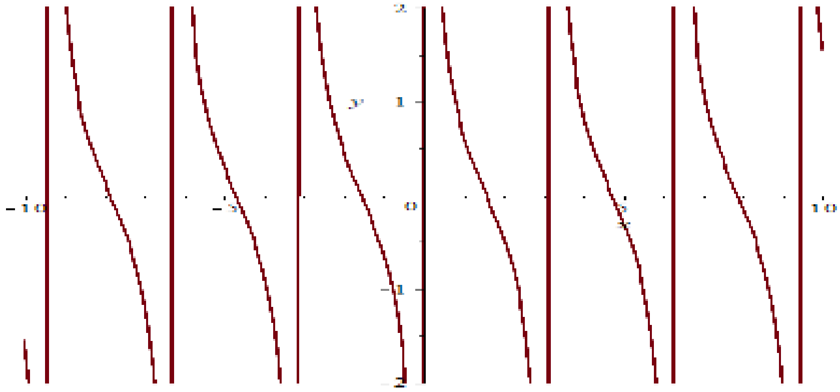
$plot([\cos(x)], x=-10..10, y=-2..2)$



$plot([\tan(x)], x=-10..10, y=-2..2)$



$plot([\cot(x)], x=-10..10, y=-2..2)$



Funksiya hosilasini toppish

$f(x) = arctg\left(\frac{x}{y}\right)$

$f_x^{'}$

> $f := \arctan(x/y)$ :

> $diff(f, x) = simplify(diff(f, x))$;

$$\frac{1}{y\left(1+\frac{x^2}{y^2}\right)} = \frac{y}{x^2+y^2}$$

**11.Limitlarni hisoblash**

$\lim\limits_{x\to 0-0} \frac{1}{1+e^{\frac{1}{x}}}$ ni hisoblash

(chapdan limitni hisoblash)

> $limit\left(\frac{1}{1+\exp\left(\frac{1}{x}\right)}, \{x=0\}, left\right)$

$$1$$

$\lim\limits_{x\to 0+0} \frac{1}{1+e^{\frac{1}{x}}}$ ni hisoblash

1

(o'ngdan limitni hisoblash)

> $limit\left(\frac{1}{1+\exp\left(\frac{1}{x}\right)}, \{x=0\}, right\right)$

$$0$$

$\lim\limits_{x\to 1}(1-x)tg\left(\frac{\pi x}{2}\right)$ ni hisoblash

> $limit\left((1-x)\cdot\tan\left(\frac{\text{pi}\cdot x}{2}\right), \{x=1\}\right)$

$$0$$

## 12. Funksiyani tekshirish va grafigini qurish

$f(x)=\frac{x^3}{x-1}$ funksiyani tekshiraylik

> $f := \frac{x^3}{x-1};$

$$f := \frac{x^3}{x-1}$$

> $fsolve(f);$

$$0.$$

> $discont(f, x);$

$$\{1\}$$

> $singular(f, x);$

$$\{x=1\}, \{x=\infty\}, \{x=-\infty\}$$

> $extrema(f, \{\ \});$

$$\left\{0, \frac{27}{4}\right\}$$

> $f1diff := diff(f, x);$

$$f1diff := \frac{3x^2}{x-1} - \frac{x^3}{(x-1)^2}$$

> $solve(f1diff);$

$$\frac{3}{2}, 0, 0$$

> $f2diff := diff(f1diff, x);$

$$f2diff := \frac{6x}{x-1} - \frac{6x^2}{(x-1)^2} + \frac{2x^3}{(x-1)^3}$$

> $solve(f2diff);$

$$0, \frac{3}{2}+\frac{I\sqrt{3}}{2}, \frac{3}{2}-\frac{I\sqrt{3}}{2}$$

> $limit\left(\frac{x^3}{x-1}, x=1, left\right);$

$$-\infty$$

> $limit\left(\frac{x^3}{x-1}, x=1, right\right);$

$$\infty$$

> $limit\left(\frac{x^3}{x-1}, x=\infty\right);$

$$\infty$$

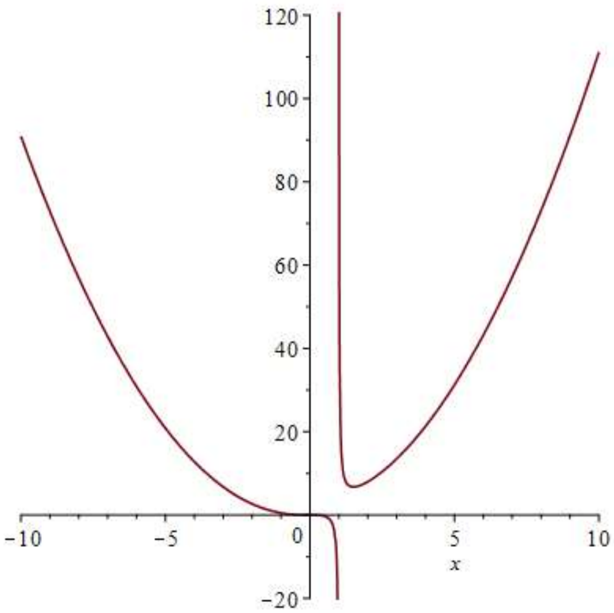
> $limit\left(\frac{x^3}{x-1}, x=-\infty\right);$

$$\infty$$

> $limit\left(\frac{x^3}{x\cdot(x-1)}, x=\infty\right);$

$$\infty$$

> $plot\left(\frac{x^3}{x-1}, x=-10..10, discont=true\right);$




## 13. Yuqori tartibli hosila

$f(x,y)=\frac{x-y}{x+y}$ ; $\frac{\partial^2 f}{\partial x^2}$ ni hisoblash

> $restart; f := (x-y)/(x+y):$

> $Diff(f, x\$2) = simplify(diff(f, x\$2));$

$$\frac{\partial^2}{\partial x^2}\left(\frac{x-y}{x+y}\right) = -\frac{4y}{(x+y)^3}$$

## 14. Ko'p o'zgaruvchili funksiya ekstremumi

$$f(x,y)=\frac{x-y}{x+y}$$

> $extrema(f, \{\ \}, \{x,y\}, 's'); s;$

$$\left\{0, -\frac{9}{8}\right\}$$

$$\left\{\{x=0, y=-1\}, \{x=0, y=0\}, \{x=0, y=1\}, \left\{x=-\frac{1}{2}, y=-1\right\}, \left\{x=-\frac{1}{2}, y=0\right\}, \left\{x=-\frac{1}{2}, y=1\right\}, \left\{x=\frac{1}{2}, y=-1\right\}, \left\{x=\frac{1}{2}, y=0\right\}, \left\{x=\frac{1}{2}, y=1\right\}\right\}$$

> $subs([x=1/2, y=1], f);$

$$-\frac{9}{8}$$

> $subs([x=1/2, y=0], f);$

$$-\frac{1}{8}$$

## 15.Shartli ekstremumni aniqlash

$$f(x;y;z)=-x+2y+3z$$

$$x+2y-3z\le 4 \qquad 5x-6y+7z\le 8 \qquad 9x+10z\le 11$$

> $restart : with(simplex) :$
> $f := -x+2*y+3*z :$
> $cond := \{x+2*y-3*z\le 4, 5*x-6*y+7*z\le 8, \quad 9*x+10*z\le 11\} :$
> $maximize(f, cond, NONNEGATIVE);$

$$\left\{x=0, y=\frac{73}{20}, z=\frac{11}{10}\right\}$$

## 16.Karrali integrallarni hisoblash

$$\int_0^2 dy \int_0^1 x^2 y^3 dx$$

> $int(int(x\text{^}2*y\text{^}3, x=0..1), y=0..2);$

$$\frac{4}{3}$$

$$\int_0^{\frac{\pi}{2}} dy \int_y^{\frac{\pi}{2}-y} \sin(x+2y)\,dx$$

> $restart : with(student) :$
> $J := Doubleint(\sin(x+2*y), x=y..Pi/2-y, y=0..Pi/2);$

$$J := \int_0^{\frac{1}{2}\pi} \int_y^{\frac{1}{2}\pi - y} \sin(x + 2y)\, dx\, dy$$

> $J := value(\%);$

$$J := \frac{2}{3}$$

$$\int_0^2 dz \int_{-1}^1 dx \int_{x^2}^1 (4+z) dy$$

> $J := Tripleint(4 + z, y = x\hat{}2 ..1, x = -1 ..1, z = 0 ..2);$

$$J := \int_0^2 \int_{-1}^1 \int_{x^2}^1 (4 + z)\, dy\, dx\, dz$$

> $J := value(\%);$

$$J := \frac{40}{3}$$

## 17.Qatorlar

$\sum_{n=1}^{N} \frac{1}{(3n-2)(3n+1)}$ summani toping

> $restart : a[n] := 1/((3*n-2)*(3*n + 1));$

$$a_n := \frac{1}{(3n-2)(3n+1)}$$

> $S[N] := Sum(a[n], n = 1 ..N) = sum(a[n], n = 1 ..N);$

$$S_N := \sum_{n=1}^{N} \frac{1}{(3n-2)(3n+1)} = -\frac{1}{9\left(N + \frac{1}{3}\right)} + \frac{1}{3}$$

> $S := limit(rhs(S[N]), N = +\ infinity);$

$$S := \frac{1}{3}$$

>
$Sum((-1)\hat{}(n + 1)*n\hat{}2*x\hat{}n, n = 1 ..infinity) =$
$sum((-1)\hat{}(n + 1)*n\hat{}2*x\hat{}n, n = 1 ..infinity);$

$$\sum_{n=1}^{\infty} (-1)^{n+1} n^2 x^n = -\frac{x(x-1)}{(x+1)^3}$$

> $Sum((1 + x)\hat{}n/((n + 1)*n!), n = 0 ..infinity) =$
$sum((1 + x)\hat{}n/((n + 1)*n!), n = 0 ..infinity);$

$$\sum_{n=0}^{\infty} \frac{(x+1)^n}{(n+1)\, n!} = \frac{e^{x+1} - 1}{x+1}$$

## 18.Funksiyani qatorga yoyish

$e^{-x} \cdot \sqrt{x+1}$ qatorga yoying

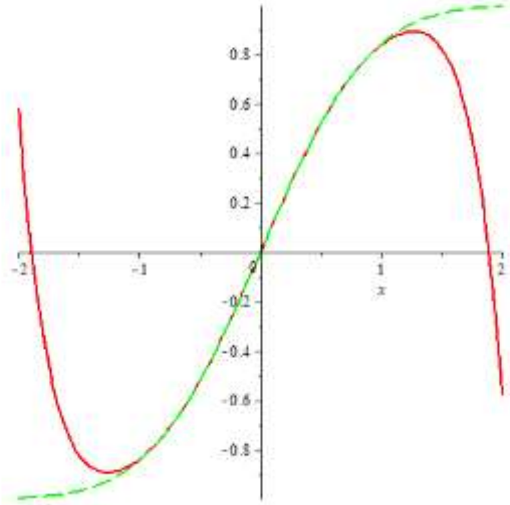
> $f(x) = series(\exp(-x)*\mathrm{sqrt}(x + 1), x = 0, 5);$

$$f(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \frac{13}{48}x^3 - \frac{79}{384}x^4 + O(x^5)$$

> $taylor(\mathrm{erf}(x), x, 8) : p := convert(\%, polynom);$

$$p := \frac{2x}{\sqrt{\pi}} - \frac{2}{3}\frac{x^3}{\sqrt{\pi}} + \frac{1}{5}\frac{x^5}{\sqrt{\pi}} - \frac{1}{21}\frac{x^7}{\sqrt{\pi}}$$

$plot(\{\mathrm{erf}(x), p\}, x=-2..2, thickness=[2, 2], \quad linestyle=[1, 3], color=[red, green])$




> $f := \exp(-x); x1 := -\mathrm{Pi}; x2 := \mathrm{Pi} :$

$$f := e^{-x}$$

$$x1 := -\pi$$

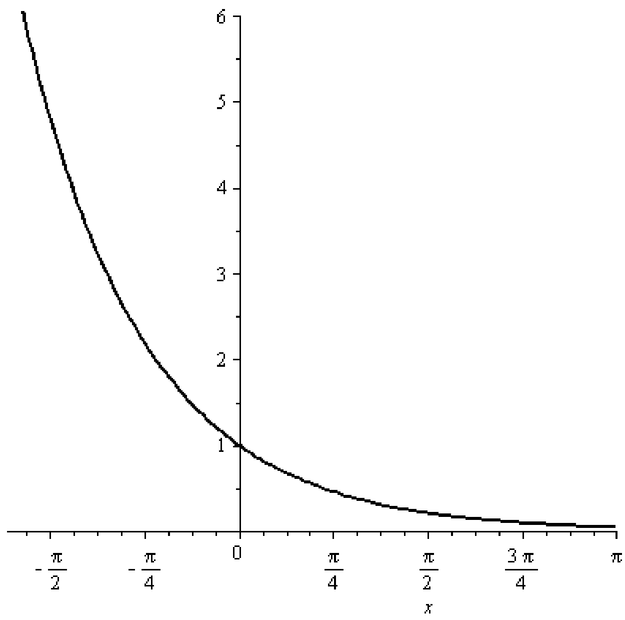
> $fr1 := fourierseries(f, x, x1, x2, 2);$

$$fr1 := fourierseries(e^{-x}, x, -\pi, \pi, 2)$$

> $fr2 := fourierseries(f, x, x1, x2, 4) :$

> $fr3 := fourierseries(f, x, x1, x2, 8) :$

> $plot(\{f, fr1, fr2, fr3\}, x = x1 .. x2, color = [black, blue, green, red], thickness = 2, linestyle = [1, 3, 2, 2]);$




## 19.Differensial tenglamalarni Maple dasturi yordamida hisoblash

$$\left(e^x+1\right)e^y y' + e^x\left(1+e^y\right)=0$$

Ko`rinishdagi differensial tenglamani Maple dasturi yordamida hisoblaylik.

```
> restart;
> de:=(e^x+1)*e^y(x)*diff(y(x),x)+e^x*(1+e^y(x));
```

$$de := (e^x+1)\, e^{y(x)} \left(\frac{\partial}{\partial x} y(x)\right) + e^x\,(1+e^{y(x)})$$

```
>
> dsolve(de,y(x));
```

$$y(x) = -\frac{\ln\left(\dfrac{-e^x-1}{-1+e^{(\_C1+x)}+e^{\_C1}}\right)+\_C1\,\ln(e)}{\ln(e)}$$

Koshi masalasining yechimini topishni ko`rib chiqaylik:

$$\cos x \sin y dy = \cos y \sin x$$

$$y(\pi)=\pi$$

```
> restart;de:=cos(x)*sin(y(x))*diff(y(x),x)=cos(y(x))*sin(x);
> cond:=y(3)=3;dsolve({de,cond},y(x));
>
```

$$de := \cos(x)\sin(y(x))\left(\frac{\partial}{\partial x} y(x)\right) = \cos(y(x))\sin(x)$$

$$cond := y(3)=3$$

$$y(x) = \arccos(\cos(x))$$

> **restart;**

$y''+y=2x \qquad y(0)=0, \quad y\left(\frac{\pi}{2}\right)=0$ tenglama yechimini chizmada ko`rib chiqaylik:

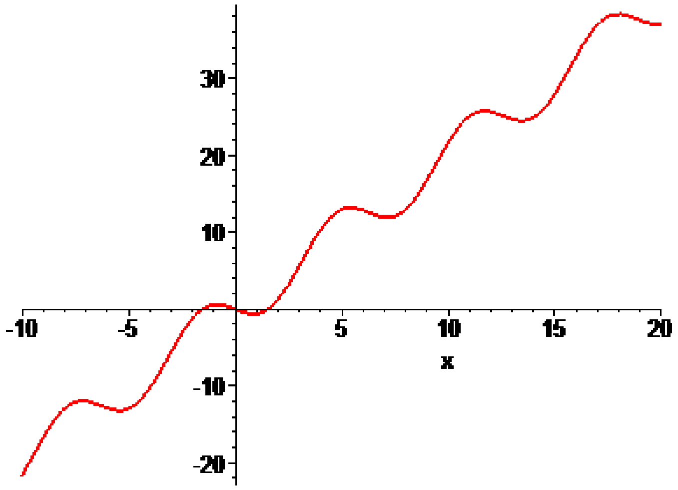
**de:=diff(y(x),x$2)+y(x)=2*x;cond:=y(0)=0,y(1.55)=0;dsolve({de,cond},y(x));**

$$de := \left(\frac{\partial^2}{\partial x^2}\,y(x)\right)+y(x)=2\,x$$

$$cond := y(0)=0,\ y(1.55)=0$$

$$y(x)=-\frac{31}{10}\,\frac{\sin(x)}{\sin\left(\frac{31}{20}\right)}+2\,x$$

> **y1:=rhs(%):plot(y1,x=-10..20,thickness=2);**




Differensial tenglamalar sistemasini ko`raylik.

$$\begin{cases} x' = -x+y+z, \\ y' = x-y+z, \\ z' = x+y-z. \end{cases}$$

> **restart;sys:=diff(x(t),t)=-x(t)+y(t)+z(t),diff(y(t),t)=x(t)-y(t)+z(t);diff(z(t),t)=x(t)+y(t)-z(t);**

$$sys := \frac{\partial}{\partial t}\,x(t)=-x(t)+y(t)+z(t),\ \frac{\partial}{\partial t}\,y(t)=x(t)-y(t)+z(t)$$

$$\frac{\partial}{\partial t}\,z(t)=x(t)+y(t)-z(t)$$

> **sol1:=dsolve([sys],[x(t),y(t),z(t)]);**

$$sol1 := \left\{ z(t) = z(t),\ y(t) = \int \left( \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} z(t) \right) + 2\, z(t) \right) dt + \_C1 \right) e^{(-2t)}\, dt + \_C2,\ x(t) = e^{(-2t)} \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} z(t) \right) + 2\, z(t) \right) dt + \_C1\, e^{(-2t)} + \int \left( \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} z(t) \right) + 2\, z(t) \right) dt + \_C1 \right) e^{(-2t)}\, dt + \_C2 - z(t) \right\}$$

> **sol2:=dsolve([sys],[y(t),x(t),z(t)]);**

$$sol2 := \left\{ z(t) = z(t),\ x(t) = \int \left( \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} z(t) \right) + 2\, z(t) \right) dt + \_C1 \right) e^{(-2t)}\, dt + \_C2,\ y(t) = e^{(-2t)} \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} z(t) \right) + 2\, z(t) \right) dt + \_C1\, e^{(-2t)} + \int \left( \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} z(t) \right) + 2\, z(t) \right) dt + \_C1 \right) e^{(-2t)}\, dt + \_C2 - z(t) \right\}$$

> **sol3:=dsolve([sys],[z(t),x(t),y(t)]);**

$$sol3 := \left\{ y(t) = y(t),\ z(t) = -e^{(-2t)} \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} y(t) \right) + 2\, y(t) \right) dt + \left( \frac{\partial}{\partial t} y(t) \right) - \_C1\, e^{(-2t)} + y(t),\ x(t) = \left( \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} y(t) \right) + 2\, y(t) \right) dt + \_C1 \right) e^{(-2t)} \right\}$$

> > **simplify(rhs(sol1[1]));simplify(rhs(sol2[1]));simplify(rhs(sol3[1]));**

$$z(t)$$

$$z(t)$$

$$\left( \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} y(t) \right) + 2\, y(t) \right) dt + \_C1 \right) e^{(-2t)}$$

> **simplify(rhs(sol1[2]));simplify(rhs(sol2[2]));simplify(rhs(sol3[2]));**

$$e^{(-2t)} \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} z(t) \right) + 2\, z(t) \right) dt + \_C1\, e^{(-2t)}$$

$$+ \int \left( \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} z(t) \right) + 2\, z(t) \right) dt + \_C1 \right) e^{(-2t)}\, dt + \_C2 - z(t)$$

$$e^{(-2t)} \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} z(t) \right) + 2\, z(t) \right) dt + \_C1\, e^{(-2t)}$$

$$+ \int \left( \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} z(t) \right) + 2\, z(t) \right) dt + \_C1 \right) e^{(-2t)}\, dt + \_C2 - z(t)$$

$$y(t)$$

> **simplify(rhs(sol1[3]));simplify(rhs(sol2[3]));simplify(rhs(sol3[3]));**

$$\int \left( \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} z(t) \right) + 2\, z(t) \right) dt + \_C1 \right) e^{(-2t)}\, dt + \_C2$$

$$\int \left( \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} z(t) \right) + 2\, z(t) \right) dt + \_C1 \right) e^{(-2t)}\, dt + \_C2$$

$$-e^{(-2t)} \int e^{(2t)} \left( \left( \frac{\partial}{\partial t} y(t) \right) + 2\, y(t) \right) dt + \left( \frac{\partial}{\partial t} y(t) \right) - \_C1\, e^{(-2t)} + y(t)$$

Differensial tenglamalarning boshlang`ich shartlarni qanoatlantiruvchi yechimini topish.

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{z-1}{z}, \\ \dfrac{dz}{dx} = \dfrac{1}{y-x}, \end{cases} \quad y(0) = -1,\ z(0) = 1.$$

> **restart;**

> **sys:=diff(y(x),x)=(z(x)-1)/z(x),diff(z(x),x)=1/(y(x)-x),y(0)=-1,z(0)=1;**

$$sys := \frac{\partial}{\partial x} y(x) = \frac{z(x)-1}{z(x)},\ \frac{\partial}{\partial x} z(x) = \frac{1}{y(x)-x},\ y(0) = -1,\ z(0) = 1$$

> **sol:=dsolve({sys},{y(x),z(x)});**

$$sol := \{ z(x) = e^{(-x)},\ y(x) = -\frac{1}{e^{(-x)}} + x \}$$

Natijani tekshirib ko`raylik.

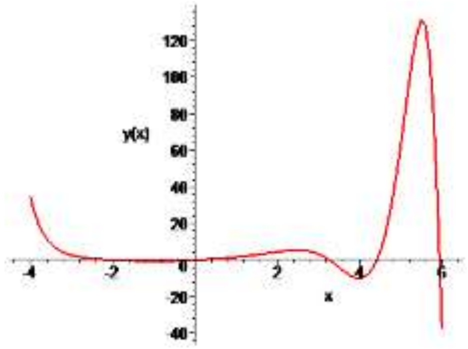
> **odetest(sol,{sys});**

$$\{0, y(0)+1, z(0)-1\}$$

```
> restart;with(DEtools):
> de3:=diff(y(x),x$3)-x*abs(diff(y(x),x))+x^2*y(x)=0;
```

$$de3 := \left(\frac{\partial^3}{\partial x^3} y(x)\right) - x\left|\frac{\partial}{\partial x} y(x)\right| + x^2\, y(x) = 0$$

```
> DEplot(de3,{y(x)},x=-
4..6,[[y(0)=0,D(y)(0)=1,(D@@2)(y)(0)=1]],stepsize=0.1,linecolor=red,thickness
=2);
```



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## MUNDARIJA

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