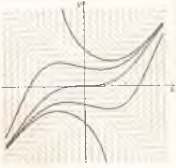
N. DILMURODOV

DIFFERENSIAL TENGLAMALAR KURSI

I
jild



$y' = x^2 - y^2$

Mazkur kitob differensial tenglamalar kursini o`rganish uchun yozilgan qo`llanmaning birinchi jildididir.

Unda differensial tenglamalarga olib keluvchi masalalar hamda birinchi tartibli va yuqori tartibli differensial tenglamalar nazariysi keltirilgan.

Bundan tashqari, bilimlarni chuqrlashtirish va mustahkamlash uchun masalalar berilgan. Bu masalalarning ko`pini yechish uchun ko`rsatmalar va/yoki javoblar ham berilgan.

O'quv-uslubiy qo`llanma "matematika" va "amaliy matematika va informatika" yo`nalishlari bo`yicha tahsil oluvchi bakalavriat talabalari uchun differensial tenglamalar kursi dasturini to`la qamrab olgan. Kitobdan differensial tenglamalarni mustaqil o`rganmoqchi bo`lgan barcha xohlovchilar unumli foydalanishlari mumkin.

Ushbu o'quv-uslubiy qo'llanma O'zbekiston Respublikasi
Matbuot va axborot agentligi, Oliy va o`rta maxsus ta'lim
vazirligi hamda Qarshi davlat universiteti tomonidan
tuzilgan uch yoqlama shartnoma rejasiga asosan
nashrga tavsiya etilgan.

Taqrizchilar: Qarshi davlat universiteti umumiy matematika kafedrasi dotsenti **J.Toshev**
Qarshi muhandislik iqtisodiyot instituti oliy matematika kafedrasi dotsenti **T.Meyliyev**

Mundarija

So`z boshi

Mazkur ikki jildli kitob "matematika" va "amaliy matematika va informatika" yo`nalishlari bo`yicha tahsil oluvchi bakalavriat talabalari uchun differensial tenglamalar kursi dasturining barcha mavzularini o`z ichiga qamrab olgan bo`lib, u shu kursni o`rganish uchun o`quv qo`llanma sifatida yozilgan.

Kitobning birinchi jildida differensial tenglamalarga olib keluvchi masalalar, birinchi va yuqori tartibli differensial tenglamalar nazariyasi batafsil va to`la bayon etilgan. Bu yerda noan'anaviy material – ekstremum prinsiplari ham keltirilgan.

Nazariyani tushuntiruvchi ko`pdan-ko`p misollar to`la yechimlari bilan birgalikda keltirilgan. Bundan tashqari, har bir paragraf oxirida mustaqil yechish uchun masalalar taklif etilgan. Bu masalalarning yechimlari va javoblari ham berilgan.

Qo`shimcha misol va masalalrni muallifning "Differensial tenglamalardan mustaqil ishlar" (Qarshi, 2010) kitobidan topish mumkin. Bundan tashqari, kitob oxirida keltirilgan adabiyotlardan ham foydlanish maqsadga muvofiq bo`ladi.

Muallif qo`llanmadagi o`quv materiallarini aprobatsiyadan o`tkazishda yordam bergan barcha shogirdlari va talabalaridan hamda kasbdoshlaridan, bundan tashqari, taqrizchilardan ham, minnatdor ekanligini mamnunlik bilan e'tirof etadi.

Kitob haqidagi fikr va mulohazalaringizni nosir_d@mail.ru elektron manzilga yozsangiz, muallif sizdan minnatdor bo`ladi.

Asosiy belgilashlar ro`yxati

$\forall$ — har qanday, ixtiyoriy, har bir (umumiylik kvantori).

$\exists$ — mavjud, kamida bitta mavjud (mavjudlik kvantori).

$\Rightarrow$ — kelib chiqadi (implikatsiya belgisi).

$\Leftrightarrow$ — teng kuchli (ekvivalent).

$\overset{def}{\Leftrightarrow}$ — ta'rifga ko`ra ekvivalent (teng kuchli).

$\overset{def}{=}$ — ta`rifga ko`ra teng.

$\{x \in E \mid P(x)\}$ — E to`plamning $P(x)$ xossaga ega bo`lgan barcha x elementlari to`plami.

$\mathbb{N}$ — natural sonlar to`plami.

$\mathbb{R}$ — haqiqiy sonlar to`plami.

$\mathbb{C}$ — kompleks sonlar to`plami.

$\mathbb{R}^n$ — n o`lchamli haqiqiy Evklid fazosi.

$c, c_1, c_2, \ldots$ — ixtiyoriy o`zgarmaslar (doimiylar).

const — o`zgzrmas (doimiy).

$(a,b) \overset{def}{=} \{x \in \mathbb{R} \mid a < x < b\}$ $(a < b)$ — interval.

$[a,b] \overset{def}{=} \{x \in \mathbb{R} \mid a \le x \le b\}$ $(a < b)$ — segment.

$(a,b] \overset{def}{=} \{x \in \mathbb{R} \mid a < x \le b\}$ $(a < b)$ — yarim segment.

$[a,b) \overset{def}{=} \{x \in \mathbb{R} \mid a \le x < b\}$ $(a < b)$ — yarim segment.

$\mathbb{R}_+ \overset{def}{=} [0, +\infty)$.

I — sonli oraliq (ichi bo`sh bo`lmagan bog`lanishli sonli to`plam).

D — soha, ya`ni ochiq va bog`lanishli to`plam.

$\max E$ — E sonli to`plamning maksimumi (eng katta elementi).

$\min E$ — E sonli to`plamning minimumi (eng kichik elementi).

$\sup E$ — E sonli to`plamning supremumi (yuqori chegaralarning eng kichigi, aniq yuqori chegara).

$\inf E$ — E sonli to`plamning infimumi (quyi chegaralarning eng kattasi, aniq quyi chegara).

$\|\ \|$ — norma (yoki matritsa) belgisi.

∂E — E to`plamning chegarasi.

E^c — E to`plamning (qaralayotgan fazogacha) to`ldiruvchisi.

$B_\delta(a)$ — δ radiusli a markazli (ochiq) shar.

$B_\delta = B_\delta(o)$

$X \times Y$ — to`plamlarning to`g`ri (Dekart) ko`paytmasi.

$\cup$, $\cap$, $\setminus$ — mos ravishda to`plamlar birlashmasi, kesishmasi, ayirmasi.

$f: X \to Y$ — X to`plamda aniqlangan, qiymatlari $Y$ to`plamda joylashgan f funksiya (akslantirish).

$D(f)$ — f funksiyaning aniqlanish to`plami (sohasi).

$f|_E$ — f funksiyaning E to`plamga torayishi.

$f_a = f(a)$

$g \circ f$ — f va g funksiyalar kompozitsiyasi (ketma-ket bajarilishi).

$f(x) = o(g(x))$, $x \to a$ — asimptotik tenglik (kichik o): u $f(x) = \varepsilon(x) g(x)$, $\lim_{x \to a} \varepsilon(x) = 0$ ekanligini anglatadi.

$f(x) = O(g(x))$, $x \to a$, — (katta o): u $f(x)$ funksiya $g(x)$ ni a nuqtaning biror atrofida chegaralangan $h(x)$ funksiyaga ko`paytirishdan hosil bo`lishini ($f(x) = h(x) g(x)$) anglatadi.

$C(X;Y)$ — barcha uzluksiz $f: X \to Y$ funksiyalar to`plami.

$C(X) = C(X, \mathbb{R})$.

$C^k(X;Y)$ — barcha k- tartibli hosilalari (demak, undan past tartiblilari ham) uzluksiz bo`lgan $f: X \to Y$ funksiyalar sinfi.

$\mathrm{dist}(X, Y)$ — to`plamlar orasidagi masofa (distance – masofa).

$\dim X$ — X fazoning o`lchami (dimension – o`lcham).

$\deg P$ — P ko`phadning darajasi (degree – daraja).

$\mathrm{M}_{n \times n}(\mathbb{R})$ — haqiqiy sonlardan tuzilgan $n \times n$ o`lchamli matritsalar to`plami.

$\mathrm{M}_{n \times n}(\mathbb{C})$ — kompleks sonlardan tuzilgan $n \times n$ o`lchamli matritsalar to`plami.

$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}, \boldsymbol{h}, \boldsymbol{f}, \boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{q}, \ldots$ (qalin harflar) — vektorlar.

MYaT — mavjudlik va jagonalik teoremasi.

DT — differensial tenglama.

ODT — oddiy differensial tenglama.

🗝 — masala (misol) yechilishining, teorema (jumla) isbotining boshlanishi belgisi.

🔑 — masala (misol) yechilishining, teorema (jumla) isbotining tugallanganligi belgisi.

0. KIRISH

0.1. Differensial tenglamalarga olib keluvchi masalalar

Fizikada, biologiyada, ekologiyada, iqtisodiyotda, amaliyotda va shunga o'xshash sohalarda uchraydigan ba'zi hodisa va jarayonlarni o'rganish uchun matematik model tuzilganda differensial tenglama(lar) deb ataluvchi tenglama(lar) hosil bo'lib, ularning yechimlarini topish kerak bo'ladi.

Differensial tenglama(lar) ning aniq va qat'iy ta'rifini keyinchalik keltiramiz. Hozircha to'la va qat'iy bo'lmagan quyidagi ta'rifni qabul qilamiz: «Noma'lum funksiyaning hosilalari (yoki hosilasi) qatnashgan tenglama differensial tenglama deyiladi». Differensial tenglamalar sistemasida ikki yoki undan ortiq noma'lum funksiya qatnashadi. Differensial tenglamalar ikki turga bo'linadi: oddiy differensial tenglamalar va xususiy hosilali differensial tenglamalar. Oddiy differensial tenglamada noma'lum funksiya bir dona erkli o'zgaruvchiga bog'liq, xususiy hosilali tenglamada esa noma'lum funksiya ikki yoki undan ortiq argumentlarga bog'liq bo'ladi.

Masalan, ushbu

$y'' - \ln(2-x)\cdot y^2 + 1 + \sin x = 0$ ($y = y(x)$ – bir o'zgaruvchining noma'lum funksiyasi),

$x'' + x'^2 - 2\sin t = 0$ ($x = x(t)$ – bir o'zgaruvchining noma'lum funksiyasi),

$y''' + y''^4 + xy^2y' - 2e^x y - \operatorname{tg} x = 0$ ($y = y(x)$ – bir o'zgaruvchining noma'lum funksiyasi), tenglamalar oddiy differensial tenglamalar,

$\begin{cases} x'' - x' + ty - 1 = 0 \\ y'' - e^t x'y' - t^2 x = 0 \end{cases}$ ($x = x(t)$ va $y = y(t)$ – bir o'zgaruvchining noma'lum funksiyalari) sistema oddiy differensial tenglamalar sistemasi,

$u'_x + uu'_y = 0$ ($u = u(x, y)$ – ikki o'zgaruvchining noma'lum funksiyasi),

$u''_{xx}+u''_{yy}+u''_{zz}=0$ $(u=u(x,y,z)$ – uch o'zgaruvchining noma'lum funksiyasi), tenglamalar esa xususiy hosilali differensial tenglamalardir.

Oddiy differensial tenglamalarni yechishga keltiriladigan ba'zi bir masalalar bilan tanishaylik.

*1. **Boshlang'ich funksiyani topish.*** Biror (a,b) intervalda $y=y(x)$ noma'lum funksiyani uning ma`lum $f(x)$ hosilasiga ko'ra topish ushbu

$$y'=f(x) \tag{0.1.1}$$

tenglamani yechish demakdir. Matematik analiz kursida bu tenglama $f(x)$ uzluksiz bo'lgan holda yechilgan. Bu holda yechim

$$y=\int f(x)dx+c\,,\ c-\text{const},$$

formula bilan beriladi; bu yerda va bundan keyin $\int f(x)dx$ bilan $f(x)$ funksiyaning biror tayin boshlang'ich funksiyasini belgilaymiz (analiz kursida u barcha boshlang`ich funksiyalarni anglatgan).

Endi matematik modellari differensial tenglamalarni yechishga keltiriladigan ba'zi masalalar bilan tanishaylik.

*2. **Radioaktiv yemirilish.*** $m(t)$ bilan radioaktiv moddaning t paytdagi massasini belgilaylik. Masala ana shu funksiyani topishdan iborat. Fizikadan ma'lumki, radioaktiv moddaning yemirilish tezligi $-\dfrac{dm(t)}{dt}$ ($\dfrac{dm(t)}{dt}$ hosila o'sish tezligini ifodalaydi) mavjud modda miqdoriga to'g'ri proporsional, ya'ni

$$-\frac{dm(t)}{dt}=km(t) \quad \text{yoki qisqaroq} \quad m'=-km\,; \tag{0.1.2}$$

bu yerda o`zgarmas $k, k>0$, – proporsionallik koeffitsienti. Demak, $m=m(t)$ noma`lum funksiya (0.1.2) tenglamani qanoatlantiradi. Bu tenglamada noma'lum funksiyaning m' hosilasi qatnashgan. Biz (0.1.2) tenglamadan $m(t)$ funksiyani topishimiz kerak. (0.1.2) tenglamani yechish qiyin emas. Uning har ikkala tomonini e^{kt} ga ko'paytiraylik:

$$m'e^{kt} + mke^{kt} = 0$$

Oxirgi tenglikni

$$(me^{kt})' = 0$$

ko'rinishda yozamiz. Ma'lumki, oraliqda hosilasi nolga teng bo'lgan funksiya o'zgarmas. Shuning uchun oxirgi tenglikdan $me^{kt} = c$ $(c = \text{const})$, ya'ni $m = ce^{-kt}$ ekanligini hosil qilamiz. Ravshanki, $m = ce^{-kt}$ funksiya (0.1.2) tenglmani qanoatlantiradi, ya'ni (0.1.2) da m ning o'rniga ce^{-kt} ni qo'ysak, u ayniyatga aylanadi. Demak, (0.1.2) tenglamaning hamma yechimlari $m = ce^{-kt}$ ko'rinishda va faqat shu ko'rinishda bo'ladi. Agar boshlang'ich, ya'ni $t = 0$ paytdagi massa m_0 bo'lsa, $c = m_0$ bo'ladi. Shunday qilib, radioaktiv moddaning massasi ushbu

$$m = m_0 e^{-kt} \tag{0.1.3}$$

qonuniyatga ko'ra o'zgaradi. Massa miqdori vaqt o'tishi blan eksponensial tezlik bilan 0 ga intiladi

Yarim yemirilish davri T deb dastlabki radioaktiv moddaning yarmi yemirilishi uchun ketgan vaqt oralig'iga aytiladi. Demak,

$$\frac{m_0}{2} = m_0 e^{-kT}, \text{ ya'ni } T = \frac{\ln 2}{k} \text{ yoki } k = \frac{\ln 2}{T}.$$

Oxirgi formuladan T ga ko'ra (uni o'lchash nisbatan oson) k ni topish uchun foydalanish mumkin.

Yuqoridagi (0.1.3) formulaning yana bir tatbig'ini e'tirof etaylik. Ma'lumki, tirik organizmlarda C^{12} turg'un uglerod bilan birgalikda oz miqdorda C^{14} radioaktiv izotop ham bo'ladi. Atmosferaning yuqori qatlamlarida γ-nurlar hosil qiluvchi C^{14} izotoplar tirik organizmlarda yutiladi. Tirik organizmlarda biologik o'zgarish jarayonlari natijasida C^{14} miqdori o'zgarmas va biror m_0 ga teng bo'ladi. Organizm o'lishi bilanoq unda radioaktiv izotopning yutilishi to'xtaydi va C^{14} ning miqdori kamaya boshlaydi. C^{14} ning yarim yemirilish davri $T \approx 5570$ (yil). Bundan, $k = \frac{\ln 2}{T} \approx \frac{0,6931}{5570} \approx 1,24 \cdot 10^{-4} \approx \frac{1}{8000}$ (1/yil). Demak, agar $m(t)$ bilan C^{14} izotopining organizm o'lgan paytdan boshlab hisoblangan t - yildagi massasini belgilasak, u holda

$$m'(t) = -\frac{1}{8000} m(t) \text{, ya'ni } m(t) = m_0 e^{-t/8000}$$

bo'ladi. Bundan, agar $m(t)$ aniqlangan (uni C^{14} chiqaradigan β-zarrachalar soni orqali topish mumkin) bo'lsa, u holda organizmning o'lganidan keyin o'tgan t vaqtni yillarda ushbu

$$t = 8000 \ln \frac{m_0}{m(t)}$$

formula orqali topish mumkinligi kelib chiqadi. Hosil bo'lgan bu formula o'lgan organizmlarning yoshini aniqlashga imkon beradi.

*3. **Populyatsiya masalasi.*** t vaqtdagi populyatsiya sonini $N(t)$ bilan belgilaylik. Populyatsiya soni katta bo'lganda $N = N(t)$ funksiyani differensiallanuvchi deb hisoblash mumkin. Maltus modelida populyatsiyaning $\frac{dN}{dt}$ o'sish tezligi mavjud populyatsiya soni N ga proporsional deb faraz qilinadi, ya'ni

$$\frac{dN}{dt} = kN \quad (k > 0 - \text{o'zgarmas son}). \qquad (0.1.4)$$

Bu tenglamadan, yuqoridagiga o'xshash fikr yuritib, topamiz:

$$N = N_0 e^{kt}$$

Demak, Maltus modeliga ko'ra vaqt o'tishi bilan populyatsiya soni eksponensial tezlik bilan o'sadi va $+\infty$ ga intiladi. Bu natija haqiqatga mos kelmaydi. Tushunarliki, N ortgan sari oziq-ovqat, joy va shunga o'xshash yashash uchun zarur bo'lgan manbalarning (resurslarning) chegaralanganligi tufayli populyatsiya orasida yashash uchun kurash paydo bo'ladi, ba'zilar bu raqobatda halok bo'ladi, o'sishning nisbiy tezligi $\frac{1}{N}\frac{dN}{dt}$ kamayadi. Shuning uchun (0.1.4) matematik modelni tuzatish kerak. Faraz qilish mumkinki, biror chegaraviy populyatsiya soni N_{cheg} mavjud bo'lib, populyatsiya N_{cheg} dan ko'p bo'lganda $N(t)$ kamayadi (ya'ni $N' < 0$), N_{cheg} dan kichik bo'lganda esa $N(t)$ o'sadi (ya'ni $N' > 0$). Bu farazni quyidagi tenglama qanoatlantiradi:

$$\frac{dN}{dt} = kN(N_{cheg} - N) \quad (k > 0 - \text{o'zgarmas son}). \qquad (0.1.5)$$

Bu tenglama (0.1.4) tenglamaning tuzatilishidir. Oxirgi differensial tenglama logistik tenglama deb yuritiladi.

4. Bir ekologik misol (Volterra-Lotka modeli). Faraz qilaylik, yopiq sistemada (muhitda) o'lja va yirtqichlar (ikki tur individuumlari) yashasin. $N_1 = N_1(t)$ va $N_2 = N_2(t)$ mos ravishda t paytdagi o'lja va yirtqichlar sonini belgilasin. Agar yirtqichlar bo'lmasa, o'ljalarning $N_1'(t)$ o'sish tezligi ularning soniga proporsional, ya'ni $aN_1(t)$ ga ($a > 0$) teng va eksponensial tezlik bilan o'sadi; $N_2(t)$ sondagi yirtqichlar bu o'sish tezligini $bN_2(t)N_1(t)$ ga, ya'ni uchrashishlar soniga proporsional miqdorga ($b > 0$) kamaytiradi. Demak, $N_1' = (a - bN_2)N_1$ tenglik o'rinli bo'ladi. Agar o'ljalar bo'lmasa, yirtqichlar o'zgarishining tezligi $-cN_2(t)$ bo'ladi ($c > 0$, ularning soni eksponensial kamayadi); $N_1(t)$ sondagi o'ljalarning mavjudligi natijasida bu tezlik $dN_1(t)N_2(t)$ ga ortadi ($d > 0$), ya'ni $N_2' = (-c + dN_1)N_2$ bo'ladi. Shunday qilib, bizning farazlarimizda o'lja va yirtqichlar (populyatsiyasi) soni quyidagi differensial tenglamalar sistemasi bilan boshqariladi:

$$\begin{cases} N_1' = (a - bN_2)N_1 \\ N_2' = (-c + dN_1)N_2 \end{cases} \qquad (0.1.6)$$

Bu yerda t ning o'rniga t/a, N_1 ning o'rniga $x = N_1 c/d$, N_2 ning o'rniga $y = N_2/b$ masshtablangan o'zgaruvchilarni kiritib, sistemaning ko'rinishini ixchamlash mumkin:

$$\begin{cases} x' = (1 - y)x \\ y' = \alpha(x - 1)y \end{cases} \quad (\alpha = c/a > 0) \qquad (0.1.7)$$

Bu (0.1.6) va (0.1.7) sistemalar Volterra-Lotka tenglamalari (o'lja-yirtqich modeli) deb yuritiladi.

5. ***Garmonik ossillyator tenglamasi.*** Faraz qilaylik, m massali moddiy nuqta inersial sanoq sistemasining Ox o'qi bo'ylab harakat qilayotgan bo'lsin. Unga $x = x(t)$ (t – vaqt) nuqtada bo'lganda unga x ga proporsional va nuqtani koordinatalar boshiga qaytaruvchi kuch $F = -kx$ (elastiklik kuchi; $k = \mathrm{const} > 0$ – proporsionallik koeffitsienti) ta'sir etsin. Nyutonning ikkinchi qonuniga ko'ra

$$ma = F \ \ (a = a(t) \text{ – tezlanish; } a = x'' = \frac{d^2x}{dt^2}).$$

Demak,

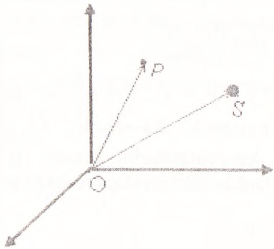
$$mx'' = -kx \text{ yoki } x'' + \omega^2 x = 0 \ \ (\omega = \sqrt{k/m} \ ) \qquad (0.1.8)$$

Shunday qilib. harakatdagi nuqtaning koordinatasini aniqlovchi $x=x(t)$ funksiya garmonik ossillyator tenglamasi deb ataluvchi (0.1.8) tenglamani qanoatlantiradi. Bu tenglamada $x = x(t)$ noma'lum funksiyaning ikkinchi tartibli hosilasi x'' qatnashgan.

Bu yerda shuni e'tirof etaylikki, (0.1.8) differensial tenglama bilan nafaqat elastik prujinaga osilgan moddiy nuqta (jism) harakati, balki matematik mayatnikning kichik tebranishlari, $L - C$ yopiq tebranish konturidagi elektromagnit to'lqinlar o'zgarishi ham boshqariladi.

6. ***Yerning Quyosh ta'siridagi harakati.*** Quyoshning massasini M bilan, Yernikini esa m bilan belgilaymiz va ularni moddiy nuqtalar deb hisoblaymiz. Fazoda inersial sanoq sistemasini

(koordinatalar sistemasini) kiritib, unga nisbatan Quyoshning koordinatalarini $S(s_1,s_2,s_3)$ ($s_i = s_i(t)$, t vaqt, i=1,2,3), Yernikini esa $P(p_1,p_2,p_3)$ ($p_i=p_i(t)$, t–vaqt, i=1,2,3) bilan belgilaymiz (0.1-rasm).



0.1-расм

Butun olam tortishish qonuniga ko'ra Yerga Quyoshning

$$\vec{F} = G\frac{Mm}{r^2}\cdot\frac{\overrightarrow{PS}}{r} \qquad (r = \left|\overrightarrow{PS}\right|,\ G - \text{gravitatsion doimiy})$$

tortish kuchi ta'sir etadi. Yerning harakat tenglamalari Nyutonning ikkinchi qonuniga ko'ra $m\vec{a} = \vec{F}$ ($\vec{a} = (p_1'', p_2'', p_3'')$ – Yerning tezlanish vektori) yoki koordinatalarda

$$\begin{cases} p_1'' = GM\dfrac{s_1 - p_1}{r^3} \\ p_2'' = GM\dfrac{s_2 - p_2}{r^3} \\ p_3'' = GM\dfrac{s_3 - p_3}{r^3} \end{cases} \qquad (0.1.9)$$

ko'rinishga ega. Quyoshning harakat tenglamalari shunga o'xshash yoziladi:

$$
\begin{cases}
s_1'' = Gm\dfrac{p_1 - s_1}{r^3} \\
s_2'' = Gm\dfrac{p_2 - s_2}{r^3} \\
s_3'' = Gm\dfrac{p_3 - s_3}{r^3}
\end{cases}
\qquad (0.1.10)
$$

Endi $x = p_1 - s_1$, $y = p_2 - s_2$, $z = p_3 - s_3$ deb, yangi x, y, z noma'lumlarni kiritaylik. Ravshanki, (x, y, z) – Yerning Quyoshga nisbatan koordinatalarini beradi. (0.1.9) va (0.1.10) sistemalarni hadma-had ayirib quyidagi tengliklarni hosil qilamiz:

$$
\begin{cases}
x'' = -\mu\dfrac{x}{r^3} \\
y'' = -\mu\dfrac{y}{r^3} \qquad (\mu = G(M + m)) \\
z'' = -\mu\dfrac{z}{r^3}
\end{cases}
\qquad (0.1.11)
$$

Demak, Yerning Quyoshga nisbatan holatini aniqlovchi $x = x(t)$, $y = y(t)$, $z = z(t)$ funksiyalar (0.1.11) differensial tenglamalar sistemasini qanoatlantiradi.

7. Egri chiziqlar oilasi differensial tenglama yechimi sifatida. Faraz qilaylik, bir parametrli silliq chiziqlar oilasi ushbu

$$\varphi(x, y, c) = 0$$

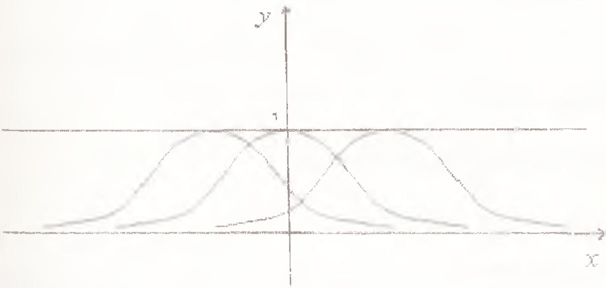
tenglama bilan berilgan bo'lsin. Bu yerda $\varphi(x, y, c)$ – silliq funksiya; x, y – haqiqiy o'zgaruvchilar, c haqiqiy parametr chiziqlar oilasining a'zolarini belgilaydi; c ning ixtiyoriy joiz qiymatida berilgan tenglama $y = y(x)$ silliq chiziqni aniqlaydi deb faraz qilamiz: $\varphi(x, y(x), c) \equiv 0$. Bu ayniyatni x bo'yicha differensiallaymiz: $\varphi_x'(x, y(x), c) + \varphi_y'(x, y(x), c) \cdot y'(x) \equiv 0$. Endi

$$\begin{cases} \varphi(x, y(x), c) \equiv 0 \\ \varphi'_x(x, y(x), c) + \varphi'_y(x, y(x), c) \cdot y'(x) \equiv 0 \end{cases}$$

sistemadan c ni yo'qotib, $x, y(x)$ va $y'(x)$ miqdorlar orasida $F(x, y(x), y'(x)) \equiv 0$ bog'lanishni topamiz. Demak, berilgan chiziqlar oilasining har qanday $y = y(x)$ a'zosi $F(x, y(x), y'(x)) = 0$ differensial tenglamanı qanoatlantiradi, ya'ni bu tenglamaning yechimi bo'ladı.

Misol. Ushbu

$$y - \frac{1}{(x-c)^2 + 1} = 0 \qquad (0.1.12)$$



0.2-rasm.

chiziqlar oilasi (0.2-rasm.) qanoatlantiruvchi differensial tenglamani tuzaylik. Hosilani hisoblaymiz:

$$y' + \frac{2(x-c)}{((x-c)^2+1)^2} = 0 .$$

Berilgan tenglamaga ko'ra bu tenglikdan

$$y' + 2y^2(x-c) = 0 \text{ yoki } x - c = -\frac{y'}{2y^2}$$

ekanligini topamiz. Oxirgi tenglikni (0.1.12)ga qo'yamiz, keraklı soddalashtirishlarni bajaramiz va izlangan differensial tenglamanı hosil qilamiz:

$$y'^2+4y^3(y-1)=0. \qquad (0.1.13)$$

Shunday qilib, (0.1.12) chiziqlar oilasi (0.1.13) differensial tenglamani qanoatlantiradi.

Matematik modellari differensial tenglamalarga keluvchi ko`pgina masala va jarayonlar bilan, masalan, quyidagi kitobdan tanishish mumkin:
Амелькин В. В. Дифференциальные уравнения в приложениях. -М.:Наука, 1987.

Masalalar

1. Tengalamani yeching $y'(x)=\sin(ax)\cdot\cos(bx)$.

2. Ushbu

$$f(x)=\left(\int_0^x e^{-s^2}ds\right)^2, \quad g(x)=\int_0^1 \frac{e^{-x^2(s^2+1)}}{s^2+1}ds$$

funksiyalar uchun

$$f'(x)+g'(x)=0, \ x\in\mathbb{R},$$

ayniyatni isbotlang. Undan foydalanib,

$$\int_0^{+\infty} e^{-s^2}ds=\frac{\sqrt{\pi}}{2}$$

tenglikni isbotlang.

3. Matematik modeli differensial tenglamalarni yechishga keltiriluvchi fizik jarayonlarga misollar keltiring.

I. BIRINCHI TARTIBLI DIFFERENSIAL TENGLAMALAR

1.1. Differensial tenglama va uning yechimi tushunchalari

$F(x,y,y')=0$ ***ko`rinishdagi tenglama.*** $F(x,y,p)$ – biror $G\subset\mathbb{R}^3$ sohada aniqlangan uch haqiqiy o'zgaruvchining uzluksiz haqiqiy funksiyasi bo'lsin, ya'ni $F:G\to\mathbb{R}$, $F\in C(G,\mathbb{R})$ (yoki qisqaroq: $F\in C(G)$). Biz bu funksiya p o'zgaruvchiga tub ma'noda bog'liq, ya'ni x va y lar tayinlanganda F funksiya p o`zgaruvchining funksiyasi sifatida o`zgarmasga aylanmaydi deb faraz qilamiz. Ushbu

$$F(x, y(x), y'(x)) = 0$$

yoki qisqaroq

$$F(x, y, y') = 0 \qquad (1.1.1)$$

tenglama $y = y(x)$ noma'lum funksiyaga nisbatan birinchi tartibli oddiy differensial tenglama deb ataladi.

Yechim uzluksiz differensiallanuvchi funksiyalar sinfida izlanadi.

I bilan sonlar o'qidagi biror oraliqni (ya'ni bog'lanishli va kamida bitta ichki nuqtaga ega bo'lgan sonli to'plamni) belgilaylik. Analizdan ma'lumki, oraliq ushbu

$$(-\infty, +\infty),\ (-\infty, b),\ (-\infty, b],\ [a, b),\ (a, b],\ (a, b),\ [a, b],$$
$$(a, +\infty),\ [a, +\infty)$$

sonli to'plamlarning biridir; bunda $a < b$.

Agar I oraliqda aniqlangan $y = \varphi(x)$ haqiqiy funksiya uchun

1^0. $\varphi(x) \in C^1(I)$, ya'ni $\varphi'(x)$ hosila I oraliqda uzluksiz,

2^0. $\forall x \in I \quad F(x, \varphi(x), \varphi'(x)) = 0$, ya'ni $y = \varphi(x)$ funksiya I oraliqda (1) tenglamani (qanoatlantiradi) ayniyatga aylantiradi

shartlar bajarilsa, u holda $y = \varphi(x)$ funksiya (1.1.1) **tenglamaning I oraliqda aniqlangan yechimi** deyiladi.

Eslatma 1. Ta'rifdagi 2^0 shartning o'rinli bo'lishi uchun ushbu $\forall x \in I \quad (x, \varphi(x), \varphi'(x)) \in G$ shartning bajarilishi, ya'ni $(x, \varphi(x), \varphi'(x))$, $x \in I$, nuqtalarning (1.1.1) tenglama aniqlangan sohaga tegishli bo'lishi zarurdir. Demak, (1.1.1) tenglama aniqlangan G soha $y = \varphi(x)$ yechimning aniqlanish oralig'i I ni, uning va $y' = \varphi'(x)$ hosilaning o'zgarish to'plamlarini ma'lum ma'noda chegaralaydi.

Eslatma 2. Ta'rifda $y = \varphi(x)$ yechim oraliqda aniqlangan ekanligi muhimdir, ya'ni yechim faqat oraliqda qaraladi.

Eslatma 3. Agar I oraliqning chegaraviy nuqtasi I ga tegishli bo'lsa, u holda bu nuqtadagi hosila sifatida mos bir tomonli hosila tushuniladi. Masalan, $I=[a,b)$ oraliqda aniqlangan $y=\varphi(x)$ funksiyaning $x=a$ nuqtadagi $\varphi'(a)$ hosilasi sifatida shu nuqtadagi $\varphi'(a+0)$ o'ng hosila qabul qilinadi.

Yechim oshkormas ko'rinishda ham berilishi mumkin. Faraz qilaylik, $\Phi(x,y)=0$ tenglama biror I oraliqda biror $y=\varphi(x)\in C^1$ funksiyani oshkormas ko'rinishda aniqlasin va bu $y=\varphi(x)$ funksiya I da (1.1.1) tenglamaning yechimi bo'lsin. U holda $\Phi(x,y)=0$ munosabat (1.1.1) tenglamaning (I oraliqda) oshkormas ko'rinishdagi yechimi deyiladi.

Agar parametrik ko'rinishda berilgan $x=x(t),\ y=y(t),\ t\in\tilde{I}$, funksiya uchun $x'(t)\neq 0,\ t\in\tilde{I}$, bo'lib, ushbu

$$1).\ y(t)\in C^1(\tilde{I}),\ x(t)\in C^1(\tilde{I}),$$

$$2).\ F\left(x(t),y(t),\frac{y'(t)}{x'(t)}\right)\equiv 0,\ t\in\tilde{I},$$

shartlar ham bajarilsa, u holda $x=x(t),\ y=y(t)$ funksiyalar (1.1.1) differensial tenglamaning parametrik ko'rinishda berilgan yechimi deyiladi. Ba'zi hollarda yechimni shu ko'rinishda yozish qulay bo'ladi.

Differensial tenglama yechimining grafigi **integral chiziq** deb ataladi. Ravshanki, integral chiziq silliq chiziqdir. Biz keyinroq integral chiziq tushunchasining ma'nosini kengaytiramiz.

Misol 1. Ushbu

$$xy'^2-2yy'+x=0$$

differensial tenglamada $F(x,y,p)=xp^2-2yp+x$, $G=\mathbb{R}^3$.

a). $y=x$ funksiya berilgan differensial tenglamaning $(-\infty,+\infty)$ intervalda yechimidir. Haqiqatdan ham,

1^0. $\varphi(x)=x$ funksiya $(-\infty,+\infty)$ intervalda uzluksiz differensiallanuvchi, chunki $y'=\varphi'(x)=1\in C(\mathbb{R})$;

2^0. ixtiyoriy $x\in(-\infty,+\infty)$ nuqtada

$$xy'^2-2yy'+x=x-2x+x=0.$$

b) Lekin $y=2x-1$ funksiya hech qanday $I\subset\mathbb{R}$ oraliqda berilgan differensial tenglamaning yechimi bo'la olmaydi. Chunki bu holda $y'=\varphi'(x)=2\in C(I)$, ammo

$$xy'^2-2yy'+x=x\cdot 4-2(2x-1)\cdot 2+x=-3x+4=0$$

tenglik I oraliqda ayniyat emas u I ning ko'pi bilan bitta nuqtasida qanoatlanishi mumkin ($x=4/3$ bo'lganda) xolos, I da esa cheksiz ko'p nuqtalar mavjud. ♠

Misol 2. Ushbu

$$(1+x^2)y'-xy+x=0 \tag{1.1.2}$$

differensial tenglama uchun $F(x,y,p)=(1+x^2)p-xy+x$ funksiya $G=\mathbb{R}^3$ da aniqlangan va silliq. Ushbu

$$y=1+c\sqrt{1+x^2} \tag{1.1.3}$$

funksiya (1.1.2) tenglamaning $(-\infty,+\infty)$ oraliqda yechimidir; bu yerda $c-$ ixtiyoriy o'zgarmas son. Haqiqatan ham, bu funksiya $x\in(-\infty,+\infty)$ oraliqda uzluksiz differensiallanuvchi,

$$y'(x)=\frac{cx}{\sqrt{1+x^2}}\in C\big((-\infty,+\infty)\big),$$

va u berilgan tenglamani $(-\infty,+\infty)$ oraliqda qanoatlantiradi:

$$(1+x^2)y'-xy+x=(1+x^2)\frac{cx}{\sqrt{1+x^2}}-x\left(1+c\sqrt{1+x^2}\right)+x=0.$$

Qaralayotgan (1.1.2) tenglamaning ixtiyoriy $y=y(x)$, $x\in(-\infty,+\infty)$, yechimi (1.1.3) ko'rinishda bo'lishini ko'rsataylik. (1.1.2) tenglamaning ixtiyoriy $y=y(x)$ yechimi berilgan bo'lsin:

$$(1+x^2)y'(x)-xy(x)+x=0,\quad x\in(-\infty,+\infty).$$

Bu yechimga ko'ra ushbu

$$Y(x)=\frac{y(x)-1}{\sqrt{1+x^2}}$$

funksiyani tuzaylik. Uning hosilasi $(-\infty,+\infty)$ intervalda aynan nolga teng:

$$Y'(x)=\frac{y'(x)\sqrt{1+x^2}-\dfrac{x}{\sqrt{1+x^2}}(y(x)-1)}{1+x^2}=$$

$$=\frac{(1+x^2)y'(x)-xy(x)+x}{(1+x^2)^{3/2}}=\frac{0}{(1+x^2)^{3/2}}=0$$

Analizdan ma'lumki, oraliqda hosilasi nolga teng funksiya o'zgarmas. Demak, tuzilgan funksiya o'zgarmasdan iborat:

$$Y=c_0,\ c_0=\text{const}\in\mathbb{R},$$

ya'ni

$$\frac{y(x)-1}{\sqrt{1+x^2}}=c_0,\ \ x\in(-\infty,+\infty).$$

Bundan $y(x)=1+c_0\sqrt{1+x^2}$ ekanligini topamiz.

Shunday qilib, (I.1.3) formula (I.1.2) tenglamaning barcha yechimlarini va faqat ularnigina ifodalaydi, ya'ni (I.1.2) ning har qanday yechimi (I.1.3) dan c ning biror xususiy qiymatida hosil bo'ladi va shuning bilan birgalikda (I.1.3) formula c ning ixtiyoriy tayin qiymatida (I.1.2) ning yechimini aniqlaydi. Demak, (I.1.2) differensial tenglamaning integral chiziqlari ushbu $y=1+c\sqrt{1+x^2}$ chiziqlar oilasidan iborat ($c-$ ixtiyoriy o'zgarmas). ♦

$y'=f(x,y)$ ***ko'rinishdagi tenglama.*** Faraz qilaylik, (I.1.1) tenglama y' hosilaga nisbatan yechilgan bo'lsin:

$$y'=f(x,y). \tag{I.1.4}$$

Bu yerdagi $f(x,y)$ funksiya tekislikning biror $D\subset\mathbb{R}^2$ sohasida aniqlangan uzluksiz haqiqiy funksiya deb faraz qilinadi, ya'ni $f\in C(D)$. (I.1.4) tenglama **hosilaga nisbatan yechilgan birinchi tartibli differensial tenglama** (yoki **normal ko'rinishdagi** birinchi tartibli differensial tenglama) deyiladi.

Yuqorida keltirilgan misolda (I.1.2) tenglamani osongina hosilaga nisbatan yechilgan holga keltirish mumkin:

$$y' = \frac{xy - x}{1 + x^2}.$$

Umumiy holda (I.1.1) tenglamani (I.1.4) ko'rinishga keltirish murakkab masala. Bunday masalalar analizda o'rganiladi. Biz (I.1.1) tenglamadan y' ni topishda to'xtalmasdan birdaniga (I.1.4) tenglama berilgan deb faraz qilamiz.

Agar $y = \varphi(x)$ funksiya I oraliqda aniqlangan bo'lib, u

1^0. $\varphi(x) \in C^1(I)$,

2^0. $\forall x \in I \quad \varphi'(x) = f(x, \varphi(x))$

shartlarni qanoatlantirsa, $y = \varphi(x)$ funksiya $y' = f(x, y)$ (I.1.4) tenglamaning I oraliqda (aniqlangan) yechimi deyiladi.

Eslatma. Bu yerda yechimdan 1^0 shart o'rniga $\varphi(x)$ning I da differensiallanuvchi bo'lishini talab etsak, u uzluksiz differensiallanuvchi ham bo'ladi, chunki $u = \varphi(x)$ yechimning I da uzluksizligi (bu uning differensiallanuvchiligidan kelib chiqadi) hamda $f(x, y)$ funksiyaning D da uzluksizligiga va 2^0 shartga ko'ra $\varphi'(x) = f(x, \varphi(x))$ ham I da uzluksizdir.

Yuqoridagi misollardan ko'rinadiki, differensial tenglama cheksiz ko'p yechimga ega bo'lishi mumkin.

Ba'zan barcha yechimlarni bitta formula bilan berish mumkin bo'ladi. Agar $y = \varphi(x, c)$ funksiya c o'zgarmasning ixtiyoriy joiz qiymatida (I.1.1) yoki (I.1.4) differensial tenglamaning (grafigi) D sohada joylashgan yechimi bo'lib, tenglamaning D da joylashgan har qanday yechimi shu $y = \varphi(x, c)$ formuladan c ning biror joiz qiymatida hosil bo'lsa, u holda $y = \varphi(x, c)$ oila berilgan tenglamaning D sohadagi **umumiy yechimi** deyiladi. Umumiy yechim oshkormas ko'rinishda $\Phi(x, y, c) = 0$ tenglama bilan, yoki parametrik $x = x(p, c)$, $y = y(p, c)$ (p – yechimdagi parametr, c esa yechimlarni belgilaydi) ko'rinishda ham berilishi mumkin.

Yuqoridagi misol 2 da biz $(1 + x^2)y' - xy + x = 0$ tenglamaning $\mathbb{R}^2$ dagi umumiy yechimi $y = 1 + c\sqrt{1 + x^2}$ (bunda

c – ixtiyoriy o'zgarmas) formula bilan berilishini ko'rsatgan edik.

Tenglamaning bitta (xususiy) yechimini ajratish uchun yechimdan qo'shimcha shart talab qilish kerak.

Masalalar

1. Differensial tenglamani yeching $y' = \dfrac{e^x}{x}$.

2. Ushbu $y' = |x|$ tenglamani yeching.

3. $y = |2x-1| + 1$ funksiya hech qanday differensial tenglamaning $(0;1)$ intervalda yechimi bo'la olmaydi. Nega?

4. Ushbu $y = \sqrt{x^2+1}$ funksiya hech qanday oraliqda $y' = xy^2 - 1$ tenglamaning yechimi bo'la olmaydi. Shuni isbotlang.

1.2. Koshi masalasi

Hosilaga nisbatan yechilgan ushbu

$$y' = f(x,y) \tag{1.2.1}$$

differensial tenglamani qaraylik; bu yerda $f \in C(D)$, D – tekislikdagi soha. (1.2.1) tenglamaning

$$y(x_0) = y_0, \ (x_0, y_0) \in D, \tag{1.2.2}$$

shartni qanoatlantiruvchi (berilgan x_0 nuqtada berilgan y_0 qiymatni qabul qiluvchi) va biror I, $x_0 \in I$, oraliqda aniqlangan $y = \varphi(x)$ yechimini topish **Koshi masalasi** yoki **boshlang'ich masala** deyiladi va u,

$$\begin{cases} y' = f(x,y) \\ y|_{x_0} = y_0 \end{cases} \quad \text{yoki} \quad y' = f(x,y),\ y|_{x_0} = y_0 \tag{K}$$

ko'rinishda yoziladi. Bu yerda $y|_{x_0}$ yozuvi $y = y(x)$ noma'lum funksiyaning $x = x_0$ nuqtadagi qiymatini anglatadi: $y|_{x_0} = y(x_0)$.

(1.2.2) shart **boshlang'ich shart** yoki **Koshi sharti** deb yuritiladi.

Agar shunday $I\ni x_0$ oraliq topilsaki, bu oraliqda biror $y=y(x)$ funksiya (I.2.1) ning yechimi bo'lib, (I.2.2) shartni ham qanoatlantirsa, u holda (K) masalaning yechimi mavjud deyiladi, bu $y=y(x)$ funksiya esa (K) masalaningning I oraliqda aniqlangan yechimi deb yuritiladi.

(K) masalaga nisbatan tabiiy ravishda quyidagi savollar tug'iladi:

1. (K) masala x_0 ga yaqin x larda aniqlangan biror yechimga egami?

2. Agar yechim mavjud bo`lsa, u yagonami?

3. Qaysi eng katta I, $I\ni x_0$, oraliqda (K) masalaning yechimi mavjud?

Misol 1. *a*) Ushbu

$$y' = \sqrt[3]{y^2},\ y(0)=0,$$

Koshi masalasi $y=0$ yechim bilan birgalikda $y=\left(\frac{x}{3}\right)^3$ yechimga ham ega (tekshirib ko'ring). Shunday qilib, a) masalaning yechimi yagona emas.

b) Ushbu

$$y' = y^2,\ \ y|_0 = 1.$$

masalaning yechimini topaylik. Faraz qilaylik, $y=y(x)$ yechim mavjud bo'lsin. Bu yechim $(\in C^1)$ $x=0$ ning kichik atrofida noldan farqli, chunki $y(0)=1$. Berilgan tenglamadan $\frac{dy}{y^2}=dx$ tenglikni topib, uni 0 dan x gacha integrallab, quyidagini topamiz: $\frac{1}{y(0)}-\frac{1}{y(x)}=x$. Bundan berilgan $y|_0=1$ boshlang'ich shartga ko'ra $y=\frac{1}{1-x}$ ekanligini hosil qilamiz. Topilgan bu funksiya qaralayotgan masalaning $(-\infty,1)$ oraliqda aniqlangan yechimidir (tekshirib ko'ring). Qaralayotgan masalaning yechimi yagona. Ravshanki, bu yechimni $(-\infty,1)$dan kattaroq (kengroq) oraliqda mavjud bo'la olmaydi. $x \to 1-0$ da yechim $+\infty$ ka ketib qoladi.

Agar $G \subset D$ sohaning har bir nuqtasidan (1.2.1) differensial tenglamaning yagona integral chizig'i o'tsa, ya'ni har qanday $(x_0, y_0) \in G$ uchun (K) Koshi masalasi yagona yechimga ega bo'lsa, u holda G soha (1.2.1) tenglama uchun (tenglamaning) **yagonalik sohasi** deyiladi.

Misol 2. Biz 1.1- paragrafda (misol 2)

$$(1+x^2)y' - xy + x = 0, \text{ ya'ni } \quad y' = \frac{xy - x}{1+x^2}$$

tenglamaning $\mathbb{R}^2$ sohadagi umumiy yechimi

$$y = 1 + c\sqrt{1+x^2}$$

formula bilan berilishini isbotlagan edik. Ixtiyoriy $(x_0, y_0) \in \mathbb{R}^2$ nuqta orqali qaralayotgan differensial tenglamaning yagona integral chizig'i o'tadi. Haqiqatan ham, $y(x_0) = y_0$, ya'ni $y_0 = 1 + c\sqrt{1+x_0^2}$ shartdan yagona c topiladi:

$$c = \frac{y_0 - 1}{\sqrt{1+x_0^2}}.$$

Demak, ixtiyoriy $(x_0, y_0) \in \mathbb{R}^2$ nuqtadan differensial tenglamaning ushbu

$$y = 1 + \frac{y_0 - 1}{\sqrt{1+x_0^2}}\sqrt{1+x^2}$$

yagona yechimi o'tadi. Shunday qilib, qaralayotgan $y' = \dfrac{xy - x}{1+x^2}$ tenglamaning yagonalik sohasi butun tekislik $\mathbb{R}^2$ dan iborat.

Misol 3. Ushu $y' = y^2$ differensial tenglamaning yagonalik sohalarini topaylik.

Bu tenglamaning $y > 0$ va $y < 0$ yarim tekisliklardagi umumiy yechimi $y=\dfrac{1}{c-x}$ formula bilan beriladi (c – ixtiyoriy o'zgarmas). Haqiqatan ham, agar $y=y(x)$ funksiya $y>0$ yoki $y<0$ yarim tekislikdagi yechim bo'lsa, u holda

$$\frac{dy}{y^2}=dx \quad \Rightarrow \quad -\frac{1}{y}=x-c \quad \Rightarrow \quad y=\frac{1}{c-x}.$$

Osongina tekshirib ko'rish mumkinki, $y=\dfrac{1}{c-x}$ formula ixtiyoriy c uchun berilgan differensial tenglamaning yechimi hamdir. Ravshanki, $y_0\neq 0$ bo'lganda

$$y_0=\frac{1}{c-x_0}$$

tenglama yagona $c=c_0=\dfrac{1}{y_0}+x_0$ yechimga ega va $y=\dfrac{1}{c_0-x}$ funksiya uchun

$$y_{|x_0}=y_0, \qquad y'(x)=y^2(x)$$

ham bo'ladi. Demak, $y>0$ va $y<0$ yarim tekisliklar berilgan tenglama uchun yagonalik sohalaridir.

Bu yerda shunga e'tibor beraylikki, topilgan

$$y=\frac{1}{c_0-x}$$

yechimning aniqlanish sohasi (x_0,y_0) boshlang'ich qiymatlarga bog'liq; yechim $y_0>0$ bo'lganda $x\in(-\infty;c_0)$ oraliqda, $y_0<0$ bo'lganda esa $x\in(c_0;+\infty)$ ($c_0=\dfrac{1}{y_0}+x_0$) oraliqda aniqlangan.

Ko'rinib turibdiki, $y=0$ berilgan tenglamaning yechimi. Ravshanki, $y>0$ va $y<0$ yarim tekisliklardan boshlangan har qanday yechim hech qachon $y=0$ yechim bilan kesishmaydi. Demak, $y=0$ to'g'ri chiziqning nuqtalaridan ham bittadan yechim

($y=0$ yechim) o'tadi. Shunday qilib, $G=\mathbb{R}^2$ soha berilgan tenglama uchun yagonalik sohasi bo'ladi.

Topilgan $y=\frac{1}{c-x}$ bir parametrli yechimlar oilasi qaralayotgan $y'=y^2$ tenglamaning shu $G=\mathbb{R}^2$ sohadagi umumiy yechimini ifodalamaydi, chunki $y=0$ yechim $y=\frac{1}{c-x}$ dan c ning hech qanday qiymatida hosil bo'lmaydi. Ushbu $y=\frac{c}{1-cx}$ formula esa (c – ixtiyoriy o'zgarmas) berilgan tenglamaning $\mathbb{R}^2$ dagi $y=-\frac{1}{x}$ yechimidan boshqa barcha yechimlarini ifodalaydi.

G sohaning (1.2.1) differensial tenglama uchun yagonalik sohasi bo'lishi uchun yetarli shartlarni mavjudlik va yagonalik teoremasi beradi (1.9.- paragrafga qarang).

1.3. Geometrik talqin

Ushbu

$$y'=f(x,y) \tag{1.3.1}$$

differensial tenglamani qaraylik; bu yerda f funksiyani D sohada yetarlicha silliq deb hisoblaymiz.

(1.3.1) tenglamaning $y(x_0)=y_0$ boshlang'ich shartni qanoatlantiruvchi yechimini topish $(x_0,y_0)\in D$ nuqta orqali (1.3.1) tenglamaning integral chizig'ini o'tkazish demakdir.

Har bir $(x_0,y_0)\in D$ nuqtaga shu nuqtadan o'tuvchi va $k=f(x_0,y_0)$ burchak koeffitsientiga ega bo'lgan to'g'ri chiziqni mos qo'yib, bu to'g'ri chiziqni «kichik kesma» (ya'ni yo'nalish) bilan tasvirlaylik. Natijada D sohada (1.3.1) differensial tenglamaga mos keluvchi yo'nalishlar maydoni hosil bo'ladi. Har bir «kichik kesma» mos nuqtadagi **maydon yo'nalishi** deyiladi.

D sohadagi egri chiziq (1.3.1) tenglamaning integral chizig'i bo'lishi uchun u o'zining ixtiyoriy nuqtasida maydonning shu nuqtadagi yo'nalishiga urinishi yetarli va zarurdir ($y'(x_0)=f(x_0,y_0)=k$ - integral chiziqning (x_0,y_0) nuqtasidagi urinmasining burchak koeffitsienti; bu yo'nalish Ox o'qi bilan $\alpha=\mathrm{arctg}k$ burchak tashkil etadi).

Maydon yo'nalishlarini ko'rsatuvchi kesmachalarnı D sohada «yetarlicha zich (qalin)» tasvirlab va bu kesmachalarga urinuvchi egri chiziqni chizib, berilgan differensial tenglamaning integral chizig'ini taqriban qurish mumkin.

Integral chiziqlarni aniqroq chizish maqsadida yechimlarning ekstremum va burilish nuqtalarini topish mumkin. Yechimlarning statsionar (kritik) va, demak, ekstremum nuqtalari $f(x,y)=0$ tenglamani qanoatlantiradi. Burilish nuqtalarida $y''=0$ bo'lishi kerak. $y=y(x)$ yechim uchun $y'(x)=f(x,y(x))$ ayniyat bajariladi. Bu tenglikni differensiallab, y'' ni topamiz:

$$y''=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot y',\ \text{ya'ni}\ \ y''=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot f.$$

Demak, burilish nuqtalari

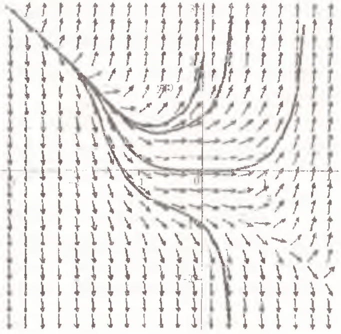
$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot f=0$$

tenglamani qanoatlantirdi.

Misol 1. Ushbu $y'=x^3+y^3$ differensial tenglama uchun yo'nalishlar maydoni va 4 dona integral chiziq 1.1-rasmda ko'rsatilgan.

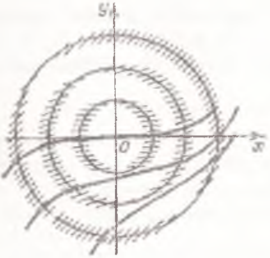
Yo'nalishlar maydonini qurishda (tasvirlashda) ba'zan izoklinalardan foydalanish maqsadga muvofiq bo'ladi.

Barcha nuqtalarida maydon yo'nalishi bir xil burchak koeffitsientiga ega bo'lgan to'plam **izoklina** deyiladi. Izoklinalar $f(x,y)=k$ (k - const) tenglama bilan beriladi. $f(x,y)=k$ izoklinaning har bir nuqtasida maydon yo'nalishi x lar o'qining musbat yo'nalishi bilan bir xil $\alpha=\mathrm{arctg}k$ burchak tashkil etadi



1.1-rasm. $y' = x^2 + y^2$ tenglamaning yo'nalishlar maydoni va yechimlari grafiklari

Misol 2. Ushbu $y' = x^2 + y^2$ differensial tenglama uchun izoklinalar $x^2 + y^2 = k$, k – const. tenglikdan topiladi. Izoklinalar markazi koordinatalar boshida joylashgan konsentrik aylanalardan ($k > 0$) va $(0;0)$ nuqtadan ($k = 0$) iborat. Masalan, $x^2 + y^2 = 1$ izoklina nuqtalarida yo'nalishlar maydoni x lar o'qi bilan bir xil $\alpha = \mathrm{arctg}\,k = \mathrm{arctg}\,1 = 45^\circ$ li burchak tashkil etadi. Bir nechta izoklina va ulardagi maydon yo'nalishlarini chizamiz hamda bu yo'nalishlarga urintirib egri chiziqlar (yechimlar grafigi)ni o'tkazamiz (1.2-rasm). Bu egri chiziqlarning ko'rinishi orqali qaralayotgan differensial tenglamaning yechimlari haqida tasavvur hosil qilamiz.



1.2-rasm

Masalalar

1. Ushbu $y' = \dfrac{|x^2 - y^2|}{x^2 - y^2}$ differensial tenglamaning yo'nalishlar maydonini quring. Yechimlarni tasvirlang.

2. Ushbu $y' = x - y^2$ differensial tenglamaning yo'nalishlar maydonini quring, Yechimlarni tasvirlang.

1.4. Differensiallarda yozilgan tenglamalar

Yuqorida qaralgan

$$y' = f(x, y) \tag{1.4.1}$$

tenglamada x va y o'zgaruvchilari teng huquqli emas: x – erkli o'zaruvchi, y esa – uning funksiyasi, ya'ni erksiz o'zgaruvchi. Buning natijasida, masalan, integral chiziq Oy o'qiga parallel urinmaga ega bo'la olmaydi. Shuning uchun (1.4.1) bilan birgalikda ushbu

$$\frac{dx}{dy} = \frac{1}{f(x, y)} \tag{1.4.2}$$

«to'ntarilgan» tenglamani ham qaraylik. Bu tenglama D sohaning $f(x, y)$ funksiya nolga aylanmagan qismida (1.4.1) tenglamaga teng kuchli bo'ladi. (1.4.2) tenglamada $x = x(y)$ noma'lum funksiya (1.4.1) tenglamadagi $y = y(x)$ noma'lum funksiyaning teskarisidir.

Differensial tenglamani differensiallarda ham yozish mumkin:

$$M(x, y)dx + N(x, y)dy = 0, \qquad (1.4.3)$$

bu yerda $\{M, N\} \subset C(D)$ deb hisoblanadi. (1.4.3) differensial tenglama o'zgaruvchilari teng huquqli qatnashgan yoki simmetrik ko'rinishdagi tenglama deb ham yuritiladi.

Agar $(x_0, y_0) \in D$ nuqtada $M(x_0, y_0) = N(x_0, y_0) = 0$ bo'lsa, (x_0, y_0) nuqta (1.4.3) tenglamaning maxsus nuqtasi deyiladi. Maxsus bo'lmagan nuqta regulyar nuqta deyiladi. Regulyar nuqtada M va N funksiyalarning birortasi, aytaylik, N (yoki M) noldan farqli bo'lgani uchun bu nuqtaning biror atrofida ham $N \neq 0$ (yoki $M \neq 0$). Demak, ixtiyoriy regulyar nuqtaning yetarlicha kichik atrofida (1.4.3) tenglama

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}, \qquad \left(\text{yoki } \frac{dx}{dy} = -\frac{N(x, y)}{M(x, y)}\right)$$

ko'rinishga keladi. Shuning uchun tenglama har qanday regulyar nuqtada yo'nalishni aniqlaydi (yo'nalish ordinatalar o'qiga parallel bo'lishi mumkin). Bu nuqtada integral chiziq shu nuqtadagi yo'nalishga urinadi.

Misol 1. Ushbu

$$xdx + ydy = 0$$

differensial tenglama (0;0) maxsus nuqtadan boshqa barcha nuqtalarda yo'nalishlar maydonini aniqlaydi. Bu tenglamani quyidagicha yechish mumkin

$$2xdx + 2ydy = 0,\ d(x^2) + d(y^2) = 0,\ d(x^2 + y^2) = 0.$$

Demak, $x^2 + y^2 = c$.

Berilgan $(x_0, y_0) \neq (0;0)$ nuqtadan o'tuvchi integral chiziq aylanadan iborat:

$$x^2 + y^2 = x_0^2 + y_0^2.$$

Bu aylana o'zining ixtiyoriy nuqtasining biror atrofida $y=\varphi(x)$ yoki $x=\psi(y)$ formula bilan oshkor holda berilishi mumkin. Lekin uni butunligicha oshkor ko'rinishda berib bo'lmaydi. ♣

Masalalar

1. $x=\cos t, y=\sin t$ ushbu $xdx+ydy=0$ tenglamaning parametrik ko'rinishdagi yechimi ekanligini isbotlang.
2. Ushbu $xdy-ydx=0$ tenglama uchun yo'nalishlar maydonini tasvirlang. Tenglamani yeching.

I.5. O'zgaruvchilari ajraladigan tenglamalar

1. Eng sodda differensial tenglamadan boshlaylik:

$$y'=f(x),\quad f\in C(I)\,. \tag{I.5.1}$$

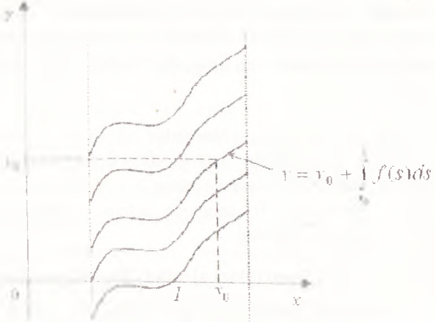
Bu tenglamaning yechimlari, analizdan ma'lumki, $f(x)$ funksiyaning boshlang'ich funksiyalaridan iborat bo'ladi:

$$y=\int f(x)dx+c\,,\ c-\text{const} \tag{I.5.2}$$

Bu yechim qaralayotgan (I.5.1) tenglamadagi f funksiyaning uzluksizlik oralig'i I da aniqlangan. Tenglamaning $y(x_0)=y_0$ $(x_0\in I)$ shartni qanoatlantiruvchi yechimi yagona va u integral hisobning asosiy teoremasi, ya'ni (I.5.2) formulaga ko'ra

$$y=y_0+\int_{x_0}^{x}f(s)ds$$

ko'rinishda ifodalanadi. Demak, (I.5.1) tenglama uchun yagonalik sohasi $I\times\mathbb{R}$ polosadan iborat (I.3-rasm).



1.3-rasm. $y' = f(x)$ tenglama yechimlari

2. Endi ushbu

$$y' = g(y),\ g \in C(I), \tag{1.5.3}$$

tenglamani qaraylik. Faraz qilaylik, g funksiya $I = (c, d)$ intervalda nolga aylanmasin. $g \in C(I)$ (uzluksiz) bo'lganligi uchun u o'z ishorasini saqlaydi. Aniqlik uchun $y \in I$ bo'lganda $g(y) > 0$ deylik.

Agar $y = y(x)$ funksiya (1.5.3) tenglamaning $y(x_0) = y_0$ ($y_0 \in I$) shartni qanoatlantiruvchi biror yechimi bo'lsa, u holda

$$\frac{dy(x)}{dx} = g(y(x)), \qquad y(x_0) = y_0$$

bo'ladi. Demak,

$$\frac{dy(x)}{g(y(x))} = dx\ , \text{ yoki } \frac{dy(s)}{g(y(s))} = ds\ .$$

Oxirgi tenglikning har ikkala tomonini $s = x_0$ dan $s = x$ gacha integrallaymiz:

$$\int_{x_0}^{x} \frac{dy(s)}{g(y(s))} = x - x_0 \quad \text{yoki} \quad \int_{y_0}^{y(x)} \frac{dz}{g(z)} = x - x_0 \tag{1.5.4}$$

Ushbu

$$\Phi_{y_0}(y) = \int_{y_0}^{y} \frac{dz}{g(z)}, \quad (\{y_0, y\} \subset (c, d)) \tag{1.5.5}$$

belgilashni kiritib, (1.5.4) tenglikni

$$\Phi_{y_0}(y(x)) = x - x_0, \quad (y_0 = y(x_0)) \tag{1.5.6}$$

ko'rinishga keltiramiz. Demak, qaralayotgan $y = y(x)$ yechim ushbu

$$\Phi_{y_0}(y) = x - x_0, \quad (y_0 = y(x_0)) \tag{1.5.7}$$

tenglamani qanoatlantiradi. (1.5.7) tenglama $y = y(x)$ funksiyani oshkormas ko'rinishda aniqlaydi. Bu $y = y(x)$ funksiya (1.5.3) differensial tenglamani qanoatlantiradi. Haqiqatdan ham,

$$\frac{\partial \Phi_{y_0}(y)}{\partial y} = \frac{1}{g(y)} > 0$$

bo'lgani uchun teskari funksiya haqidagi teoremaga ko'ra $u = \Phi_{y_0}(y)$ funksiya teskari funksiya $y = \Phi_{y_0}^{-1}(u)$ ga ega:

$$u = \Phi_{y_0}(\Phi_{y_0}^{-1}(u)), \qquad y = \Phi_{y_0}^{-1}(\Phi_{y_0}(y))$$

Teskari $y = \Phi_{y_0}^{-1}(u)$ funksiya

$$\Phi_{y_0}(c) < u < \Phi_{y_0}(d)$$

oraliqda aniqlangan bo'ladi.

(1.5.7) tenglikdan topilgan

$$y = \Phi_{y_0}^{-1}(x - x_0),$$

funksiya (1.5.3) tenglamani qanoatlantiradi (tekshirib ko'ring). Demak, yechim

$$\Phi_{y_0}(c) < x - x_0 < \Phi_{y_0}(d)$$

bo'lganda mavjud. Shunday qilib, $y = y(x)$ yechimning aniqlanish sohasi (1.5.5) ga ko'ra

$$x_0 - \int_{y_0}^{c} \frac{dz}{g(z)} < x < x_0 + \int_{y_0}^{d} \frac{dz}{g(z)}$$

intervaldan iborat bo'lib, u

$$\int_{y_0}^{c} \frac{dz}{g(z)}, \quad \int_{y_0}^{d} \frac{dz}{g(z)} \tag{1.5.8}$$

integrallar uzoqlashuvchi bo'lgan holdagina $(-\infty,+\infty)$ oraliqda aniqlangan bo'ladi.

Biz quyidagi teoremani isbotladik.

Teorema. *Aytaylik, (1.5.3) differensial tenglamada* $g \in C((c,d))$ *va* $g(y) \neq 0$ $(y \in (c,d))$ *bo'lsin. U holda* $\{(x,y) \mid -\infty < x < +\infty,\ c < y < d\}$ *polosaning ixtiyoriy* (x_0, y_0) *nuqtasidan (1.5.3) tenglamaning yagona* $y = y(x)$ *integral chizig'i o'tadi va bu yechim (1.5.7) formula bilan oshkormas ko'rinishda beriladi. Bunda yechim* $(-\infty,+\infty)$ *oraliqda aniqlangan bo'lishi uchun (1.5.8) integrallarning uzoqlashuvchi bo'lishi yetarli va zarurdir.*

Endi (1.5.3) tenglamadagi g funksiyaning (c,d) intervalda nolga aylangan holida to'xtalaylik. Faraz qilaylik, $g(y)$ funksiya (c,d) ning yagona $y = \bar{y} \in (c,d)$ nuqtasida nolga aylansin. Bu holda (1.5.3) tenglamaning $y(x) = \bar{y}$ o'zgarmas yechimi mavjud. Bundan boshqa $y = y(x)$ yechim uchun (1.5.3) tenglamadan

$$\frac{dy(x)}{g(y(x))} = x \quad \text{yoki} \quad \Phi_{y_0}(y(x)) = x - x_0,\ \bar{y} \neq y(x_0) = y_0$$

(x_0, y_0) nuqtadan chiqqan (integral chiziqning) yechimning chekli x da $\bar{y}$ ga aylanishi ushbu

$$\int_{\bar{y}-0}^{c} \frac{dz}{g(z)}, \quad \int_{\bar{y}+0}^{d} \frac{dz}{g(z)} \tag{1.5.9}$$

xosmas integrallarning yaqinlashuvchiligi bilan aniqlanadi.

Agar (1.5.9) xosmas integrallarning birortasi yaqinlashuvchi bo'lsa, $y=\bar{y}$ to'g'ri chiziqning ixtiyoriy nuqtasidan kamida ikkita integral chiziq o'tadi (yechimning yagonalik xossasi buziladi).

$y=\bar{y}$ to'g'ri chiziq atrofida integral chiziqlarning turli hollardagi tabiatini ko'rsatuvchi grafiklarni quring.

3. Ushbu

$$y' = f(x)g(y)$$

tenglama **o'zgaruvchilari ajraladigan differensial tenglama** deb ataladi. Bu yerda $f \in C((a,b))$ va $g \in C((c,d))-$ berilgan funksiyalar.

Agar g funksiya nolga aylanmasa, ixtiyoriy (x_0, y_0), $a < x_0 < b$, $c < y_0 < d$, nuqtadan bu tenglamaning yagona integral chizig'i o'tadi. Bu $y = y(x)$ yechim

$$\int_{y_0}^{y} \frac{dz}{g(z)} = \int_{x_0}^{x} f(x)dx$$

tenglama bilan oshkormas ko'rinishda beriladi.

Agar biror $\bar{y}$ nuqtada $g(\bar{y})=0$, lekin $g(y)\neq 0, y\neq \bar{y}$, va (1.5.9) xosmas integrallarning ikkalasi ham uzoqlashuvchi bo'lsa, bu holda ham yechimning yagonaligi saqlanadi; (1.5.9) xosmas integrallarning kamida bittasi yaqinlashuvchi bo'lgan holda esa $y=\bar{y}$ to'g'ri chiziqning har bir nuqtasidan kamida 2 ta (va, demak, cheksiz ko'p) integral chiziq o'tadi (bu to'g'ri chiziqda yotmagan ixtiyoriy nuqtadan bitta va faqat bitta integral chiziq o'tadi). Bu tasdiqlar 2-bandda bajarilgan tekshirishlardan bevosita kelib chiqadi.

Differensiallarda yozilgan ushbu

$$M(x)N(y)dx + P(x)Q(y)dy = 0 \qquad (1.5.10)$$

$$(\{M(x), P(x)\} \subset C((a,b)), \{N(y), Q(y)\} \subset C((c,d))$$

tenglama ham **o'zgaruvchilari ajraladigan tenglama** deb ataladi. Agar $P(x_0)N(y_0)\neq 0$ bo'lsa, (x_0, y_0) nuqtaning yetarlicha kichik atrofida tenglamaning har ikkala tomonini $P(x)N(y)\neq 0$ ga bo'lib, o'zgaruvchilarni ajratamiz:

$$\frac{M(x)}{P(x)}dx + \frac{Q(y)}{N(y)}dy = 0$$

Bu tenglikning har ikala tomononi integrallab, yechimni oshkormas ko'rinishda topamiz:

$$\int\frac{M(x)}{P(x)}dx + \int\frac{Q(y)}{N(y)}dy = c \quad (c - \text{const}). \qquad (1.5.11)$$

Agar $N(y_0) = 0$ ($P(x_0) = 0$) bo'lsa, $y = y_0$ ($x = x_0$) o'zgarmas yechimlar ham mavjud. Topilgan (1.5.11) yechimlar orasida bu yechimlar bo'lmasligi, ya'ni ular yo'qolishi mumkin. Mashqlar bajarganda ana shuni esda tutish lozim.

Misol. Ushbu

$$ye^x dx + (e^x + 1)dy = 0$$

tenglamani yeching.

Tenglamada o'zgaruvchilar ajraladi. Tenglamaning har ikkala tomonini $y(e^x + 1)$ ($y \neq 0$) ga bo'lib, integrallashlarni bajaramiz:

$$\frac{e^x}{e^x + 1}dx + \frac{dy}{y} = 0\ , \quad \int\frac{e^x}{e^x + 1}dx + \int\frac{dy}{y} = c_1 \ (c_1 - \text{const}),$$

$$\ln(e^x + 1) + \ln|y| = c_1, \ y = \frac{\pm e^{c_1}}{e^x + 1}.$$

Bu yerdagi $\pm e^{c_1}$ ni c ($c \neq 0$) bilan belgilab, $y = \dfrac{c}{e^x + 1}$ ($c \neq 0$) yechimni hosil qilamiz. Bu formuladan $c = 0$ da yo'qolgan $y = 0$ yechim hosil bo'ladi. Demak, berilgan tenglamaning umumiy yechimi $y = \dfrac{c}{e^x + 1}$, c – ixtiyoriy o'zgarmas, formula bilan aniqlanadi.

Masalalar

1. Ushbu $F(x,y) = x + y$ ikki erkli o'zgaruvchining funksiyasi $f(x)$ va $g(y)$ bir o'zgaruvchining funksiyalari ko'paytmasi sifatida ifodalanmaydi. Shuni isbotlang.

2. Differensial tenglamani yeching: $y' = |x| y$.

3. Ushbu $y' = \max(x,y)$, $y(0) = 0$, boshlang'ich masalaning $[0,+\infty)$ oraliqda aniqlangan notrivial ($y \neq 0$) yechimini toping.

4. Ushbu $y' + 2|y| = 1$, $y(0) = 1/4$, Koshi masalasini yeching.

5. Ushbu $(x+|x|)dx + (y+|y|)dy = 0$ differensial tenglama yechimlari grafigklarini quring.

6. Agar biror $(-a;a)$ $(a>0)$ intervalda aniqlangan $y = f(x)$ haqiqiy funksiya ushbu

$$f(u+v) = \frac{f(u)+f(v)}{1-f(u)\cdot f(v)} \quad (\{u,v,u+v\} \subset (-a;a)) \quad (*)$$

(funksional) tenglamani qanoatlantirsa va 0 nuqtada $f'(0)$ hosilaga ega bo'lsa, bu $y = f(x)$ funksiyani toping.

1.6. O'zgaruvchilariga nisbatan bir jinsli differensial tenglamalalar

Agar $y' = f(x,y)$ differensial tenglamadagi $f(x,y)$ funksiya x, y o'zgaruvchilarni mos ravishda tx, ty ($t>0$ yoki $t<0$) bilan almashtirilganda o'zgarmasa, ya'ni

$$f(tx,ty) = f(x,y),\ (x,y) \in D(f), t>0 \text{ (yoki } t<0) \quad (1.6.1)$$

bo'lsa, u holda bu $f(x,y)$ funksiya (0- tartibli) bir jinsli, mos

$$y' = f(x,y)$$

differensial tenglama esa **o'zgaruvchilariga nisbatan bir jinsli tenglama** deyiladi.

Agar $f(x,y)$ bir jinsli funksiya va $x \neq 0$ bo'lsa, u holda

$$f(x,y)=f\left(\frac{1}{x}\cdot x,\frac{1}{x}\cdot y\right)=f\left(1,\frac{y}{x}\right)=g\left(\frac{y}{x}\right),$$ bunda

$$g(t)\overset{def}{=}f(1,t).$$

Aksincha, bir o'zgaruvchining $g(t)$ funksiyasi orqali ifodalangan $f(x,y)=g\left(\frac{y}{x}\right)$ ($x\neq 0$ bo'lganda) (yoki $f(x,y)=g\left(\frac{x}{y}\right)$ ($y\neq 0$ bo'lganda)) funksiya bir jinslidir.

Bir jinsli tenglamada

$$y=xu \tag{1.6.2}$$

deb, yangi $u=u(x)$ noma'lum funksiyaga o`tamiz. U holda

$$y'=u+xu'$$

va berilgan tenglama

$$u+xu'=f(x,xu)$$

yoki, $f(x,xu)=f(1,u)=g(u)$ bo'lgani uchun,

$$xu'=g(u)-u$$

ko`rinishni oladi. Bu o`zgaruvchilari ajraladigan differensial tenglamadir. Oxirgi tenglamaning $u=u(x)$ yechimi topilgach, (1.6.2) formulaga ko`ra berilgan tenglamaning $y=xu(x)$ yechimini hosil qilamiz.

Misol. Ushbu

$$xy'=y\ln\frac{y}{x},\quad y|_{x=1}=e^2,$$

boshlang`ich masalani yeching.

⊶ Dastlab berilgan tenglamaning barcha yechimlarini topamiz. So`ngra ular orasidan ko`rsatilgan boshlang'ich shartni qanoatlantiradiganini ajratamiz. Berilgan tenglama o`zgaruvchilarga nisbatan bir jinsli. Yangi $u=u(x)$ noma'lum funksiyani $y=xu$ formula bilan kiritamiz. Zarur hisoblashlarni va shakl almashtirishlarni bajaramiz:

$$y' = u + xu',\ y' = \frac{y}{x}\ln\frac{y}{x},\ u + xu' = u\ln u,$$

$$xu' = u(\ln u - 1).$$

Oxirgi tenglamada o'zgaruvchilarni ajratamiz va integrallashlarni bajaramiz:

$$\frac{du}{u(\ln u - 1)} = \frac{dx}{x},\quad u = e^{1+cx}.$$

Tenglamani $u(\ln u - 1)$ ga bo'lishda $u = e$, yechim yo'qolishi mumkin. Lekin bu yechim $u = e^{1+cx}$ formuladan $c = 0$ da hosil bo'ladi. Demak, berilgan tenglamaning barcha yechimlari $y = xu = xe^{1+cx}$ formula bilan beriladi. $y|_{x=1} = e^2$ boshlang'ich shart qanoatlanishi uchun $e^2 = e^{1+c}$, ya'ni $c = 1$ bo'lishi kerak. Shunday qilib, berilgan Koshi masalasining yechimi umumiy yechim formulasidan $c = 1$ da hosil bo'ladi: $y = xe^{1+x}$.

Ushbu

$$y' = \frac{y}{x} + h(x)g\left(\frac{y}{x}\right)$$

ko'rinishdagi tenglamani ham $y = xu$ almashtirish yordamida yechish mumkin.

Ba'zi tenglamalarni noma'lum funksiyani almashtirish yordamida o'zgaruvchilariga nisbatan bir jinsli tenglamaga keltirish mumkin.

Masalalar

1. Ushbu

$$y' = \frac{y}{x} + h(x)g\left(\frac{y}{x}\right)$$

tenglamani umumiy holda yeching.

2. Ushbu $x^3y' = y^2 + x^4$. tenglamani yeching.

1.7. Chiziqli tenglama. Bernulli va Rikkati tenglamalari

Chiziqli tenglama. Ushbu

$$y' + p(x)y = q(x), \quad \{p, q\} \subset C(I), \qquad (1.7.1)$$

tenglama birinchi tartibli **chiziqli differensial tenglama** deyiladi. Bu yerdagi $q(x)$ ozod had deb ataladi. Ozod had nolga teng bo'lganda hosil bo'luvchi ushbu

$$y' + p(x)y = 0 \qquad (1.7.2)$$

tenglama (1.7.1) ga mos **bir jinsli tenglama** deb ataladi. (1.7.2) o'zgaruvchilari ajraladigan tenglamaning umumiy yechimini topish oson. Biz bu yerda umumiy yechimni boshqa usulda topamiz.

Lemma. *Ushbu*

$$y' + p(x)y = 0 \qquad (1.7.2)$$

differensial tenglamaning umumiy yechimi

$$y = c\exp\left(-\int_{x_0}^{x} p(s)ds\right) \qquad (1.7.3)$$

ko'rinishda bo'ladi; bunda $x_0 \in I$ – tayinlangan nuqta.

⊸ Osongina tekshirib ko'rish mumkinki, (1.7.3) formula bilan berilgan funksiya (1.7.2) differensial tenglamaning I oraliqda aniqlangan yechimi. Endi ixtiyoriy yechimning (1.7.3) ko'rinishda ekanligini isbotlaymiz. Ixtiyoriy $y(x)$ yechimni olaylik:

$$y'(x) + p(x)y(x) = 0.$$

Bu tenglikni $\exp\left(\int_{x_0}^{x} p(s)ds\right) > 0$ ga ko'paytirib,

$$\left(y(x)\cdot\exp\left(\int_{x_0}^{x} p(s)ds\right)\right)' = 0$$

munosabatni hosil qilamiz. Demak, analizdan ma'lum teoremaga ko'ra

$$y(x)\cdot\exp\left(\int_{x_0}^{x} p(s)ds\right) = c \;\; (c - \text{const}) \;\Rightarrow\; y(x) = c\exp\left(-\int_{x_0}^{x} p(s)ds\right). \;\text{⊸}$$

Tushunarliki, (I.7.3) formuladagi c o'zgarmas son noma'lum funksiyaning x_0 nuqtadagi qiymatiga teng, ya'ni $c = y(x_0)$. Ravshanki, $I \times (-\infty, +\infty)$ polosaning har bir (x_0, y_0) nuqtasidan (2) tenglamaning yagona integral chizig'i o'tadi. U

$$y = y_0 \exp\left(-\int_{x_0}^{x} p(s)ds \right)$$

formula bilan beriladi. Demak, $I \times (-\infty, +\infty)$ polosa (I.7.2) tenglamaning yagonalik sohasi.

Shunday qilib, (I.7.3) formula bir jinsli tenglama (I.7.2) ning $I \times (-\infty, +\infty)$ sohadagi umumiy yechimini beradi, ya'ni (I.7.2) tenglamaning $I \times (-\infty, +\infty)$ sohadagi barcha yechimlari va ulargina (I.7.3) formula bilan aniqlanadi..

Bir jinsli bo'lmagan (I.7.1) tenglamaning yechimini

$$y = v(x) \cdot \exp\left(-\int_{x_0}^{x} p(s)ds \right) \tag{I.7.4}$$

ko'rinishda izlaylik (mos bir jinsli tenglamaning umumiy yechimi (I.7.2) dagi ixtiyoriy o'zgarmasni «variatsiyalab», ya'ni o'zgartirib. (I.7.1) ning yechimini quramiz); bu – **Lagranj metodi**. (I.7.4) ni (I.7.1) ga qo'yib,

$$v'(x) \cdot \exp\left(-\int_{x_0}^{x} p(s)ds \right) = q(x)$$

tenglikni hosil qilamiz. Bundan

$$v'(x) = q(x)\exp\left(\int_{x_0}^{x} p(s)ds \right),$$

$$v(x) = c + \int_{x_0}^{x} q(\tau)\exp\left(\int_{x_0}^{\tau} p(s)ds \right) d\tau \tag{I.7.5}$$

(I.7.5) ni (I.7.4) ga qo'yib, (I.7.1)ning umumiy yechimini hosil qilamiz:

$$y = c\exp\left(-\int_{x_0}^{x} p(s)ds\right) +$$

$$+\exp\left(-\int_{x_0}^{x} p(s)ds\right)\int_{x_0}^{x} q(\tau)\exp\left(-\int_{x_0}^{\tau} p(s)ds\right)d\tau \quad . \tag{I.7.6}$$

(I.7.6) formuladagi birinchi qo'shiluvchi bir jinsli tenglama (I.7.2) ning umumiy yechimini, ikkinchi qo'shiluvchi esa (I.7.1) ning xususiy (biror) yechimini beradi.

Shunday qilib, bir jinsli bo'lmagan (I.7.1) tenglamaning umumiy yechimi uning xususiy yechimiga mos bir jinsli tenglama (I.7.2) ning umumiy yechimini qo'shishdan hosil bo'ladi.

Tenglamaning yagona (birorta) xususiy yechimini ajratish uchun qo'shimcha shart qo'yish kerak. (I.7.1) tenglamaning $x_0 \in I$ nuqtada berilgan y_0 qiymatni qabul qiluvchi, ya'ni

$$y(x_0) = y_0 \tag{I.7.7}$$

shartni qanoatlantiruvchi yechimi (I.7.6) formuladan osongina topiladi. Bu yechim bitta va u

$$y = y_0\exp\left(-\int_{x_0}^{x} p(s)ds\right) + \exp\left(-\int_{x_0}^{x} p(s)ds\right)\int_{x_0}^{x} q(\tau)\exp\left(-\int_{x_0}^{\tau} p(s)ds\right)d\tau \tag{I.7.8}$$

formula bilan ifodalanadi. Demak, $I \times (-\infty, +\infty)$ polosa (I.7.1) tenglamaning yagonalik sohasidir. Ravshanki, (I.7.1),(I.7.7) boshlang'ich masalaning (I.7.8) yechimi I oraliqda, ya'ni (I.7.1) tenglamada berilgan funksiyalarning uzluksizlik oralig'ida aniqlangan.

Bernulli tenglamasi deb ushbu

$$y' = p(x)y + q(x)y^{\alpha} \quad (\alpha \neq 1, \alpha \neq 0, \{p(x), q(x)\} \subset C(I)) \tag{I.7.9}$$

ko'rinishdagi tenglamaga aytiladi. Bernulli tenglamasi $u(x) = y^{1-\alpha}$ almashtirish yordamida

$$u' = (1-\alpha)p(x)u + (1-\alpha)q(x)$$

chiziqli tenglamaga keltiriladi.

Bernulli tenglamasini **Eyler-Bernulli usuli** deb ataluvchi usul bilan ham yechish mumkin. Bu usulga ko'ra yechim $y=uv$ ko'rinishda izlanadi, bunda u,v – hozircha noma'lum funksiyalar. $y=uv$ ni berilgan tenglamaga qo'yamiz:

$$u'v+uv'-p(x)uv=q(x)y^{\alpha},$$

$$\big(u'-p(x)u\big)v+uv'=q(x)u^{\alpha}v^{\alpha}.$$

Endi $u'-p(x)u=0$, ya'ni $u=\exp\left(\int p(x)dx\right)$ deb, v uchun $uv'=q(x)u^{\alpha}v^{\alpha}$ tenglamaga kelamiz. v ga nisbatan oxirgi tenglamani yechib, topilgan u,v larga ko'ra Bernulli tenglamasining $y=uv$ yechimini topamiz.

Rikkati tenglamasi deb

$$y'=a(x)y^2+b(x)y+c(x) \qquad (1.7.10)$$

ko'rinishdagi tenglamaga aytiladi; bunda $\{a(x),b(x),c(x)\}\subset C(I), a(x)\neq 0, b(x)\neq 0$ deb hisoblanadi ($a(x)=0$ bo'lganda chiziqli tnglama hosil bo'ladi, $c(x)=0$ bo'lganda esa – Bernulli tenglamasi). Rikkati tenglamasi differensial tenglamalar nazariyasida alohida o'rin tutadi. Bu tenglama amaliy masalalarni yechishda ko'p uchraydi.

Agar Rikkati tenglamasidagi koeffitsientlar o'zgarmas, ya'ni $a(x)=a=\text{const}$, $b(x)=b=\text{const}$, $c(x)=c=\text{const}$ bo'lsa, u o'zgaruvchilari ajraladigan tenglamaga aylanadi va yechimi kvadraturalarda ifodalanadi.

Agar Rikkati tenglamasining biror $y=\varphi(x)$ xususiy yechimi ma'lum bo'lsa, uning umumiy yechimini topish mumkin. Buning uchun tenglamada $y=\varphi(x)+u$ almashtirishni bajarish kerak, bunda u – yangi noma'lum funksiya. U holda ushbu

$$u'=(2a(x)\varphi(x)+b(x))u+a(x)u^2$$

Bernulli tenglamasini hosil qilamiz. Oxirgi tenglamani yechish uchun $u=\frac{1}{z}$ deb yangi $z=z(x)$ noma'lum funksiyani kiritish mumkin; bunda chiziqli tenglama hosil bo'ladi. Demak, yangi noma'lum funksiya $z = z(x)$ ni

$$y = \varphi(x) + \frac{1}{z}$$

formula bilan kiritib, z ga nisbatan chiziqli tenglamaga kelamiz.

Quyidagi ikki holda xususiy yechim osongina topiladi:

$c(x) = -a(x)d^2 - b(x)d$ bo'lganda $y = \varphi(x) = d$ xususiy yechim;

$c(x) = -a(x)x^2 - b(x)x + 1$ bo'lganda esa

$y = \varphi(x) = x$ xususiy yechim.

Umumiy holda Rikkati tenglamasining yechimi kvadraturalarda ifodalanmaydi. Ba'zi maxsus hollardagina u kvadratularda yechiladi. Shu hollarning ba'zilarini keltiraylik:

1) $y' = f(x)(ay^2 + by + c)$, 2) $y' = a\frac{y^2}{x^2} + b\frac{y}{x} + c$,

3) $y' = ay^2 + \frac{b}{x}y + \frac{c}{x^2}$

(uchala holda ham a, b, c koeffitsientlar o'zgarmas sonlardan iborat). 1) tipdagi tenglamalarda o'zgaruvchilar ajraladi, 2) tipdagilari o'zgaruvchilariga nisbatan bir jinsli tenglamalar, 3) tipdagilari esa $y = z / x$ almashtirish yordamida o'zgaruvchilari ajraladigan tenglamaga keltiriladi.

Noma'lum funksiyani almashtirish yoramida Rikkati tenglamasining ko'rinishini soddalashtirish mumkin. y noma'lum funksiya o'rniga yangi $v = \alpha(x)y$ noma'lum funksiyani kiritib, ushbu

$$v' = \frac{a(x)}{\alpha(x)}v^2 + \left(b(x) + \frac{\alpha'(x)}{\alpha(x)}\right)v + c(x)\alpha(x)$$

tenglamani hosil qilamiz. Endi $\alpha(x)$ sifatida $a(x)$ ni tanlab ($\alpha(x) = a(x)$), tenglamada noma'lum funksiya kvadrati oldidagi koeffitsientni birga tenglashtiramiz:

$$v' = v^2 + \bar{b}(x)v + \bar{c}(x),$$

bu yerda $\quad \tilde{b}(x) = b(x) + \dfrac{a'(x)}{a(x)}, \quad \tilde{c}(x) = c(x)a(x).$

Agar v noma'lum funksiya o'rniga yangi $u = \beta(x) + y$ noma'lum funksiyani kiritsak, $\beta(x)$ ni tanlash evaziga tenglamada noma'lum funksiya qatnashgan hadni yo`qotish mumkin.

Rikkati tenglamasining yana bir xususiy holida to'xtalaylik. Agar $a(x) = a = \text{const}, \ b(x) \equiv 0,$

$c(x) = cx^m \quad (c, m$ – o'zgarmaslar) bo'lsa, ushbu

$$\frac{dy}{dx} = ay^2 + cx^m \tag{I.7.11}$$

maxsus Rikkati tenglamasi deb ataluvchi tenglamaga kelamiz. Bu tenglama yechimlarining kvadraturalarda ifodalanishi yoki ifodalanmasligi to'la o'rganilgan. $m = 0$ bo'lganda (I.7.11) - o'zgaruvcilari ajraladigan tenglama, $m = -2$ bo'lganda esa u $y = -1/u$ almashtirish bilan ushbu $\dfrac{du}{dx} = a + c\dfrac{u^2}{x^2}$ bir jinsli tenglamaga keltiriladi. Demak, bu hollarda (I.7.11) maxsus Rikkati tenglamasi kvadraturalarda yechiladi. m ning (I.7.11) tenglama kvadraturalarda yechiladigan boshqa qiymatlarini topish maqsadida tenglamaning ko`rinishini o`zgartirmaydigan quyidagi almashtirishni qaraylik: y funksiyaning o`rniga $y_1$ ni, $x$ argumentning o`rniga esa x_1 ni

$$y = \frac{1}{x^2 y_1} - \frac{1}{ax}, \quad x = x_1^{m+3} \tag{I.7.12}$$

formulalar bilan kiritaylik. Natijada ushbu

$$\frac{dy_1}{dx_1} = a_1 y_1^2 + c_1 x_1^{m_1} \tag{I.7.13}$$

tenglamani hosil qilamiz. bunda $a_1 = \dfrac{c}{m+3}, \ c_1 = -\dfrac{a}{m+3}, \ m_1 = -\dfrac{m+4}{m+3}$. m va m_1 lar orasidagi munosabatni

$$\frac{1}{m_1+2}=1+\frac{1}{m+2} \qquad (1.7.14)$$

ko`rinishda yozish mumkin. Yuqoridagiga o`xshash almashtirishni (1.7.13) tenglmada bajarib

$$\frac{dy_2}{dx_2}=a_2y_2^2+c_2x_2^{m_2}$$

tenglamaga kelamiz, bunda (1.7.14) ga ko`ra

$$\frac{1}{m_2+2}=1+\frac{1}{m_1+2}=2+\frac{1}{m+2}.$$

Bunday almashtirishlarni k marta bajarib, m o`rniga m_k,

$$\frac{1}{m_k+2}=k+\frac{1}{m+2},\ k=1,2,3,\ldots, \qquad (1.7.15)$$

ko`rsatkichni hosil qilamiz. Agar (1.7.11) tenglamadan boshlab yuqoridagi almashtirishlarni teskari tartibda bajarsak, $m_{-1}, m_{-2}, \ldots, m_{-k}$ ko`rsatkichli tenglamalarni hosil qilamiz, bunda

$$\frac{1}{m_{-k}+2}=-k+\frac{1}{m+2},\ k=1,2,3,\ldots \qquad (1.7.16)$$

Agar $m=-2$ bo`lsa, $m_1=m_{-1}=-2$ yuqoridagi almashtirishlar $m=-2$ ko`rsatkichni o`zgartirmaydi. Agar biror $k$ da $m_k=0$ yoki $m_{-k}=0$ bo`lsa, hosil bo`lgan tenglama va, demak, dastlabki (1.7.11) tenglama ham kvadraturalarda yechiladi. Demak, (1.7.15) va (1.7.16) ga ko`ra, agar m ko`rsatkich

$$\frac{1}{m+2}=-k+\frac{1}{2},\ \text{ya'ni}\ m=\frac{4k}{1-2k},k=\pm1,\pm2,\ldots,$$

shartni $\left(\frac{m}{2m+4}-\text{butun son}\right)$ bajarsa, (1.7.11) maxsus Rikkati tenglamasi kvadraturalarda yechiladi. m ning qolgan boshqa qiymatlarida bu tenglamaning yechimi elementar funksiyalarning chekli sondagi integrallari orqali ifodalanmasligini Liuvill 1841 yil isbotlagan.

Masalalar

1. Agar

$$y' + p(x)y = q(x), \quad \{p,q\} \subset C(I).$$

chiziqli tenglamaning ikki dona $y = y_1(x)$ va $y = y_2(x)$ yechimlari ma'lum bo'lsa, uning umumiy yechimi

$$y = y_1(x) + c\left(y_2(x) - y_1(x)\right), \ c - const,$$

formula bilan berilishini ko'rsating.

2. p va q funksiyalar $t \in [0, +\infty)$ oraliqda aniqlangan va uzluksiz bo'lsin. Agar $x = x(t)$ va $u = u(t)$ funksiyalari uchun

$$x' + p(t)x = q(t), x(0) = x_0,$$

$$u' + p(t)u \geq q(t), u(0) \geq x_0,$$

bo'lsa, u holda $t \in [0, +\infty)$ oraliqda $u(t) \geq x(t)$ tengsizlik o'rinli. Shuni isbotlang.

3. Tenglamani yeching $x(e^y - y') = a \quad (a = \text{const} \neq 1)$.

4. Tenglamani yeching $(x^2 - 1)y' \sin y - 2x \cos y = x^2$.

5. Ushbu

$$y' = c(y + a)(y + b), \ (a, b, c - \text{const})$$

Rikkati tenglamasida $z = 1/(y + a)$ almashtirishni bajaring.

6. Agar $y(x)$, $y_1(x)$, $y_2(x)$, $y_3(x)$ – funksiyalar

$$y' = p(x)y^2 + q(x)y + r(x)$$

($p(x), q(x), r(x)$ – uzluksiz funksiyalar) Rikkati tenglamasining yechimlari bo'lsa, ushbu

$$\frac{y_2(x) - y(x)}{y_2(x) - y_1(x)} : \frac{y_3(x) - y(x)}{y_3(x) - y_1(x)}$$

nisbatning o'zgarmas ekanligini ko'rsating.

7. Rikkati tenglamasining har qanday yechimi uning uchta yechimi orqali ifodalanishini isbotlang.

8. Tenglamani yeching $y' = y^2 + x^{-4/3}$.

I.8. To'la differensialli tenglama va integrallovchi ko'paytuvchi

To'la differensialli tenglama. x va y o'zgaruvchilar teng huquqli qatnashgan differensial (differensiallarda yozilgan) tenglamani qaraylik:

$$M(x,y)dx + N(x,y)dy = 0 \quad (\{M,N\} \subset C(D)). \qquad (I.8.1)$$

Agar D sohada (I.8.1) tenglamaning chap tomonidagi differensial ifoda **potensial** deb ataluvchi biror $u(x,y) \in C^1(D)$ funksiyaning to'la differensialidan iborat, ya'ni

$$du = M(x,y)dx + N(x,y)dy \qquad (I.8.2)$$

bo'lsa, u holda (I.8.1) tenglama D sohada **to'la (to'liq) differensialli tenglama** deyiladi.

To'la differensialli tenglama uchun differensial ta'rifi va (I.8.2) tenglikka ko'ra

$$\frac{\partial u}{\partial x} = M, \qquad \frac{\partial u}{\partial y} = N \qquad (I.8.3)$$

va, demak, bu tenglama $du(x,y) = 0$ ko'rinishga keladi. Bu holda, ravshanki, $y = \varphi(x) \in C^1(I)$ (yoki $x = \psi(y) \in C^1(I)$) funksiya (I.8.1) ning yechimi bo'lishi uchun I oraliqda $u(x,\varphi(x)) \equiv \text{const}$ (yoki $u(\psi(y),y) \equiv \text{const}$) ayniyat bajarilishi kerak.

Demak, oshkormas funksiya haqidagi teoremaga ko'ra to'la differensialli (I.8.1) tenglamaning har qanday regulyar nuqtasi atrofida integral chiziqlari ushbu

$$u(x,y) = c \quad (c - \text{const}) \qquad (I.8.4)$$

tenglama bilan oshkormas ko'rinishda beriladi.

Bu yerda shuni e'tirof etaylikki, u potensial ixtiyoriy additiv o'zgarmas aniqligida topiladi, chunki ixtiyoriy o'zgarmas c uchun u bilan birgalikda $u + c$ ham potensial bo'ladi.

$Mdx + Ndy$ differensial ifoda biror funksiyaning to'la differensialidan iborat bo'lishi uchun shartlar matematik analiz kursida o'rganiladi. Biz bu yerda analizda isbotlanadigan quyidagi teoremani keltiramiz.

Teorema. *Aytaylik, D soha bir bog'lamli,* $\dfrac{\partial M}{\partial y}$ *va* $\dfrac{\partial N}{\partial x}$ *hosilalar D da mavjud va uzluksiz bo'lsin. U holda (I.8.1) tenglamaning D to'la differensialli bo'lishi uchun D ning har bir nuqtasida*

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \tag{I.8.5}$$

shartning bajarilishi yetarli va zarurdir. Bu shart bajarilganda

$$u(x,y)=\int_{(x_0,y_0)}^{(x,y)} M(x,y)dx+N(x,y)dy \tag{I.8.6}$$

funksiyaning to'la differensiali $Mdx+Ndy$ *ifodadan iborat bo'ladi; bu yerdagi integral* (x_0,y_0) *va* (x,y) *nuqtalarni D da tutashtiruvchi ixtiyoriy yo'l bo'ylab olingan 2 – tur egri chiziqli integraldir ((I.8.6) integralning qiymati integrallash yo'liga bog'liq bo'lmaydi).*

Shunday qilib, to'la differensialli tenglamani integrallash (I.8.6) formulaga ko'ra $u(x,y)$ potensialni topishga keltirildi, chunki to'la differensialli tenglamaning yechimi (I.8.4) formula bilan oshkormas ko'rinishda beriladi.

Misol 1. Ushbu

$$(x^2+2y)dx+(2x+y^2)dy=0 \tag{I.8.7}$$

tenglama $D=\mathbb{R}^2$ tekislikda to'la differensiallidir, chunki bu holda

$$M=x^2+2y,\ N=2x+y^2 \quad \text{va} \quad \frac{\partial M}{\partial y}=2=\frac{\partial N}{\partial x}.$$

$u=u(x,y)$ potensialni topish uchun (I.8.3) shartlardan foydalanamiz;

$$u'_x=x^2+2y,\ u'_y=2x+y^2;$$

$$u'_x=x^2+2y \Rightarrow u=\frac{x^3}{3}+2xy+\varphi(y),\ u'_y=2x+\varphi'(y);$$

$$u'_y = 2x + y^2 = 2x + \varphi'(y) \Rightarrow \varphi'(y) = y^2 \Rightarrow \varphi(y) = \frac{y^3}{3};$$

$$u = \frac{x^3}{3} + 2xy + \varphi(y) = \frac{x^3}{3} + 2xy + \frac{y^3}{3}.$$

Qaralayotgan misolda u funksiyani differensial xossalaridan foydalanib, bevosita topsa ham bo'ladi:

$$(x^2 + 2y)dx + (2x + y^2)dy = x^2dx + y^2dy + 2ydx + 2xdy =$$

$$= d\left(\frac{x^3}{3}\right) + d\left(\frac{y^3}{3}\right) + 2d(xy) =$$

$$= d\left(\frac{x^3}{3} + \frac{y^3}{3} + 2xy\right).$$

Demak, (1.8.7) tenglamaning yechimi

$$\frac{x^3}{3} + \frac{y^3}{3} + 2xy = \frac{c}{3}$$

yoki

$$x^3 + y^3 + 6xy = c$$

oshkormas ko'rinishda beriladi.

Integrallovchi ko'paytuvchi. «Agar berilgan tenglama uchun (1.8.5) shart bajarilmasa, tenglamani biror "integrallovchi" ko'paytuvchiga ko'paytirib, uni to'la differensialli ko'rinishga keltirish mumkinmi?» – degan savol tug'iladi.

Agar D sohada nolga aylanmaydigan $\mu = \mu(x, y) \in C^1(D)$ funksiya uchun

$$\mu M dx + \mu N dy = 0 \qquad (1.8.8)$$

tenglama to'liq differensialli bo'lsa, u holda μ funksiya (1.8.1) tenglamaning **integrallovchi ko'paytuvchisi** deyiladi.

Faraz qilaylik, D – bir bog'lamli soha va $\{M, N\} \subset C^1(D)$ bo'lsin. (1.8.8) dan $\mu \in C^1(D)$ integrallovchi ko'paytuvchi uchun

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \qquad (1.8.9)$$

shartni hosil qilamiz. (I.8.9) dan

$$M\frac{\partial \mu}{\partial y}-N\frac{\partial \mu}{\partial x}=\mu\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) \qquad (I.8.10)$$

Shunday qilib, μ integrallovchi ko'paytuvchi xususiy hosilali differensial tenglama (I.8.10) ning yechimi sifatida aniqlanishi kerak. Umumiy holda bu tenglamani yechish dastlabki tenglama (I.8.1) ni yechishdan oson emas. Lekin ba'zi hollarda (I.8.10) dan integrallovchi ko'paytuvchi μ ni topish uchun foydalanish mumkin.

Integrallovchi ko'paytuvchini biror $\omega=\omega(x,y)$ funksiyaning

$$\mu=\mu(\omega) \qquad (I.8.11)$$

funksiyasi sifatida izlab ko'raylik. (I.8.11)ni (I.8.10) ga qo'yamiz:

$$\left(M\frac{\partial \omega}{\partial y}-N\frac{\partial \omega}{\partial x}\right)\frac{d\mu}{d\omega}=\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\mu \qquad (I.8.12)$$

Bu tenglik qanoatlanishi uchun

$$\frac{\dfrac{\partial N(x,y)}{\partial x}-\dfrac{\partial M(x,y)}{\partial y}}{M(x,y)\dfrac{\partial \omega(x,y)}{\partial y}-N(x,y)\dfrac{\partial \omega(x,y)}{\partial x}}\equiv f(\omega) \qquad (I.8.13)$$

bo'lishi kerak, ya'ni (I.8.13) ning chap tomonidagi x va y ga bog'liq bo'lgan funksiya $\omega=\omega(x,y)$ ning funksiyasi sifatida ifodalanishi lozim. Bu holda (I.8.13) ni (I.8.12) ga qo'yib,

$$\mu=e^{\int f(\omega)d\omega} \qquad (I.8.14)$$

integrallovchi ko'paytuvchini topamiz. (I.8.13) tenglik $\mu=\mu(\omega)$ ko'rinishdagi integrallovchi ko'paytuvchining mavjudlik shartini beradi. Mashqlar bajarganda ω funksiyani $\omega(x,y)\equiv y$, $\omega(x,y)\equiv x$, $\omega(x,y)\equiv xy$, $\omega(x,y)\equiv x^2+y^2$ va hokazo ko'rinishlarda tanlashga harakat qilib ko'rish mumkin.

Umumiy holda itegrallovchi ko'paytuvchining mavjud bo'lishi uchun yetarli va zaruriy shartlar Li gruppalari nazariyasida o'rganiladi.

Misol 2. Ushbu

$$3ydx + x(\ln y + 2\ln x + 1)dy = 0 \quad (x > 0, y > 0)$$

differensial tenglamani yeching.

Bu tenglama to'la differensialli emas. $\mu = \mu(\omega)$, $\omega = \omega(x, y)$, ko'rinishdagi integrallovchi ko'paytuvchini topishga harakat qilamiz. Integrallovchi ko'paytuvchini

$$3y\mu dx + x(\ln y + 2\ln x + 1)\mu dy = 0$$

tenglamaning to'la differensialli bo'lishi shartidan, ya'ni

$$(3y\mu)'_y = (x(\ln y + 2\ln x + 1)\mu)'_x$$

tenglamadan topamiz. Zarur hisoblashlarni bajarib, oxirgi tenglamani quyidagi ko'rinishga keltiramiz:

$$(3y\omega'_y - x(\ln y + 2\ln x + 1)\omega'_x)\mu' = (\ln y + 2\ln x)\mu .$$

Bu tenglamaning ko'rinishidan kelib chiqib, $\omega = \omega(x, y)$ funksiya uchun

$$\omega'_x = -\frac{1}{x} \quad \text{va} \quad 3y\omega'_y + 1 = 0$$

shartlarni qo'yamiz va $\mu = \mu(\omega)$ funksiya uchun $\mu' = \mu$, ya'ni $\mu = e^{\omega}$ ekanligini topamiz. Endi zarur integrallashlarni va ixchamlashtirishlarni bajarib, integrallovchi ko'paytuvchini aniqlaymiz:

$$\omega'_x = -\frac{1}{x} \Rightarrow \omega = -\ln x + \psi(y),\ \omega'_y = \psi'(y)$$

$$3y\omega'_y + 1 = 0 \Rightarrow \omega'_y = -\frac{1}{3y} = \psi'(y) \Rightarrow \psi(y) = -\frac{1}{3}\ln y ;$$

$$\omega = -\ln x + \psi(y) = -\ln x - \frac{1}{3}\ln y = \ln\frac{1}{xy^{1/3}} ;$$

$$\mu = e^{\omega} = \exp(\ln\frac{1}{xy^{1/3}}) = \frac{1}{xy^{1/3}} .$$

Endi berilgan differensial tenglamani topilgan $\mu = \frac{1}{xy^{1/3}}$ integrallovchi ko'paytuvchiga ko'paytirib, ushbu

$$\frac{3}{x}y^{2/3}dx + \frac{1}{y^{1/3}}(\ln y + 2\ln x + 1)dy = 0$$

to'la differensialli tenglamani hosil qilamiz. Bu tenglamaning $u = u(x, y)$ potensiali uchun

$$u'_x = \frac{3}{x}y^{2/3}, \; u'_y = \frac{1}{y^{1/3}}(\ln y + 2\ln x + 1)$$

bo'lishi kerak. Oxirgi ikki shartdan $u = u(x, y)$ funksiyani osongina topamiz:

$$u = 3y^{2/3}\ln x + \frac{3}{2}y^{2/3}\ln y - \frac{3}{4}y^{2/3}.$$

Demak, berilgan tenglamaning umumiy yechimi

$$y^{2/3}(4\ln x + 2\ln y - 1) = c$$

tenglama bilan oshkormas ko'rinishda beriladi.

Ba'zan (I.8.1) tenglamaning integrallovchi ko'paytuvchisini topish uchun chap tomonni ikkiga ajratib, tuzilgan tenglamalarning integrallovchi ko'paytuvchilarini topib, ular orqali izlangan integrallovchi ko'paytuvchini qurish mumkin bo'ladi. Aniqrog'i, quyidagicha ish tutish mumkin. (I.8.1) tenglamani

$$Mdx + Ndy = (M_1dx + N_1dy) + (M_2dx + N_2dy) = 0$$

$$(M = M_1 + M_2, \; N = N_1 + N_2)$$

ko'rinishda yozib, $M_1dx + N_1dy = 0$ va $M_2dx + N_2dy = 0$ tenglamalar uchun μ_1 va μ_2 integrallovchi ko'paytuvchilarni topaylik; u holda

$$\mu_1M_1dx + \mu_1N_1dy = du_1 \text{ va } \mu_2M_2dx + \mu_2N_2dy = du_2$$

tengliklarga ega bo'lamiz. Ravshanki, ixtiyoriy noldan farqli uzluksiz φ_1 va φ_2 funksiyalar uchun

$$\varphi_1(u_1)\mu_1(M_1dx+N_1dy)=\varphi_1(u_1)(\mu_1M_1dx+\mu_1N_1dy)=$$
$$=\varphi_1(u_1)du_1=d\left(\int\varphi_1(u_1)du_1\right)$$
$$\varphi_2(u_2)\mu_2(M_2dx+N_2dy)=d\left(\int\varphi_2(u_2)du_2\right)$$
bo'ladi. Endi, agar φ_1 va φ_2 funksiyalarni ushbu
$$\varphi_1(u_1)\mu_1=\varphi_2(u_2)\mu_2$$
shartdan topsak, u holda ravshanki,
$$\varphi_1(u_1)\mu_1(Mdx+Ndy)=\varphi_1(u_1)\mu_1(M_1dx+N_1dy)+\varphi_2(u_2)\mu_2(M_2dx+N_2dy)$$
$$=d\left(\int\varphi_1(u_1)du_1+\int\varphi_2(u_2)du_2\right)$$
bo'ladi. Demak, berilgan (I.8.1) tenglamaning integrallovchi ko'paytuvchisi
$$\mu=\varphi_1(u_1)\mu_1=\varphi_2(u_2)\mu_2$$
funksiyadan iborat, yechimi esa
$$u\equiv\int\varphi_1(u_1)du_1+\int\varphi_2(u_2)du_2=c$$
munosabat bilan beriladi.

Masalalar

1. Ushbu
$$\frac{y}{x^2+y^2}dx-\frac{x}{x^2+y^2}dy$$
differensial ifoda bir bog'lamli bo'lmagan $D=\mathbb{R}^2\setminus(0;0)$ sohada aniqlangan, shu sohada silliq koeffitsientli va $\frac{\partial}{\partial y}\frac{y}{x^2+y^2}=\frac{\partial}{\partial x}\frac{x}{x^2+y^2}$ shart bajarilishini tekshiring. D sohada $du=\frac{y}{x^2+y^2}dx-\frac{x}{x^2+y^2}dy$ shartni qanoatlantiruvchi $u=u(x,y)$ funksiyaning mavjud emasilini isbotlang.

2. Aytaylik, D soha biror $A_0\in D$ nuqtaga nisbatan yulduzsimon bo'lsin, ya'ni ixtiyoriy $A\in D$ nuqta bilan birgalikda A_0A

[illegible] ham D da yotsin. Bundan tashqari, M va N funksiyalar D da C^1 sinfga tegishli ham bo'lsin. U holda, agar D da $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ bo'lsa, $A_0 = (x_0, y_0)$ deb,

$$u(x,y) = \int_0^1 \left[M(t)\cdot(x - x_0) + N(t)\cdot(y - y_0) \right] dt,$$

$$M(t) = M(x_0 + t(x - x_0),\ y_0 + t(y - y_0)),$$

$$N(t) = N(x_0 + t(x - x_0),\ y_0 + t(y - y_0)),$$

funksiyani tuzsak, uning uchun D da $du(x,y) = M(x,y)dx + N(x,y)dy$ bo'lishini isbotlang.

3. Differensial tenglamani yeching

$$(2x^2 - y^2 + y)dx + x(2y - y)dy = 0.$$

1.9. Koshi masalasi yechimining mavjudligi va yagonaligi

Lipshits sharti. Aytaylik, $f(x,y)$ funksiya biror $E \subset \mathbb{R}^2$ to'plamda aniqlangan bo'lsin. Agar

$$\exists L > 0 \quad \forall \{(x,y_1),(x,y_2)\} \subset E \quad |f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2|$$

shart bajarilsa, u holda $f(x,y)$ funksiya E to'plamda y ga nisbatan (yoki y bo'yicha) ***Lipshits shartini qanoatlantiradi*** deyiladi.

Jumla. *Aytaylik, D ochiq yoki yopiq soha Oy o'qiga nisbatan qavariq bo'lsin, ya'ni*

$$\{(x,y_1),(x,y_2)\} \subset D \Rightarrow \{(x, y_1 + \theta(y_2 - y_1)) \mid 0 < \theta < 1\} \subset D.$$

Agar $f(x,y)$ funksiya D da y bo'yicha uzluksiz va uning ichki nuqtalarida chegaralangan $\frac{\partial f}{\partial y}$ xususiy hosilaga ega bo'lsa, u holda $f(x,y)$ funksiya D da y ga nisbatan Lipshits shartini qanoatlantiradi.

Chekli orttirmalar haqidagi Lagranj teoremasidan foydalanib, mustaqil isbotlang.

Koshi masalasi yechiming mavjudligi va yagonaligi.

Ushbu

$$\begin{cases} y' = f(x,y) & (1.9.1) \\ y\big|_{x_0} = y_0 & (1.9.2) \end{cases} \qquad (K)$$

Koshi masalasiga qaytaylk. Bu masala – (x_0, y_0) nuqta orqali berilgan differensial tenglamaning integral chizig'ini o'tkazish masalasi.

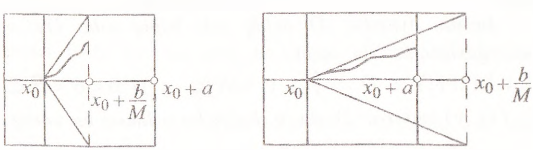
Qo'yilgan (K) Koshi masalasi yechimining mavjudligi va yagonaligi uchun yetarli shart quyidagi teoremada keltirilgan.

Teorema (MYaT). *Aytaylik,* $f(x,y)$ *funksiya ushbu*

$T = \{(x,y) \in \mathbb{R}^2 \mid \quad |x - x_0| \le a, \ |y - y_0| \le b\} \qquad (a > 0, b > 0)$

to'rtburchakda uzluksiz va y *argumentga nisbatan Lipshits shatini qanoatlantirsin.* $f(x,y)$ *funksiya chegaralangan va yopiq* $T \subset \mathbb{R}^2$ *to'plamda uzluksiz bo'lgani uchun u shu* T *da chegaralangan; demak, shunday* $M > 0$ *son mavjudki, barcha* $(x,y) \in T$ *nuqtalar uchun* $|f(x,y)| \le M$ *bo'ladi.*

$h = \min\{a, \frac{b}{M}\}$ $(h > 0)$ *deylik. U holda (K) Koshi masalasining* $I = [x_0 - h, x_0 + h]$ *oraliqda aniqlangan yechimi mavjud va bu yechim yagonadir* (1.4-rasm).



1.4-rasm.

Isbotni bir necha qismga bo'lib bajaramiz.

1. Integral tenglamaga o'tish. Faraz qilaylik, $y = y(x)$ funksiya (K) Koshi masalasining biror $I, x_0 \in I$, oraliqda aniqlangan yechimi bo'lsin. Shu oraliqda bu funksiya (I.9.1) tenglamani va (I.9.2) shartni qanoatlantiradi:

$$\begin{cases} y'(s) = f(s, y(s)), \; s \in I, \\ y(x_0) = y_0 \end{cases}$$

Bu yerdagi birinchi ayniyatning har ikkala tomoni I oraliqda uzluksiz funksiyadan iborat. Uni $s = x_0$ dan $s = x \in I$ gacha integrallaymiz va boshlang`ich shartdan foydalanamiz:

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds \;, \; x \in I$$

Demak, $y = y(x) \in C^1(I)$ yechim I oraliqda ushbu

$$y = y_0 + \int_{x_0}^{x} f(s, y(s))ds \qquad (I.9.3)$$

integral tenglamani qanoatlantiradi.

Endi faraz qilaylik, $y = y(x) \in C(I)$ funksiya (I.9.3) integral tenglamaning $I, x_0 \in I$, oraliqda yechimi bo'lsin:

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds \;, \; x \in I \,.$$

Bu ayniyatning o'ng tomonidagi funksiya I oraliqda uzluksiz differinsiallanuvchi (integral ostidagi funksiya uzluksiz funksiyalar kompozitsiyasi sifatida uzluksiz); demak, uning chap tomoni ham shu xususiyatga ega, ya'ni aslida $y(x) \in C^1(I)$. Bu ayniyatni differensiallab, $y(x) \in C^1(I)$ funksiya I oraliqda (I.9.1) differensial tenglamani qanoatlantirishini ko'ramiz, ayniyatda $x = x_0$ deb esa (I.9.2) boshlang'ich shartning bajarilishiga ham ishonch hosil qilamiz. Demak, (I.9.3) integral tenglamaning $y = y(x) \in C(I)$ yechimi (K) Koshi masalasining I oraliqda aniqlangan yechimidan iborat.

Shunday qilib, (K) Koshi masalasini yechish (I.9.3) integral tenglamaning $y \in C(I)$, $x_0 \in I$, yechimini topishga teng kuchlidir.

Endi biz (K) masala o'rniga unga ekvivalent bo'lgan (I.9.3) integral tenglamani yechamiz.

2. Yechimga ketma-ket yaqinlashishlarni qurish. (I.9.3) integral tenglamani ketma-ket yaqinlashishlar metodi yordamida yechamiz. Boshlang'ich yaqinlashish sifatida ushbu

$$y_0(x) = y_0 \qquad (I.9.4_0)$$

o'zgarmas funksiyani tanlaymiz. Endi ketma-ket quyidagi funksiyalarni kiritamiz:

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0(s))ds, \qquad (I.9.4_1)$$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(s, y_1(s))ds, \qquad (I.9.4_2)$$

....

$$y_n(x) = y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s))ds, \qquad (I.9.4_n)$$

....

Bu formulalardagi integrallarning mavjud bo'lishini, ya'ni $y_1(x), y_2(x), ..., y_n(x), ...$ ketma-ket yaqinlashishlarning aniqlangan bo'lishini ta'minlashimiz kerak.

Agar

$$|x - x_0| \le a \qquad (I.9.5)$$

bo'lsa, (I.9.4$_1$)dagi integral mavjud va $y_1(x)$ uzluksiz funksiyadan iborat (integral ostidagi funksiya uzluksiz). (I.9.4$_2$)dagi integral aniqlangan bo'lishi uchun $(s, y_1(s))$ o'zgaruvchi nuqta T to'rtburchakdan chiqib ketmasligi kerak. (I.9.4$_1$)ga ko'ra

$$|y_1(x) - y_0| = \left|\int_{x_0}^{x} f(s, y_0(s))ds\right| \le$$

$$\le \left|\int_{x_0}^{x} |f(s, y_0)|\, ds\right| \le \left|\int_{x_0}^{x} M ds\right| = M\,|x - x_0|.$$

Bundan ravshanki, $|y_1(x)-y_0|\le b$ bo'lishi uchun $M|x-x_0|\le b$, ya'ni

$$|x-x_0|\le\frac{b}{M} \tag{1.9.6}$$

shartning bajarilishi yetarli. Demak, (1.9.5) va (1.9.6) shartlar birgalikda o'rinli, ya'ni

$|x-x_0|\le h$ ($h\le a$ va $h\le\frac{b}{M}$) bo'lganda $(x,y_1(x))\in T$ bo'ladi.

Bundan keyin x o'zgaruvchi uchun ana shu $|x-x_0|\le h$ shart bajarilgan deb hisoblaymiz. Endi tushunarliki, agar $y_k(x)$ funksiya $|x-x_0|\le h$ oraliqda aniqlangan, grafigi T da joylashgan va uzluksiz bo'lsa, $y_{k+1}(x)$ funksiya ham shu xususiyatlarga ega bo'ladi, chunki

$$|y_{k+1}(x)-y_0|=\left|\int_{x_0}^{x}f(s,y_k(s))ds\right|\le$$

$$\le\left|\int_{x_0}^{x}|f(s,y_k)|ds\right|\le\left|\int_{x_0}^{x}Mds\right|=M|x-x_0|\le Mh\le b.$$

Demak, matematik induksiya prinsipiga ko'ra $y_n(x)$ $(n\in\mathbb{N})$ funksiyalarning barchasi $[x_0-h,x_0+h]$ oraliqda aniqlangan, grafiklari T da joylashgan va uzluksiz (aslida $y_n'(x)$ hosilalar ham uzluksiz, chunki (1.9.4$_n$) formulada integral ostidagi funksiya uzluksiz).

3. Ketma-ket yaqinlashishlarning tekis yaqinlashuvchiligi. Yuqorida aniqlangan $y_0(x),y_1(x),y_2(x),\dots,y_n(x),\dots$ uzluksiz funksiyalardan tuzilgan ketma-ketlikning $[x_0-h,x_0+h]$ oraliqda tekis yaqinlashuvchi ekanligini ko'rsatamiz. Buning uchun ushbu

$$y_0(x)+\big(y_1(x)-y_0(x)\big)+\big(y_2(x)-y_1(x)\big)+\dots \tag{1.9.7}$$

funksional qatorning tekis yaqinlashishini asoslash kerak. Tekis yaqinlashish haqidagi Veyershtrass alomatidan foydalanamiz: agar $|u_n(x)| \le c_n,\ x \in I,\ n \in \mathbb{N}$ va $\sum_n c_n$ sonli qator yaqinlashuvchi bo'lsa, u holda $\sum_n u_n(x)$ qator I oraliqda tekis yaqinlashuvchi bo'ladi.

Teoremaning shartiga ko'ra $f(x, y)$ funksiya y o'zgaruvchiga nisbatan Lipshits shartini qanoatlantiradi:

$$\exists L > 0 \ |f(x,u) - f(x,v)| \le L|u - v| \ \ ((x,u) \in T, (x,v) \in T). \quad (I.9.8)$$

Quyidagi baholashlarni bajaramiz.

$$|y_0(x)| = |y_0|,$$

$$|y_1(x) - y_0(x)| \le \left| \int_{x_0}^{x} |f(s, y_0)|\, ds \right| \le M\,|x - x_0| \quad (\text{chunki } T \text{ da } |f(x, y)| \le M), \quad (I.9.9_1)$$

$$|y_2(x) - y_1(x)| = \left| \int_{x_0}^{x} (f(s, y_1(s)) - f(s, y_0(s)))ds \right| \le$$

$$\le \left| \int_{x_0}^{x} |f(s, y_1(s)) - f(s, y_0(s))|\, ds \right| \overset{(8)}{\le} \left| \int_{x_0}^{x} L\,|y_1(s) - y_0(s)|\, ds \right| \overset{(9_1)}{\le}$$

$$\overset{(9_1)}{\le} \left| \int_{x_0}^{x} LM\,|s - x_0|\, ds \right| = ML \frac{|x - x_0|^2}{2}, \quad (I.9.9_2)$$

$$|y_3(x) - y_2(x)| \le \left| \int_{x_0}^{x} |f(s, y_2(s)) - f(s, y_1(s))|\, ds \right| \overset{(8)}{\le}$$

$$\overset{(8)}{\le} \left| \int_{x_0}^{x} L\,|y_2(s) - y_1(s)|\, ds \right| \overset{(9_2)}{\le}$$

$$\leq \left| \int_{x_0}^{x} LML \frac{|s-x_0|^2}{2} ds \right| = ML^2 \frac{|x-x_0|^3}{3!}, \qquad (I.9.9_3)$$

....

$$|y_n(x) - y_{n-1}(x)| \leq ML^{n-1} \frac{|x-x_0|^n}{n!}, \qquad (I.9.9_n)$$

....

Demak,

$$|x-x_0| \leq h \text{ bo'lganda } |y_n(x) - y_{n-1}(x)| \leq ML^{n-1} \frac{h^n}{n!}$$

va ushbu $\sum_n ML^{n-1} \frac{h^n}{n!}$ sonli qator yaqinlashuvchi bo'lgani uchun (Dalamber alomatiga ko'ra $D = 0 < 1$) (I.9.7) funksional qator I oraliqda tekis yaqinlashuvchi.

4. Ketma-ket yaqinlashishlar limitining yechim ekanligi. Biz uzluksiz funksyalardan tuzilgan $y_0(x), y_1(x), y_2(x), ..., y_n(x), ...$ ketma-ketlikning I oraliqda tekis yaqinlashuvchi ekanligini ko'rsatdik. Demak, analizdan ma'lum teoremaga ko'ra (uzluksiz funksyalarning tekis limiti uzluksiz)

$$\varphi(x) = \lim_{n \to \infty} y_n(x), \; |x-x_0| \leq h, \qquad (I.9.10)$$

limit funksiya I oraliqda uzluksiz. Uning (I.9.3) integral tenglamani qanoatlantirishini ko'rsatamiz. (I.9.4$_n$) formulaga qaytaylik:

$$y_n(x) = y_0 + \int_{x_0}^{x} f(s, y_{n-1}(s)) ds. \qquad (I.9.4_n)$$

Agar

$$n \to \infty \text{ bo'lganda } \int_{x_0}^{x} f(s, y_{n-1}(s)) ds \to \int_{x_0}^{x} f(s, \varphi(s)) ds \quad (I.9.11)$$

ekanligini ko'rsatsak, u holda (I.9.4$_n$) tenglikda limitga o'tib, (I.9.10) funksiyaning (I.9.3) integral tenglama yechimi ekanligini asoslagan bo'lamiz:

$$\varphi(x) = y_0 + \int_{x_0}^{x} f(s, \varphi(s))ds \ , \ x \in I. \tag{1.9.12}$$

(1.9.11) dagi tasdiq (1.9.8) Lipshits sharti va (1.9.10) dagi yaqinlashishning tekis ekanligidan kelib chiqadi: yetarlicha katta n nomerlar uchun barcha $x \in I$ nuqtalarda $|y_{n-1}(x)) - \varphi(x)| < \varepsilon$ bo'lganligiga ko'ra

$$\left| \int_{x_0}^{x} f(s, y_{n-1}(s))ds - \int_{x_0}^{x} f(s, \varphi(s))ds \right| \le$$

$$< \left| \int_{x_0}^{x} | f(s, y_{n-1}(s)) - f(s, \varphi(s) | ds \right| <$$

$$\le \left| \int_{x_0}^{x} L | y_{n-1}(s)) - \varphi(s) | ds \right| <$$

$$< L\varepsilon | x - x_0 | \le$$

$$\le Lh\varepsilon .$$

5. Yechimning yagonaligi. Biz (1.9.3) integral tenglamaning $y = \varphi(x)$, $x \in I$, yechimini topdik. Uning har qanday yechimi ana shu yechim bilan ustma-ust tushishini ko'rsatamiz.

(1.9.3) tenglamaning ixtiyoriy $y = \psi(x)$, $x \in I$, yechimini qaraylik. Demak.

$$\psi(x) \in C(I) \text{ va } \psi(x) = y_0 + \int_{x_0}^{x} f(s, \psi(s))ds \ , \ x \in I. \tag{1.9.13}$$

$y = \varphi(x)$ yechimga intiluvchi $y_n(x)$ ketma-ket yaqinlashishlar bilan $y = \psi(x)$ yechim orasidagi farqning modulini baholaymiz. Quyidagilarga egamiz:

$$|y_0(x)-\psi(x)| \overset{(4_0),(13)}{=} \left|\int_{x_0}^{x} f(s,\psi(s))ds\right| \le$$

$$\le \left|\int_{x_0}^{x} | f(s,\psi(s))| ds\right| \le M\,|x-x_0| \qquad (I.9.14_1)$$

$$|y_1(x)-\psi(x)| \overset{(4_1),(13)}{=} \left|\int_{x_0}^{x} (f(s,y_0(s))-f(s,\psi(s)))ds\right| \le$$

$$\le \left|\int_{x_0}^{x} | f(s,y_0(s))-f(s,\psi(s))| ds\right| \overset{(8)}{\le}$$

$$\overset{(8)}{\le} \left|\int_{x_0}^{x} L\,| y_0(s)-\psi(s)| ds\right| \overset{(14_1)}{<}$$

$$\overset{(14_1)}{<} \left|\int_{x_0}^{x} LM\,| s-x_0| ds\right| =$$

$$= ML\frac{|x-x_0|^2}{2}. \qquad (I.9.14_2)$$

. . . .

$$|y_n(x)-\psi(x)| \le ML^n\frac{|x-x_0|^{n+1}}{(n+1)!},\ x\in I. \qquad (I.9.14_n)$$

Oxirgi tengsizlikda $x \in I$ nuqtani tayinlab, $n \to \infty$ da limitga o'tamiz. U holda $\frac{a^n}{n!} \to 0$ bo'lganligi uchun quyidagilarga ega bo'lamiz:

$$|\varphi(x)-\psi(x)| \le 0\ , x\in I \Rightarrow \varphi(x)\equiv\psi(x),\ x\in I\ .$$

Eslatma. $(I.9.14_n)$ tengsizlikdan (K) Koshi masalasining yechimi $y=\varphi(x)$ va unga ketma-ket yaqinlashishlar orasidagi farqning baholanishi kelib chiqadi:

$$|y_n(x)-\varphi(x)|\le ML^n\frac{|x-x_0|^{n+1}}{(n+1)!},\ x\in I.$$

Misol. Noma'lum $y=y(x)$ funksiyaga nisbatan qo'yolgan ushbu

$$\begin{cases} y'=y \\ y(0)=1 \end{cases}$$

Koshi masalasini ketma-ket yaqinlashishlar metodi yordamida yeching.

Berilgan tenglamaning o'ng tomonidagi $f(x,y)=y$ funksiya MYaT shartlarini ixtiyoriy $T=\{(x,y)\in\mathbb{R}^2\,|\,|x|\le a,\ |y-1|\le b\}$ $(a>0,\ b>0)$ to'rtburchakda qanoatlantiradi. Ravshanki, qaralayotgan misol uchun $M=\sup\limits_T|f(x,y)|=\sup\limits_T|y|=1+b$ va $h=\min\left\{a,\dfrac{b}{1+b}\right\}<1$. Berilgan Koshi masalasi ushbu

$$y=1+\int_0^x y(s)ds$$

integral tenglamaga ekvivalent. Bu tenglamani ketma-ket yaqinlashishlar metodi yordamida yechamiz:

$$y_0(x)=1,$$

$$y_1(x)=1+\int_0^x y_0(s)ds=1+\int_0^x 1ds=1+x,$$

$$y_2(x)=1+\int_0^x y_1(s)ds=1+\int_0^x(1+s)ds=1+x+\frac{x^2}{2!},$$

$$y_3(x)=1+\int_0^x y_2(s)ds=1+\int_0^x\left(1+s+\frac{s^2}{2!}\right)ds=1+x+\frac{x^2}{2!}+\frac{x^3}{3!},$$

....

$$y_n(x)=1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!},$$

....

Teoremada isbotlandiki, bu $y_n(x)$ ketma-ket yaqinlashishlar berilgan masalaning yagona yechimi bo'lmish $y=e^x$ funksiyaga $|x|\le h$, $h=\min\left\{a,\frac{b}{1+b}\right\}<1$, oraliqda tekis yaqinlashadi:

$$e^x=1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!}+...$$

Oxirgi formula, analizdan ma'lumki, aslida ixtiyoriy $x\in\mathbb{R}$ uchun o'rinli (bu yerdagi qator ixtiyoriy chegaralangan oraliqda tekis yaqinlashuvchi bo'ladi). ♁

Eslataylikki, agar $D\subset\mathbb{R}^2$ sohaning har bir nuqtasidan $y'=f(x,y)$ differensial tenglamaning yagona integral chizig'i (yechimi) o'tsa, D soha shu tenglama uchun yagonalik sohasi deyiladi.

Quyidagi teorema isbotlangan teoremaning bevosita natijasidir.

Teorema (soha uchun MYaT). *Agar $f(x,y)$ funksiya $D\subset\mathbb{R}^2$ sohada uzluksiz va ixtiyoriy $(x_0,y_0)\in D$ nuqtaning yetarlicha kichik atrofida y bo'yicha Lipshits shartini qanoatlantirsa, u holda D soha $y'=f(x,y)$ differensial tenglamaning yagonalik sohasi bo'ladi. Xususan, agar D sohada $f(x,y)$ funksiya va uning $f'_y(x,y)$ hosilasi uzluksiz bo'lsa, D soha $y'=f(x,y)$ differensial tenglama uchun yagonalik sohasidir.*

Misol. Ushbu $y'=x^2y^3-\ln y+1$ differensial tenglama uchun biror yagonalik sohasini ko'rsating.

8━ Ravshanki, $f(x,y)=x^2y^3-\ln y+1$ funksiya va uning $f'_y(x,y)=3x^2y^2-\frac{1}{y}$ hosilasi $y>0$ yarim tekislikda uzluksiz. Demak shu $y>0$ yarim tekislik berilgan tenglama uchun yagonalik sohasidir, ya'ni $y>0$ yarim tekislikning har bir nuqtasidan $y'=x^2y^3-\ln y+1$ differensial tenglamaning yagona yechimi o`tadi.

Agar $y'=f(x,y)$ tenglamaning o`ng tomonidan faqatgina uzluksiz bo`lishni talab qilsak, u holda berilgan nuqtadan o'tuvchi yechim mavjud bo`ladi, lekin umumiy holda yechimning yagonalik xossasi bo'lmaydi.

***Teorema** (Peano). Agar $f(x,y)$ funksiya D sohada uzluksiz bo'lsa, D sohaning har bir nuqtasidan $y'=f(x,y)$ tenglamaning kamida bitta yechimi o'tadi.*

Bu teoremani isbotlamaymiz. Isboti analizning nozik tushunchalariga tayanadi.

Agar $f(x,y)$ funksiyadan uzluksizlikdan boshqa shart talab qilmasak, nuqtadan o`tuvchi yechimning yagonaligi haqida hech narsa deb bo`lmaydi.

M. A. Lavrent'ev tekislikda uzluksiz bo'lgan shunday $f(x,y)$ funksiya qurganki, u orqali tuzilgan $y'=f(x,y)$ differensial tenglamaning $\forall(x_0,y_0)\in\mathbb{R}^2$ nuqta orqali cheksiz ko`p yechimi o'tadi.

Masalalar

1. Ushbu

$$f(x)=\begin{cases} x\ln x, \text{ agar } x>0 \text{ bo'lsa,}\\ 0\ , \text{ agar } x=0 \text{ bo'lsa.}\end{cases}$$

fuksiyaning ixtiyoriy $[0,b]$ $(b>0)$ yoki $[a,+\infty)$ $(a>0)$ ko`rinishdagi oraliqda ( $x$ bo`yicha) Lipshits shartini qanoatlantirmasligini ko'rsating.

2. Faraz qilaylik, $f(x,y)\in C^1(\mathbb{R}^2)$ va $f(x,0)\equiv 0$ bo`lsin. $y=\sin x$ funksiya $y'=f(x,y)$ tenglamaning biror $(-a;a)$ $(a>0)$

oraliqda yechimi bo`lishi mumkinmi?

3. $y_0:[-1,1]\to\mathbb{R}$ uzluksiz funksiya bo'lsin. Ushbu $y_n(x)=\int_0^x\sqrt[3]{y_{n-1}(s)}ds$, $n\in\mathbb{N}$, ketma-ket yaqinlashishlar ushbu $y'=\sqrt[3]{y}$, $y(0)=0$, Koshi masalasining yechimiga intilishini isbotlang ($f(y)=\sqrt[3]{y}$ funksiya ixtiyoriy $[-a,a]$ $(a>0)$ segmentda Lipshits shartini qanoatlantirmaydi, MYaTni qo`llab bo`lmaydi).

I.10. Davomsiz yechimlar

Ушбу

$$y'=f(x,y) \qquad (I.10.1)$$

tenglamani qaraylik; bu yerda $f\in C(D)$, $f'_y\in C(D)$ deb faraz qilamiz. Bu farazga ko`ra $\forall(x_0,y_0)\in D$ uchun ushbu

$$\begin{cases} y'=f(x,y) \\ y|_{x_0}=x_0 \end{cases} \qquad (K)$$

Koshi masalasi biror $[x_0,x_0+h_0]$ $(h_0>0)$ segmentda aniqlangan yagona $y=\varphi(x)$ yechimga ega.

Agar $y=\varphi(x)$ funksiya (1) differensial tenglamaning $I=[a;b]$ oraliqda, $y=\psi(x)$ funksiya esa uning $J=[a;b^*]$, $b\le b^*$, yoki $J=[a;b^*)$, $b<b^*$, oraliqda aniqlangan yechimi bo`lib, ular I da ustma-ust ham tushsa, u holda $y=\psi(x)$ yechim $y=\varphi(x)$ **yechimning** I dan J gacha **o'ngga davomi (davom ettirilishi)** deb ataladi.

Yechimning boshqa tur oraliqlardan o'ngga hamda chapga davomi shunga o'xshash aniqlanadi.

Yechimning o'ngga davom ettirishni amalga oshirishdan avval yechimlarni yelimlash (biriktirish) bilan bog`liq bo'lgan bir jumlani keltiramiz.

Jumla 1. *Agar* $y=\varphi(x)$ *funksiya (I.10.1) differensial* *tenglamaning* $[x_0, x_1]$ *segmentda,* $y=\varphi_1(x)$ *esa uning* $[x_1, x_2]$ *segmentda aniqlangan yechimlari bo'lib,* $\varphi(x_1)=\varphi_1(x_1)$ *shart ham bajarilsa, u holda bu yechimlarning yelimlanishi (biriktirilishi) bo'lgan*

$$\psi(x)=\begin{cases}\varphi(x), \text{ agar } x\in[x_0,x_1] \text{ bo'lsa}\\ \varphi_1(x), \text{ agar } x\in[x_1,x_2] \text{ bo'lsa}\end{cases}$$

funksiya (1.10.1) differensial tenglamaning $[x_0, x_2]$ *segmentda aniqlangan yechimini beradi, ya'ni* $y=\psi(x)$ *yechim* $y=\varphi(x)$ *yechimning* $[x_0, x_1]$ *segmentdan* $[x_1, x_2]$ *segmentgacha davomidan iborat.*

⊶ Berilganga ko'ra

$$\psi(x)\in C^1([x_0,x_1]);\ \psi'(x)=f(x,\psi(x)),\ x\in[x_0,x_1];$$

$$\psi(x)\in C^1([x_1,x_2]);\ \psi'(x)=f(x,\psi(x)),\ x\in[x_1,x_2];$$

$$\psi(x_1-0)=\varphi(x_1)=\varphi_1(x_1)=\psi(x_1+0).$$

Shuning uchun

$$\psi'(x_1-0)=\varphi'(x_1)=f(x_1,\varphi(x_1))=f(x_1,\varphi_1(x_1))=\varphi_1'(x_1)=\psi'(x_1+0).$$

Demak, $\psi'(x_1)$ mavjud, $\psi(x)\in C^1([x_0,x_2])$ va $\psi(x)$ funksiya (1.10.1) differensial tenglamaning $[x_0,x_2]$ segmentda yechimi. ⊷

Yechimning o'ng uchi bo'lmish

$(x_1; y_1)\overset{def}{=}(x_0+h_0, \varphi(x_0+h_0))\in D$ nuqtaga ko'ra

$$\begin{cases} y'=f(x,y)\\ y\big|_{x_1}=y_1\end{cases}$$

Koshi masalasini yechib $[x_1, x_1+h_1]$ $(h_1>0)$ segmentda aniqlangan yagona $y=\varphi_1(x)$ yechimni topamiz. Yuqoridagi ikki yechimni yelimlanishi (biriktirilishi)dan ushbu

$$\varphi^*(x)=\begin{cases}\varphi(x),\ \text{agar } x\in[x_0,x_1] \text{ bo'lsa,}\\ \varphi_1(t),\ \text{agar } x\in[x_1,x_1+h_1] \text{ bo'lsa}\end{cases}$$

$$(x_1=x_0+h_0,\varphi(x_1)=\varphi_1(x_1)=y_1)$$

funksiyani quramiz. Jumla 1 ga ko'ra bu $\varphi^*(x)$ funksiya (K) masalaning $[x_0,x_1]\cup[x_1,x_1+h_1]=[x_0,x_1+h_1]$ segmentda aniqlangan yechimidir. U $[x_0,\ x_0+h_0]$ da aniqlangan $y=\varphi(x)$ yechimning $[x_0,x_1+h_1]$ segmentgacha (o`ngga) davomidir. Bu yechimni (funksiyani) yana $y=\varphi(x)$ bilan belgilaymiz; endi bu $y=\varphi(x)$ funksiya (K) masalaning $[x_0,x_1+h_1]$ segmentda aniqlangan yechimi. Yechimning bu davomi bir qiymatli aniqlanadi. Endi bu yechimni yana o'ngga davom ettiramiz va h.k. Yechimning chapga davomi yuqoridagiga o'xshash amalga oshiriladi.

Davom ettirilgan yechim grafigi D da joylashgan har qanday K kompaktdan tashqariga chiqib ketadi, chunki shunday yetarlicha kichik $h_K>0$ mavjudki, K ning ixtiyoriy nuqtasidan chiqqan integral chiziq o`ngga $h_K>0$ masofaga davom etadi. Chekli marta yechimni $K$ ning nuqtalaridan o`ngga (hamda chapga) davom ettirib, K dan tashqariga chiqib ketamiz.

Shunday qilib, (ξ,η) intervalda aniqlangan yechim hosil bo`ladi ($\xi=-\infty$ yoki/va $\eta=+\infty$ bo'lishi ham mumkin). Bu yechim $(K)$ masalaning ($D$ sohadagi) **davomsiz yechimi** deyiladi. Davomsiz yechimning aniqlanish sohasi eng katta bo`ladi. Bu masalani biz differensial tenglamalar sistemasini o`rganganimizda to`la qarab chiqamiz.

Masalalar

1. $y'=f(x,y)$ tenglamada $f(x,y)\in C^1(\mathbb{R}^2)$ bo`lsin. Bu tenglamaning $\mathbb{R}$ da chegaralangan har qanday yechimi $(-\infty,+\infty)$ gacha davom etishini ko`rsating.

2. $(1+x^2+y^{2002})y'=1$ tenglamaning barcha yechimlari $(-\infty,\infty)$ da chegaralangan. Shuni isbotlang.

1.11. Hosilaga nisbatan yechilmagan tenglama uchun yechimning mavjudlik va yagonalik teoremasi

Quyidagi tenglamani qaraylik:

$$F(x, y, y') = 0; \qquad (1.11.1)$$

bu yerda $F(x, y, p)$ funksiya $G \subset \mathbb{R}^3$ sohada aniqlangan, uzluksiz ($F \in C'(G)$) va p ga tub ma'noda bog'liq deb hisoblanadi.

Berilgan $(x_0, y_0) \in D \subset \mathbb{R}^2$ (bu yerda D – G sohaning $\mathbb{R}^2 = \{(x, y)\}$ dagi ortogonal proyeksiyasi) nuqta orqali bir nechta (hatto, cheksiz ko`p) integral chiziq o'tishi mumkin. Chunki $F(x, y, y') = 0$ tenglamani y' ga nisbatan yechib, umumiy holda bir nechta $y' = f_i(x, y)(i = 1, 2, ...)$ qiymatlarni hosil qilamiz. Endi, agar har bir $f_i(x, y)$ fuksiya (x_0, y_0) nuqta atrofida yechimning mavjudlik va yagonaligi haqidagi teorema shartlarini qanoatlantirsa, u holda har bir $y' = f_i(x, y)$ differensial tenglama uchun $y(x_0) = y_0$ shartni qanoatlantiruvchi yagona $y = y(x)$ yechim mavjud va bu yechim uchun $y'(x_0) = f_i(x_0, y_0)$ bo'ladi. Agar $y'(x_0) = p_0$ bo'lsa, $y = y(x)$ yechim (integral chiziq) (x_0, y_0) nuqtadan p_0 yo'nalishda o'tadi (chiqadi) deyiladi.

Ushbu

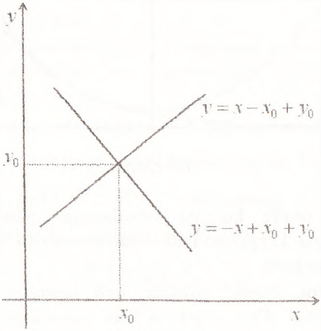
$$\begin{cases} F(x, y, y') = 0 \\ y(x_0) = y_0 \end{cases}$$

masala yechimining yagonaligi berilgan (x_0, y_0) nuqtadan har qanday yo`nalish bo'ylab ko'pi bilan bitta integral chiziqning o'tishini anglatadi.

Agar berilgan (x_0, y_0) nuqtadan biror yo'nalish bo'ylab kamida ikkita yechim (integral chiziq) o'tsa, bu nuqta yagonalik (yechimning yagonaligi) buzilgan nuqta deb ataladi.

Misol 1. Ushbu $(y')^2 - 1 = 0$ tenglama uchun yagonalik xossasi ixtiyoriy (x_0, y_0) nuqtada o'rinli, chunki (x_0, y_0) nuqtadan ikkita integral chiziq ikki xil (turli) yo'nalish bo'ylab o'tadi. (1.5-rasm):

$$\frac{dy}{dx} = \pm 1, \quad y = x - x_0 + y_0, \quad y = -x + x_0 + y_0$$

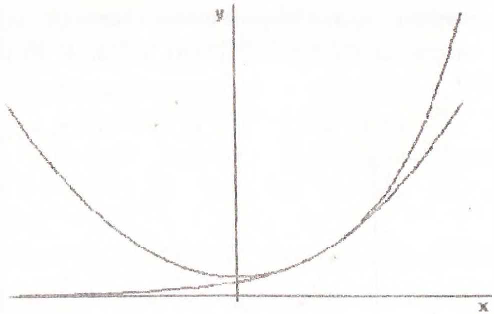


1.5-rasm.

Misol 2. Ushbu $(y')^2 - (y+2x)y' + 2xy = 0$ tenglamani y' ga nisbatan yechib, $y' = y$, $y' = 2x$ differensial tenglamalarni hosil qilamiz. Ularni yechib, $y = ce^x$ va $y = x^2 + c$ yechimlar oilasini topamiz. $y = 2x$ to'g'ri chiziqning nuqtalarida yechimning yagonalik xossasi buzilgan, chunki bu to'g'ri chiziqning ixtiyoriy $(x_0, 2x_0)$ nuqtasi orqali bir xil $y'(x_0) = 2x_0$ yo'nalish bo'ylab ikkita

$$y = x_0 e^{x - x_0}, \quad y = x^2 + 2x_0 - x_0^2$$

yechim o'tadi (1.6-rasm). Bu yechimlar qismlarini $(x_0, 2x_0)$ nuqtada tutashtirib, yana ikkita yechim tuzish mumkin.



I.6-rasm.

Bizga analiz kursida isbotlangan oshkormas funksiya haqidagi teorema kerak bo'ladi. Shu teoremani biz uchun qulay ko'rinishda keltiraylik.

Teorema. *Faraz qilaylik, uch haqiqiy o'zgaruvchining haqiqiy funksiyasi $F(x, y, p)$ uchun quyidagi shartlar o'rinli bo'lsin:*

1. $F(x_0, y_0, p_0) = 0$,

2. $(x_0, y_0, p_0) \in \mathbb{R}^3$ *nuqtaning biror atrofida* $F \in C^1$,

3. $\dfrac{\partial F(x_0, y_0, p_0)}{\partial p} \neq 0$.

U holda $(x_0, y_0) \in \mathbb{R}^2$ *nuqtaning shunday* V *va* $p_0 \in \mathbb{R}$ *nuqtaning shunday* W *atroflari (simmetrik bo'lishi shart emas) topiladiki, ixtiyoriy* $(x, y) \in V$ *uchun* $F(x, y, p) = 0$ *tenglamaning* W *ga tegishli bo'lgan yagona* $p = f(x, y)$ *yechimi mavjud*

$\left(F(x, y, f(x, y)) \equiv 0, (x, y) \in V, f(x, y) \in W, p_0 = f(x_0, y_0)\right)$.

Bundan tashqari, $f : V \to W$ *funksiya* C^1 *sinfga tegishli va*

$$\frac{\partial f(x,y)}{\partial x}=-\frac{F'_x(x,y,f(x,y))}{F'_p(x,y,f(x,y))},$$

$$\frac{\partial f(x,y)}{\partial y}=-\frac{F'_y(x,y,f(x,y))}{F'_p(x,y,f(x,y))}$$

formulalar o'rinli bo'ladi.

Quyidagi teorema (I.11.1) tenglama uchun Koshi masalasi yechimining mavjudligi va yagonaligini ta'minlovchi yetarli shartlarni beradi.

Teorema 1. *Aytaylik,* $p=p_0$ *son* $F(x_0,y_0,p)=0$ *tenglamaning biror haqiqiy ildizi bo'lsin, ya'ni* $F(x_0,y_0,p_0)=0$. *Agar* $(x_0,y_0,p_0)\in G\subset\mathbb{R}^3$ *nuqtaning biror atrofida* $F(x,y,p)$ *fuksiya* C^1 *sinfga tegishli va*

$$\frac{\partial F(x_0,y_0,p_0)}{\partial p}\neq 0$$

bo'lsa, u holda shunday $h>0$ *mavjudki,* $x_0-h\le x\le x_0+h$ *oraliq'ida (I.11.1) diffrensial tenglamaning* $y(x_0)=y_0,\ y'(x_0)=p_0$ *shartlarni qanoatlantiruvchi yagona* $y=y(x)$ *yechimi mavjud.*

Oshkormas funksiya haqidagi teoremaga ko'ra (x_0,y_0) nuqtaning biror V atrofida (I.11.1) tenglama $y'=f(x,y)$ ko'rinishga keltiriladi; bunda $p_0=f(x_0,y_0)$ hamda (x_0,y_0) nuqtaning yetarli kichik atrofida $f\in C^1$ va $F(x,y,f(x,y))=0$ bo'ladi. Teoremaning shartlaridan $\frac{\partial F}{\partial y}$ va $\frac{\partial F}{\partial p}$ funksiyalar (x_0,y_0,p_0) nuqtaning yetarlicha kichik atrofida uzluksiz va $\frac{\partial F}{\partial p}$ shu atrofda noldan farqli ekanligi kelib chiqadi. Shuning uchun $(x_0,y_0)\in\mathbb{R}^2$ nuqtaning yetarlicha kichik atrofida

$$\left|\frac{\partial F(x,y,f(x,y))}{\partial p}\right| \geq \text{const} > 0.$$

$\left|\frac{\partial f(x,y)}{\partial y}\right| = \left|\frac{F'_y(x,y,f(x,y))}{F'_p(x,y,f(x,y))}\right| \leq \text{const}$ va, demak, shu atrofda $f(x,y)$ funksiya y bo'yicha Lipshits shartini qanoatlantiradi. Mavjudlik va yagonalik teoremasiga ko'ra $y' = f(x,y)$ tenglamaning (x_0, y_0) nuqtadan p_0 yo'nalish bo'ylab o'tuvchi (shu nuqtadagi urinmaning burchak koeffitsiyenti p_0 ga teng, $y'(x_0) = f(x_0, y_0) = p_0$) yagona $y = \varphi(x)$ integral chizig'i (yechimi) biror yetarlicha kichik $\left[x_0 - h,\ x_0 + h\right]$ $(h > 0)$ oraliqda mavjud, $\varphi'(x) = f(x, \varphi(x))$. Ravshanki, bu $y = \varphi(x)$ funksiya uchun $0 = F\left(x, \varphi(x), f(x, \varphi(x))\right) = F\left(x, \varphi(x), \varphi'(x)\right)$. Demak, $y = \varphi(x)$ funksiya (1) tenglamaning $y(x_0) = y_0$, $y'(x_0) = p_0$ shartlarni qanoatlantiruvchi yechimi. Teoremaning mavjudlik qismi isbot bo'ldi. Endi yagonalik qismini isbotlaymiz. Faraz qilaylik $y = y(x)$ ixtiyoriy yechim bo'lsin:

$$F\left(x, y(x), y'(x)\right) = 0,\ x \in [x_0 - h, x_0 + h],\ y(x_0) = y_0,\ y'(x_0) = p_0.$$

Yetarlicha kichik $h > 0$ uchun $\left(x, y(x)\right) \in U$, $x \in [x_0 - h, x_0 + h]$, bo'ladi. Demak, oshkormas funksiya haqidagi teoremaning yagonalik qismiga ko'ra

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

bo'lishi kerak. Shunday qilib, yuqorida topilgan $y = \varphi(x)$ va berilgan $y = y(x)$ funksiyalar ushbu

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

Koshi masalasining yechimi. Bu masala yechimining yagonalik tengligidan $\varphi(x) \equiv y(x), x \in [x_0 - h, x_0 + h]$, ekanligi kelib chiqadi. ♘

Isbotlangan bu teoremadan quyidagi teorema bevosita kelib chiqadi.

Teorema 2. *Faraz qilaylik, $G \subset \mathbb{R}^3$ sohada $F(x, y, p)$ funksiya C^1 sinfga tegishli va G sohaning $F(x, y, p)$ funksiya nolga aylangan, ya'ni $F(x, y, p) = 0$ bo'lgan nuqtalarida $\frac{\partial F(x, y, p)}{\partial p}$ hosila nolga aylanmasin, ya'ni $\frac{\partial F(x, y, p)}{\partial p} \neq 0$. U holda G sohaning x, y tekisligidagi ortogonal proeksiyasida joylashgan ixtiyoriy (x_0, y_0) nuqta orqali $F(x, y, y') = 0$ differensial tenglamaning ushbu $F(x_0, y_0, \tilde{p}) = 0$ tenglamani qanoatlantiruvchi har bir $\tilde{p}$ yo'nalish bo'ylab bittadan yechimi (integral chizig'i) o'tadi.*

Misol. Yuqorida qaralgan

$$(y')^2 - (y + 2x)y' + 2xy = 0$$

tenglamada $F(x, y, p) = p^2 - (y + 2x)p + 2xy \in C^1(\mathbb{R}^2)$ va $F(x, y, p)$ funksiya nolga aylangan nuqtalar uchun $F(x, y, p) = p^2 - (y + 2x)p + 2xy = 0$, ya'ni $p = 2x$ yoki $p = y$ bo'ladi. Demak, bunday nuqtalar uchun mos ravishda

$$\frac{\partial F}{\partial p} = 2p - (y + 2x) = 2x - y \text{ yoki}$$

$\frac{\partial F}{\partial p} = 2p - (y + 2x) = y - 2x$. Agar $y \neq 2x$ bo'lsa, $\frac{\partial F}{\partial p} \neq 0$ va bunday $(x, y) \in \mathbb{R}^2$ nuqtadan qaralayotgan tenglamaning $p = 2x$ va $p = y$ yo'nalishlar bo'ylab bittadan yechimi o'tadi, ya'ni $2x < y$ va $2x > y$ sohalar (yarim tekisliklar) berilgan tenglama uchun yagonalik sohalaridir. Yuqorida $y = 2x$ to'g'ri chiziq nuqtalarida yagonalik buzilishi e'tirof etilgan edi. ♘

Agar $y=\varphi(x,c)$ funksiya c o'zgarmasning ixtiyoriy joiz qiymatida berilgan (I.11.1) differensial tenglamaning D sohada joylashgan yechimini bersa va (I.11.1) tenglamaning D da joylashgan har qanday yechimi shu $y=\varphi(x,c)$ formuladan c ning biror joiz qiymatida hosil bo'lsa, u holda $y=\varphi(x,c)$ oila berilgan tenglamaning D sohadagi umumiy yechimi deyiladi. Umumiy yechim oshkormas ko'rinishda $\Phi(x,y,c)=0$ tenglama bilan yoki parametrik ko'rinishda $x=x(p,c),\ y=y(p,c)$ (p – parametr) formulalar bilan ham berilishi mumkin.

I.12. Hosilaga nisbatan yechilmagan tenglamani yechish usullari

Ushbu $F(x,y,y')=0$ (I.11.1) differensial tenglama yechimini topishning ikki usulini keltiraylik.

I. Tenglamani y' ga nisbatan yechish usuli. Agar (I.11.1) tenglama y' ga nisbatan oshkor yechilishi mumkin bo'lsa, u holda bitta yoki bir nechta (hatto cheksiz ko'p)

$$\frac{dy}{dx}=f(x,y) \qquad (I.12.1)$$

ko'rinishdagi differensial tenglamaga kelamiz. Biz $f(x,y)$ funksiya kamida uzluksiz bo'lishi kerak deb talab qilamiz. Normal ko'rinishdagi (I.12.1) tenglama(lar)ni yechish uchun yuqoridan bizga ma'lum metodlarni qo'llaymiz (agar mumkin bo'lsa).

Misollar.

0. Ushbu $y'^2-(y+x)^2=0$ tenglamadan y' ni x va y ning ***silliq funksiyasi*** sifatida topsak, ikki dona

$$y'=y+x,\ y'=-y-x,$$

yechimni hosil qilamiz. Tenglamadan y' ni x va y ning ***uzluksiz funksiyasi*** sifatida izlasak,

$$y'=y+x,\ y'=-y-x,\ y'=|y+x|,\ y'=-|y+x|,$$

to'rtta yechim topamiz.

1. Quyidagi differensial tenglamani yeching:

$$y'^2 - 4x^2 = 0.$$

Berilgan tenglamani ushbu

$$y' = 2x,\ y' = -2x,\ y' = 2|x| \text{ va } y' = -2|x|$$

tenglamalarga ajrataylik. Bu tenglamalarni yechib, berilgan differensial tenglamaning yechimlar oilasini hosil qilamiz:

$$y = x^2 + c,\ y = -x^2 + c,\ y = x|x| + c,\ y = -x|x| + c \ (c - \text{ixtiyoriy o'zgarmas}).$$

Yuqoridagi misolni umumlashtirib, ushbu

$$a_n(x,y)(y')^n + a_{n-1}(x,y)(y')^{n-1} + \ldots + a_1(x,y)y' + a_0(x,y) = 0$$

birinchi tartibli (n – darajali) differensial tenglamani qaraylik. Bu yerda $a_n(x,y), a_{n-1}(x,y), \ldots, a_0(x,y)$ uzluksiz fuksiyalar va $a_n(x,y) \neq 0$ deb hisoblanadi. Agar qaralayotgan tenglamani y' ga nisbatan yechib,

$$y' = f_1(x,y),\ y' = f_2(x,y), \ldots,\ y' = f_k(x,y) \qquad (I.12.2)$$

tenglamalarni topolsak, u holda berilgan tenglamaning yechimlari (I.12.2) tenglamalar yechimlarining birlashmasidan iborat bo'ladi.

2. Parametr kiritish metodi.

a). Aytaylik, (I.12.1) tenglama y o'zgaruvchiga nisbatan oshkor ko'rinishda yechilsin:

$$y = f(x,y') \quad (f \in C^1,\ f'_{y'} \neq 0 \text{ deb faraz qilinadi}). \qquad (I.12.3)$$

Oshkormas funksiya haqidagi teoremadan ko'rsatilgan shartlarda (I.12.3) differensial tenglama yechimlari C^2 sinfga tegishli ekanligi kelib chiqadi: (I.12.3) dan $y' = g(x,y),\ g \in C^1 \Rightarrow y \in C^2$.

Qaralayotgan (I.12.3) tenglamada $y' = p$ ($dy = pdx$) deb p parametrni kiritamiz va

$$y = f(x,p) \qquad (I.12.4)$$

munosabatni hosil qilamiz. Agar x ni p ning funksiyasi sifatida topolsak, uni (I.12.4) ga qo'yib, y ni p ning funksiyasi sifatida ifodalaymiz va yechimni parametrik ko'rinishda topgan bo'lamiz. Bu ishni bajarish uchun (I.12.4) ning har ikkala tomonini differensiallaymiz:

$$dy = f'_x dx + f'_p dp; \quad p dx = f'_x dx + f'_p dp .$$

Oxirgi tenglikdan

$$(p - f'_x) dx = f'_p dp \tag{I.12.5}$$

munosabatni topamiz. Endi yana qo'shimcha ravishda

$$f'_x \neq p \tag{I.12.6}$$

deb faraz qilamiz. (I.12.6) shartda (I.12.5) tenglamani

$$\frac{dx}{dp} = \frac{f'_p}{p - f'_x} \tag{I.12.7}$$

ko'rinishga keltirib, uni yechamiz va $x = \varphi(p, c)$ bog'lanishni hosil qilamiz. Bu bog'lanishni (I.12.4) ga qo'yib, (I.12.3) differensial tenglamaning yechimi ushbu

$$\begin{cases} x = \varphi(p, c) \\ y = f(\varphi(p, c), p) \end{cases} \tag{I.12.8}$$

parametrik (p – parametr, c – ixtiyoriy o'zgarmas) ko'rinishda bo'lishi kerakligini topamiz. Endi ixtiyoriy o'zgarmas c uchun (I.12.8) formulalar bilan parametrik ko'rinishda berilgan funksiya (I.12.3) differensial tenglamaning yechimi ekanligini ko'rsatamiz. Buning uchun ixtiyoriy tayinlangan c da (I.12.8) dagi birinchi tenglamadan $p = p(x)$ bog'lanishni topib ($\frac{\partial \varphi}{\partial p} \neq 0$), uni ikkinchisiga qo'yishdan hosil bo'lgan $y = y(x)$ funksiyaning hosilasi uchun $\frac{dy}{dx} = p$ bo'lishini ko'rsatish kifoya. $\frac{dy}{dx} = p$ ekanligi esa ravshan. (I.12.8) dagi ikkinchi tenglikdan (I.12.5) formulaga ko'ra

$$\frac{dy}{dx} = f'_x + f'_p \cdot \frac{dp}{dx} = f'_x + f'_p \cdot \frac{p - f'_x}{f'_p} = p .$$

Shunday qilib, qo'yilgan $f'_p \neq 0$ va $f'_x \neq p$ shartlarda (I.12.3) differensial tenglamaning umumiy yechimi (I.12.8) formulalar bilan parametrik ko'rinishda ifodalanadi.

Agar $f'_x(x,p)=p$ tenglik $x=x(p)\in C^1$ funksiyani aniqlasa va $f''_p(x(p),p)\equiv 0$ ham bo`lsa, u holda (I.12.5) tenglama, chunki, $x=x(p)$ yechimga ega. Buni (I.12.4) formulaga qo`yib, qaralayotgan (I.12.3) differensial tenglamaning $\{x=x(p),$ $y=f(x(p),p)\}$ parametrik yechimini hosil qilamiz. Endi (I.12.5) tenglamani $x-x(p)$ ga "qısqartirib", so`ngra yuqoridagidek ish ko'rib (I.12.3) tenglamaning bir parametrli yechimlar oilasini (agar mavjud bo`lsa) topamiz.

$f'_x(x,p)=p$ tenglama biror $p=p^*$ (− o'zgarmas son) yechimga ega bo'lgan holi $f(x,p)=x\psi(p)+\chi(p)$ bo'lganda, ya'ni $y=x\psi(y')+\chi(y')$ − Lagranj tenglamasi holida uchrashi mumkin; bu tenglamani biz keyinroq o'rganamiz. $f'_x\equiv p$ holi esa Klero tenglamasi bandida o'rganiladi.

Misol. Ushbu

$$x^3y'^2+x^2yy'-1=0$$

differensial tenglamani yechaylik.

⊶ Berilgan tenglama y ga nisbatan osongina yechiladi:

$$y=\frac{1}{x^2y'}-xy'. \tag{I.12.6}$$

Yuqorida aytilganidek, $y'=p$, ya'ni $dy=pdx$ deb, p parametrni kiritamiz. U holda

$$y=\frac{1}{x^2p}-xp. \tag{I.12.7}$$

x ni p ning funksiyasi sifatida topish maqsadida, bu tenglikni differensiallaymiz va kerakli shakl almashtirishlarni bajaramiz:

$$dy=d\left(\frac{1}{x^2p}\right)-d(xp),\quad pdx=-\frac{2}{x^3p}dx-\frac{1}{x^2p^2}dp-xdp-pdx,$$

$$\left(\frac{1}{x^2p^2}+x\right)\left(dp+\frac{2p}{x}dx\right)=0.$$

Oxirgi tenglamadan

$$\frac{1}{x^3p^2}+x=0,\ dp+\frac{2p}{x}dx=0$$

ya'ni

$$x=-\sqrt[3]{\frac{1}{p^2}},\ x=\frac{c}{\sqrt{|p|}}$$

bog`lanishlarni topamiz. Ularni (1.12.7) ga qo`yib, y ni p ning funksiyasi sifatida topamiz:

$y=2\sqrt[3]{p}$, $y=\frac{1}{c^2}\frac{|p|}{p}-c\sqrt{p}$. Yechimlarni parametrik ko`rinishda hosil qildik:

$$\begin{cases} x=-\sqrt[3]{\dfrac{1}{p^2}} \\ y=2\sqrt[3]{p} \end{cases},\quad \begin{cases} x=\dfrac{c}{\sqrt{|p|}} \\ y=\dfrac{|p|}{p}\left(\dfrac{1}{c^2}-c\sqrt{|p|}\right) \end{cases}.$$

Bu yerda p ni yo`qotib, yechimlarni oshkor ko`rinishda ham ifodalash mumkin:

$$x=-\frac{4}{y^2},\quad y=\frac{c}{x}-\frac{1}{c}.$$

b). (1.12.1) tenglama x ga nisbatan yechilgan

$x=f(y,y')$ ($f\in C^1$, $f'_{y'}\neq 0$ deb faraz qilnadi)

ko`rinishga bo`lsa ham, uning parametrik yechimlari yuqoridagi usul bilan topiladi.

c). Umumiy holda parametr kiritishning nazariyasida to`xtalmaymiz.

Masalalar

1. Quyidagi differensial tenglamani $y'=p$ parametr kiritish usuli bilan yeching: $yy'^2+x^3y'-x^2y=0$.

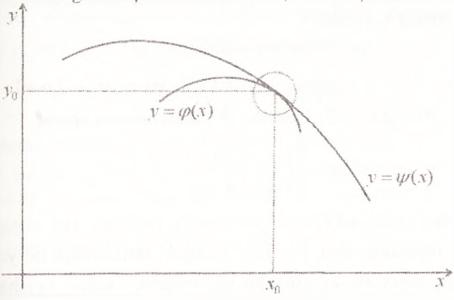
2. Ushbu $y'^2+y^2=1$ ($y=y(x)$) differensial tenglamani u parametrni $y'=\cos u$, $y=\sin u$ formulalar bilan kiritib, yeching.

I.13. Maxsus yechimlar

Ushbu

$$F(x, y, y') = 0 \qquad (I.13.1)$$

differensial tenglamani qaraylik; bu yerda ham avvalgi paragrafdagidek $F \in C^1$ deb hisoblanadi. (I.13.1) ***tenglamaning maxsus yechimi*** deb uning shunday $y = \psi(x)$ yechimiga aytiladiki, bu yechim grafigining har qanday nuqtasidan shu integral chiziqqa (grafikka) urinib boshqa bir $y = \varphi(x)$ yechim grafigi ham o'tadi; bunda $y = \psi(x)$ va $y = \varphi(x)$ yechimlar urinish nuqtasining ixtiyoriy atrofida ustma-ust tushmasliklari kerak. Shunday qilib, maxsus yechim grafigi yechimning yagonalik xossasi buziladigan nuqtalardan tuziladi (I.7-rasm).



I.7-расм

Eslatamizki, $y = \varphi(x)$ va $y = \psi(x)$ funksiyalar grafiklarining (x_0, y_0) nuqtada urinishi ularning shu nuqtada umumiy urinmaga ega ekanligini, ya'ni

$$\varphi(x_0) = \psi(x_0) = y_0, \; \varphi'(x_0) = \psi'(x_0)$$

shartlarning bajarilishini anglatadi.

$F \in C^1$ bo`lgani uchun I.12 paragrafdagi teoremadan ravshanki, (I.13.1) differensial tenglamaning maxsus yechimi grafigidagi ixtiyoriy (x, y) nuqta p ning biror qiymatida ushbu

$$\begin{cases} F(x, y, p) = 0 \\ \dfrac{\partial F(x, y, p)}{\partial p} = 0 \end{cases} \qquad (1.13.2)$$

sistemani qanoalantiradi.

Bu (1.13.2) sistemani qanoatlantiruvchi barcha (x, y) juftliklar to'plami (1.13.1) **differensial tenglamaning** $p-$**diskriminant chizig'i** deyiladi. Demak, maxsus yechim $p-$diskriminant chiziqning tarkibida bo'ladi. $p-$diskriminant chiziqda yechimning yagonalik xossasi buziladigan nuqtalardan tashqari yana boshqa nuqtalar ham bo'lishi mumkin.

$p-$diskriminant chiziq mos differensial tenglamaning yechimi bo'lmasligi mumkin. Quyidagi misol bu tasdiqni asoslaydi.

Misol 1. Ushbu

$$yy'^2 - 2y' + 1 = 0$$

tenglama uchun $F(x, y, p) = yp^2 - 2p + 1$ va $F'_p(x, y, p) = 2yp - 2$. Demak, $p-$diskriminant chiziq

$$\begin{cases} yp^2 - 2p + 1 = 0 \\ 2yp - 2 = 0 \end{cases}$$

sistemadan p ni yo'qotish yordamida topiladi. Bu sistemaning ikkinchi tenglamasidan $p = 1/y$ ni topib, birinchisiga qo'yamiz va $y = 1$ $p-$diskriminant chiziqqa ega bo'lamiz. Lekin, ravshanki, bu $y = 1$ funksiya, ya'ni $p-$diskriminant chiziq, qaralayotgan differensial tenglamaning yechimi emas.

$p-$diskriminant chiziq mos differensial tenglamaning yechimi bo'lganda ham u maxsus yechim bo'lmasligi mumkin. Bu fikrni asoslash uchun quyidagi misolni keltiramiz.

Misol 2. Ushbu

$$y'^2 + 4y^2(y-1) = 0$$

differensial tenglamaning $p-$diskriminant chizig'i va maxsus yechim(lar)ini topaylik. Berilgan tenglama uchun $F(x,y,p)=p^2+4y^3(y-1)$, $F'_p(x,y,p)=2p$. Demak, p – diskriminant chiziq

$$\begin{cases} p^2+4y^3(y-1)=0 \\ 2p=0 \end{cases}$$

sistemadan topiladi. Bu sistemadagi ikkinchi tenglamadan $p=0$ ni topib, birinchisiga qo'ysak, $y^3(y-1)=0$ tenglamaga kelamiz. Demak, p – diskriminant chiziq $y=0$ va $y=1$ shoxlardan iborat. Ravshanki, ularning har biri berilgan differensial tenglamaning yechimidir. Endi bu yechimlarni maxsus yechim bo'lish yoki bo'lmaslikka tekshiraylik. Buning uchun tenglamaning boshqa yechimlarini topamiz. Berilgan tenglamani y' ga nisbatan yechib, ikki dona o'zgaruvchilari ajraladigan tenglama hosil qilamiz: $y'=\pm 2y\sqrt{y(1-y)}$. Bu tenglamalardan ushbu

$$y=\frac{1}{(x+c)^2+1},\quad y=0,\quad y=1$$

yechimlarni topamiz (17-bet, 0.2-rasm). Ravshanki, $y=1$ yechim maxsus yechim, chunki bu yechim grafigining barcha nuqtalarida yechimning yagonalik xossasi buziladi: $y=1$ yechimning ixtiyoriy $(x_0,1)$ nuqtasidan bu yechimga urinib $y=\frac{1}{(x-x_0)^2+1}$ boshqa yechim ham o'tadi. $y=0$ yechim grafigining nuqtalarida esa yechimning yagonalik xossasi saqlanadi, ya'ni $y=0$ yechim maxsus yechim emas.

Shunday qilib, (I.13.1) differensial tenglamaning maxsus yechim(lar)ini topishni quyidagilarni ketma-ket bajarib amalga oshirish mumkin:

p – diskriminant chiziqni aniqlash, ya'ni (I.13.2) sistemani yechib, undan p ni yo'qotish;

p – diskriminant chiziq shoxlari orasidan differensial tenglama

yechimlarini ajratish;
3. topilgan yechimlar orasidan maxsus yechimlarni aniqlash.

I.14. Lagranj va Klero tenglamalari

Lagranj tenglamasi deb ushbu

$$y = x\psi(y') + \chi(y') \qquad (I.14.1)$$

ko'rinishdagi differensial tenglamaga aytiladi; bu yerda $\{\psi, \chi\} \subset C^1(I, \mathbb{R}), I \subset \mathbb{R}$ – biror oraliq, va $\psi(p) \not\equiv p$ deb faraz qilinadi.

Lagranj tenglamasini parametr kiritish usuli bilan yechamiz. Umumiy qoidadan kelib chiqib, $y' = p$ deb parametr kiritamiz va (I.14.1) tenglamadan

$$y = x\psi(p) + \chi(p)$$

bog'lanishni topamiz. Endi, agar x ni p ning funksiyasi sifatida topsak, u holda yechimni parametrik ko'rinishda topgan bo'lamiz. Buning uchun oxirgi tenglikni differensiallaymiz va

$$p = \psi(p) + x\psi'(p)\frac{dp}{dx} + \chi'(p)\frac{dp}{dx}$$

yoki

$$p - \psi(p) = \left(x\psi'(p) + \chi'(p)\right)\frac{dp}{dx}$$

tenglamani hosil qilamiz. Bu yerdan $\forall p \in I$ uchun $p - \psi(p) \neq 0$ deb, quyidagini topamiz:

$$\frac{dx}{dp} = \frac{\psi'(p)}{p - \psi(p)}x + \frac{\chi'(p)}{p - \psi(p)}.$$

Bu tenglama $x = x(p)$ funksiyaga nisbatan chiziqli birinchi tartibli differensial tenglamadir. Uning yechimi kvadraturalarda topiladi. Uni yechib, umumiy yechimini

$$x = a(p)c + b(p) \quad (a(p), b(p) \text{ – aniq funksiyalar})$$

ko'rinishda topamiz. Demak, berilgan (I.14.1) Lagranj tenglamasining parametrik ko'rinishdagi umumiy yechimi

$$\begin{cases} x = a(p)c + b(p) \\ y = \big(a(p)c + b(p)\big)\psi(p) + \chi(p) \end{cases} \qquad (p - \text{parametr})$$

formulalar bilan beriladi.

Agar $p - \psi(p) = 0$ tenglama yechimga ega bo'lsa, uning ixtiyoriy $p = p^*$ yechimidan Lagranj tenglamasining $y = x\psi(p^*) + \chi(p^*)$ yechimini ham topamiz.

Misol. Ushbu

$$y = \frac{1}{2}xy' + \ln y'$$

Lagranj tenglamasini yechaylik.

$y' = p$ parametrni kiritib,

$$y = \frac{1}{2}xp + \ln p$$

bog'lanishni hosil qilamiz. Bu tenglikni differensiallab, $dy = pdx$ ekanligini hisobga olib, $x = x(p)$ noma'lum funksiyaga nisbatan chiziqli differensial tenglama hosil qilamiz (bisda $p > 0$):

$$\frac{dx}{dp} = \frac{1}{p}x + \frac{2}{p^2}.$$

Bu tenglamaning umumiy yechimi osongina topiladi:

$$x = cp - \frac{1}{p}.$$

Endi berilgan tenglamaning parametrik ko'rinishdagi umumiy yechimini yozamiz:

$$\begin{cases} x = cp - \dfrac{1}{p} \\ y = \dfrac{1}{2}(cp - \dfrac{1}{p})p + \ln p \end{cases}$$

Klero tenglamasi deb

$$y = xy' + \chi(y') \tag{1.14.2}$$

ko'rinshdagi differensial tenglamaga aytiladi; bu yerda $\chi \in C^{1}(I,\mathbb{R})$ deb faraz qilinadi. Klero tenglamasini yechish uchun $y' = p$ parametrni kiritamiz. U holda y ning p parametrga bog'lanish qonunini hosil qilamiz:

$$y = xp + \chi(p)$$

x ning p parametrga bog'lanish qonuniyatini topish uchun Klero tenglamasi (1.14.2) ning har ikkala tomonini x bo'yicha differensiallaymiz:

$$p = p + x\frac{dp}{dx} + \chi'(p)\frac{dp}{dx} \text{ yoki } (x + \chi'(p))\frac{dp}{dx} = 0.$$

Bundan $\frac{dp}{dx} = 0$ (demak, $p = c$) yoki $x + \chi'(p) = 0$ kelib chiqadi. 1-holda $p = c$ va

$$y = cx + \chi(c) \quad \text{(to'g'ri chiziqlar oilasi)}$$

ko'rinishdagi yechimlarni topsak, 2-holda esa

$$\begin{cases} x = -\chi'(p) \\ y = -\chi'(p)\cdot p + \chi(p) \end{cases} \quad (\ p \text{ -- parametr}) \tag{1.14.3}$$

yechimni topamiz. Ko'rsatish mumkinki, agar $\chi(\cdot)$ funksiya nochiziqli bo'lsa, $y = cx + \chi(c)$ to'g'ri chiziqlar oilasining o'ramasi (1.14.3) chiziqdan iborat bo'ladi; agar $\chi(\cdot)$ funksiya chiziqli ($\chi(c) = \alpha c + \beta$) bo'lsa, u holda $y = cx + \chi(c)$ to'g'ri chiziqlarning barchasi bitta tayin (ya'ni $(-\alpha, \beta)$) nuqtadan o'tadi va ular o'rama chiziqqa ega bo'lmaydi.

1.15. Maxsus yechimni yechimlar o'ramasi sifatida topish

Maxsus yechim differensial geometriya kursida o'rganiladigan bir parametrli chiziqlar oilasining o'ramasi (yopishmasi) tushunchasi bilan bevosita bog'liq. Dastlab zarur ma'lumotlarni eslaylik.

Bizga bir parametrli silliq chiziqlar oilasi

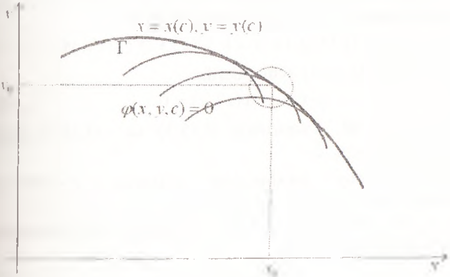
$$\varphi(x, y, c) = 0 \qquad (I.15.1)$$

tenglama bilan berilgan bo'lsin; bu yerda $\varphi(x, y, c)$ funksiya $(x, y) \in D$ ($D - \mathbb{R}^2$ tekislikdagi soha) va $c \in (c_1, c_2)$ bo'lganda C^1 sinfga tegishli, $\varphi \in C^1\big(D \times (c_1, c_2), \mathbb{R}\big)$, hamda $D \times (c_1, c_2)$ to'plamda $\varphi_x'^2 + \varphi_y'^2 \neq 0$.

Ushbu

$$\begin{cases} x = x(c) \\ y = y(c) \end{cases} \quad \big(c \in (c_1, c_2)\big) \qquad (I.15.2)$$

parametrik ko'rinishda berilgan Γ silliq chiziqni qaraylik, bunda $\{x(c), y(c)\} \subset C^1\big((c_1, c_2)\big)$ va $|x'(c)| + |y'(c)| \neq 0$. Agar bir qanday $(x_0, y_0) \in \Gamma$, $x_0 = x(c_0)$, $y_0 = y(c_0)$, $c_0 \in (c_1, c_2)$, nuqtadan shu nuqtada Γ chiziqqa urinib (I.15.1) oilaning biror chizig'i o'tsa va bu chiziq (x_0, y_0) nuqtaning ixtiyoriy atrofida Γ bilan ustma-ust tushmasa, u holda Γ chiziq (I.15.1) oilaning o'ramasi (o'rama chizig'i) deyiladi (I.8-rasm). Bu yerda Γ o'ramaning nuqtalari berilgan oilaning Γ ga urinuvchi mos chizig'ini belgilovchi parametr qiymatlari bilan patametrlangan.



I.8-rasm

(1.15.1) oilaning c – diskriminant chizig'i deb

$$\begin{cases} \varphi(x,y,c)=0 \\ \dfrac{\partial\varphi(x,y,c)}{\partial c}=0 \end{cases}$$

sistemani biror c da qanoatlantiruvchi (x,y) nuqtalar to'plamiga aytiladi.

Teorema (o'rama mavjudligining zaruriy sharti). *Berilgan (1.15.1) chiziqlar oilasining Γ o'ramasi shu oilaning c – diskriminant chizig'ida joylashadi.*

Γ o'ramadan ixtiyoriy $(x(c),y(c))\in\Gamma$ ($c\in(c_1,c_2)$) nuqtani olaylik. Bu nuqta berilgan oilaning $\varphi(x,y,c)=0$ chizig'ida yotadi va shu nuqtada bu chiziqlar urinadi, ya'ni

$$\varphi(x(c),y(c),c)=0\ \ (c\in(c_1,c_2)) \tag{1.15.3}$$

va Γ ning $\{x'(c),y'(c)\}$ urinma vektori mos chiziqning $\{\varphi'_x(x,y,c),\varphi'_y(x,y,c)\}$ normal vektoriga ortogonal (ularning skalyar ko'paytmasi nolga teng):

$$\varphi'_x(x,y,c)\cdot x'(c)+\varphi'_y(x,y,c)\cdot y'(c)=0\,. \tag{1.15.4}$$

(1.15.3) ayniyatni c bo'yicha differensiallaymiz:

$$\varphi'_x(x,y,c)\cdot x'(c)+\varphi'_y(x,y,c)\cdot y'(c)+\varphi'_c(x(c),y(c),c)=0\,.$$

Endi (1.15.3) va (1.15.4) ayniyatlardan

$$\varphi'_c(x(c),y(c),c)=0 \tag{1.15.5}$$

ekanligini topamiz. Yuqoridagi (1.15.3) va (1.15.5) tengliklar teoremani isbotlaydi.

Qo'shimcha tekshirishlar o'tkazib, c – diskriminant chiziqdan o'rama chiziq ajratib olinadi.

Faraz qilaylik, (1.13.1) differensial tenglamaning bir paramertli yechimlar oilasi

$$\Phi(x,y,c)=0 \tag{1.15.6}$$

ma'lum bo'lsin. Agar bu oila o'rama chiziqqa (o'ramaga) ega bo'lsa, u holda bu o'rama

1) differensial tenglamaning yechimidan iborat bo`ladi, chunki o'ramaning har bir $(x, y)$ nuqtasida $(x, y, y')$ uchlik (I.15.6) oilaning biror bir egri chizig'ining mos nuqtasidagi mos uchlikka teng bo`ladi.

2) (I.13.1) differensial tenglamaning maxsus yechimni berada, chunki o'ramaning har bir nuqtasidan o'ramaning o'zi va unga tegishli (I.15.6) oilaning biror chizig'i o'tadi; bu chiziqlar turli yechimlarni ifodalaydi.

(I.15.6) oilaning o`ramasini topish uchun, ma'lumki, c diskriminant chiziq deb ataluvchi chiziqlarni topib ular orasidan o`rama egri chiziqni ajratish kerak. c – diskriminant chiziqlar ushbu

$$\begin{cases} \Phi(x, y, c) = 0 \\ \dfrac{\partial \Phi(x, y, c)}{\partial c} = 0 \end{cases} \qquad (I.15.7)$$

sistemadan aniqlanadi. Umumiy holda bu sistema nafaqat o`ramani, balki u karrali nuqtalar to'plamini ham aniqlashi mumkin. (I.15.7) sistemadan c ni yo'qotib, ushbu

$$\varphi(x, y) = 0 \qquad (I.15.8)$$

c – diskriminant chiziqni topamiz. Uning o`ramadan iborat bo`lgan qismini ajratib, (I.13.1) tenglamaning maxsus yechimini aniqlaymiz.

II. YUQORI TARTIBLI DIFFERENSIAL TENGLAMALAR

II.1. Umumiy ko'rinishdagi n – tartibli differensial tenglama va uning yechimi

Aytaylik, $F(x,y,p_1,...,p_n)-\mathbb{R}^{2+n}$ $(n\in\mathbb{N}, n>1)$ fazoning biror U sohasida aniqlangan va uzluksiz haqiqiy funksiya bo'lsin. Ushbu

$$F\left(x, y(x), \frac{dy(x)}{dx},..., \frac{d^n y(x)}{dx^n}\right)=0 \text{ yoki}$$

$$F(x,y,y',...,y^{(n)})=0, \qquad \text{(II.1.1)}$$

tenglama x haqiqiy o'zgaruvchining $y=y(x)$ noma'lum funksiyasiga nisbatan **n- tartibli oddiy differensial tenglama** deyiladi; bunda F funksiya p_n o'zgaruvchiga tub ma'noda bog'liq deb hisoblanadi, ya'ni bu argumentning o'zgarishi bilan (boshqa argumentlar tayinlanganda) F funksiyaning qiymatlari ham o'zgaradi. Shunday qilib, $y=y(x)$ noma'lum funksiyaga nisbatan $n-$tartibli oddiy differensial tenglama bu – argumentning, noma'lum funksiyaning va uning $n-$tartibgacha hosilalarining ushbu x, $y(x), y'(x), y''(x), ..., y^{(n)}(x)$ qiymatlari orasidagi bog'lanish.

Ba'zi shartlarda (masalan, oshkormas funksiya haqidagi teorema shartlari bajarilganda) (II.1.1) tenglamani yuqori tartibli hosila $y^{(n)}$ ga nisbatan yechilgan ko'rinishga keltirish mumkin:

$$y^{(n)}=f(x,y,y',...,y^{(n-1)}); \qquad \text{(II.1.2)}$$

bu yerda $f\in C(D)$, $D\in\mathbb{R}^{1+n}$ – soha, deb hisoblanadi.

Misol 1. Inersial sanoq sistemasining Ox o'qida harakat qiluvchi m massali moddiy nuqtaning t paytdagi koordinatasini $x=x(t)$ bilan belgilaylik. Faraz qilaylik, moddiy nuqtaga vaqt t

ga, nuqta koordinatasi $x = x(t)$ ga va tezligi $x' = \frac{dx(t)}{dt}$ ga bog'liq bo'lgan $f(t, x, x')$ kuch ta'sir etsin. U holda Nyutonning 2 qonuniga ko'ra moddiy nuqtaning harakat tenglamasi ushbu

$$x'' = \frac{1}{m} f(t, x, x')$$

ko'rinishdagi 2-tartibli differensial tenglama bilan ifodalanadi.

$y = \varphi(x)$ funksiya berilgan bo'lsin. Agar

1^0. $y = \varphi(x)$ funksiyaning I oraliqda n - tartibli hosilasi uzluksiz, ya'ni $\varphi \in C^n(I)$;

2^0. $y = \varphi(x)$ funksiya (II.1.1) tenglamani I oraliqda qanoatlantiradi, ya'ni

$$\forall x \in I \quad F\big(x, y(x), y'(x), \ldots, y^{(n)}(x)\big) = 0;$$

shartlar o'rinli bo'lsa, berilgan $y = \varphi(x)$ funksiya (II.1.1) differensial tenglamaning **I oraliqda aniqlangan yechimi** deyiladi.

Shunday qilib, (II.1.1) differensial tenglamaning I oraliqda yechimi deb shunday funksiyaga aytiladiki, uning tenglamada qatnashgan barcha hosilalari shu oraliqda uzluksiz va uni (II.1.1) tenglamaga qo'yilganda, $x \in I$ ga nisbatan ayniyat hosil bo'ladi.

Agar

$$y = \varphi(x, c_1, c_2, \ldots, c_n) \tag{II.1.3}$$

formula $c_1, c_2, \ldots, c_n$ parametrlarning tayinlangan joiz qiymatlarida (II.1.1) tenglamaning ($\bar{D} \subset \mathbb{R}^2$ sohada joylashgan) yechimini bersa, shu bilan birgalikda (II.1.1) tenglamaning (shu sohadagi) har qanday yechimi (II.1.3) formuladan $c_1, c_2, \ldots, c_n$ larning biror joiz qiymatlarida hosil bo'lsa, u holda $y = \varphi(x, c_1, c_2, \ldots, c_n)$ oila (II.12.1) tenglamaning ($\bar{D}$ sohadagi) **umumiy yechimi** deb ataladi.

Misol 2. Quyidagi tenglamani qaraylik:

$$y'' + y = 0. \tag{II.1.4}$$

Bu yerda $F(x,y,p_1,p_2)=y+p_2$, $U=\mathbb{R}^4$. Ushbu $y=\cos x$ funksiya $(-\infty;+\infty)$ intervalda bu tenglamaning yechimi bo'ladi. Haqiqatdan ham, $y=\cos x$ va $y''=-\cos x$ ($y\in C^2(\mathbb{R})$) ifodalarni qaralayotgan tenglamaga qo'yib, $(-\infty;+\infty)$ oraliqda ayniyat hosil qilamiz:

$$(y''+y)=-\cos x+\cos x\equiv 0.$$

Na faqat $y=\cos x$ funksiya, balki ushbu $y=c_1\cos x+c_2\sin x$ funksiya ham c_1 va c_2 o'zgarmaslarning ixtiyoriy qiymatida qaralayotgan tenglamaning $(-\infty;+\infty)$ intervalda aniqlangan yechimi bo`ladi:

$$\begin{aligned} y''+y&=(c_1\cos x+c_2\sin x)''+c_1\cos x+c_2\sin x=\\ &=c_1(\cos x)''+c_2(\sin x)''+c_1\cos x+c_2\sin x=\\ &=-c_1\cos x-c_2\sin x+c_1\cos x+c_2\sin x=\\ &=0 \end{aligned}$$

Shunday qilib, $y=c_1\cos x+c_2\sin x$ funksiya (II.1.4) tenglamaning ikki parametrli (c_1,c_2 – parametrlar) yechimlar oilasini aniqlaydi. (II.1.4) tenglamaning har qanday $y=y(x)$ yechimi $y=c_1\cos x+c_2\sin x$ ko'rinishda bo'lishi II,10 paragrafdagi umumiy nazariyadan kelib chiqadi (Bu tasdiqning elementar isboti haqida masalalar bo`limiga qarang). Shunday qilib, $y''+y=0$ tenglamaning ($\mathbb{R}^2$ dagi) umumiy yechimi $y=c_1\cos x+c_2\sin x$ formula bilan ifodalanar ekan.

Umuman olganda, (II.1.1) tenglama cheksiz ko`p yechimlarga ega bo'ladi. Bitta yechimni ajratish uchun yechimdan qo`shimcha shartlar talab qilish kerak.

$n-$ tartibli differensial tenglama (II.1.1) uchun ***Koshi masalasi*** (***boshlang'ich masala***) quyidagicha qo'yiladi:

(II.1.1) tenglamaning ushbu

$$y\big|_{x_0}=y_0,\ y'\big|_{x_0}=y'_0,\ \dots,\ y^{(n-1)}\big|_{x_0}=y_0^{(n-1)} \qquad \text{(II.1.5)}$$

shartlarni (ular n ta) qanoatlantiruvchi yechimini biror $I\ni x_0$

oraliqda toping; bunda $x_0, y_0, y_0', \dots, y_0^{(n-1)}$ – berilgan sonlar va $(x_0, y_0, y_0', \dots, y_0^{(n-1)}) \in U$. (II.1.5) shartlar **Koshi shartlari** yoki **boshlang'ich shartlar** deb ataladi. Shunga e'tibor berish kerakki, bu shartlarning hammasi bitta $x = x_0$ boshlang'ich nuqtada qo'yilgan.

Misol 1 da keltirilgan moddiy nuqtaning to'g'ri chiziqli harakatini ifodalovchi tenglama uchun Koshi masalasi quyidagicha yoziladi:

$$\begin{cases} x'' = f(t, x, x'), \\ x|_{t_0} = x_0, \ x'|_{t_0} = v_0 . \end{cases}$$

Bu masala nuqtaning $t = t_0$ paytdagi x_0 koordinatasi (holati) va v_0 tezligiga ko'ra uning $x = x(t)$ harakat qonunini aniqlashni anglatadi. Bu - Koshi masalasining mexanik talqini.

Geometrik nuqtai nazaridan ushbu

$$\begin{cases} y'' = f(x, y, y') \\ y|_{x_0} = y_0, \ y'|_{x_0} = y_0' \end{cases}$$

boshlang'ich masala (x_0, y_0) nuqta orqali y_0' yo'nalishda o'tgan yechim grafigini topishni anglatadi.

Masalalar

1. Usbu $y'' + y = 0$ tenglamaning har qanday yechimi $y = c_1 \cos x + c_2 \sin x$ (c_1, c_2 – const) ko'rinishda bo'ishini isbotlang.

2. h balandlikdan v_0 boshlang'ich tezlik bilan yuqoriga tik otilgan jismning harakat tenglamasini yozing. Mos boshlang'ich masalani qo'ying va uni yeching.

II.2. Koshi masalasi yechimining mavjudligi va yagonaligi

Lipshits sharti. $f(x, y, p_1, \dots, p_{n-1})$ funksiya,

$D(f) \subset \mathbb{R}^{1+n}$, va $E \subset D(f)$ to'plam berilgan bo'lsin. Agar shunday $L > 0$ mavjud bo'lib, ixtiyoriy $(x, y, p_1, ..., p_{n-1}) \in E$ va $(x, \tilde{y}, \tilde{p}_1, ..., \tilde{p}_{n-1}) \in E$ nuqtalar uchun

$$\left| f(x, y, p_1, ..., p_{n-1}) - f(x, \tilde{y}, \tilde{p}_1, ..., \tilde{p}_{n-1}) \right| \leq$$

$$\leq L\left(|y - \tilde{y}| + |p_1 - \tilde{p}_1| + ... + |p_{n-1} - \tilde{p}_{n-1}| \right) \qquad \text{(II.2.1)}$$

tengsizlik bajarilsa, u holda $f(x, y, p_1, ..., p_{n-1})$ funksiya E to'plamda $y, p_1, ..., p_{n-1}$ o'zgaruvchilarga nisbatan **Lipshits shartini qanoatlantiradi** deyiladi.

Teorema (Lipshits sharti uchun yetarli shart). *Agar*

1. $G \subset \mathbb{R}^{1+n}$ sohaga ixtiyoriy $(x, y, p_1, ..., p_{n-1}) \in G$ va $(x, \tilde{y}, \tilde{p}_1, ..., \tilde{p}_{n-1}) \in G$ nuqtalar bilan birgalikda ularni tutashtiruvchi kesma

$$\left\{ (x, y + \theta(\tilde{y} - y), p_1 + \theta(\tilde{p}_1 - p_1), ..., p_n + \theta(\tilde{p}_n - p_n)) \,\middle|\, 0 < \theta < 1 \right\}$$

ham tegishli.

2. G sohada barcha

$$\frac{\partial f}{\partial y}, \frac{\partial f}{\partial p_1}, ..., \frac{\partial f}{\partial p_{n-1}}$$

xususiy hosilalar mavjud va chegaralangan bo'lsa, u holda f funksiya G sohada $y, p_1, ..., p_{n-1}$ o'zgaruvchilarga nisbatan Lipshits shartini qanoatlantiradi.)

⊶ Teoremaning 2- shartiga ko'ra shunday $L > 0$ son topiladki, uning uchun G sohada

$$\left| \frac{\partial f}{\partial y} \right| \leq L, \left| \frac{\partial f}{\partial p_1} \right| \leq L, ..., \left| \frac{\partial f}{\partial p_{n-1}} \right| \leq L \qquad \text{(II.2.2)}$$

tengsizliklar o'rinli bo'ladi. Ixtiyoriy $(x, y, p_1, ..., p_{n-1}) \in G$ va $(x, \tilde{y}, \tilde{p}_1, ..., \tilde{p}_{n-1}) \in G$ nuqtalar uchun teoremaning 1-sharti va chekli orttirmalar haqidagi Lagranj teoremasiga ko'ra

$$f(x, y, p_1, ..., p_{n-1}) - f(x, \tilde{y}, \tilde{p}_1, ..., \tilde{p}_{n-1}) =$$

$$= \frac{\partial f}{\partial y}(y-\bar{y}) + \frac{\partial f}{\partial p_1}(p_1-\bar{p}_1) + \ldots + \frac{\partial f}{\partial p_{n-1}}(p_{n-1}-\bar{p}_{n-1});$$

bu yerda xususiy hosilalar biror

$$(x, y+\theta^*(\bar{y}-y), p_1+\theta^*(\bar{p}_1-p_1), \ldots, p_n+\theta^*(\bar{p}_n-p_n)) \in G$$

$(0<\theta^*<1)$ nuqtada hisoblangan. Demak, (II.1.2.3) tengsizliklarga ko'ra

$$|f(x,y,p_1,\ldots,p_{n-1}) - f(x,\bar{y},\bar{p}_1,\ldots,\bar{p}_{n-1})| \le$$

$$\le \left|\frac{\partial f}{\partial y}\right||y-\bar{y}| + \left|\frac{\partial f}{\partial p_1}\right||p_1-\bar{p}_1| + \ldots + \left|\frac{\partial f}{\partial p_{n-1}}\right||p_{n-1}-\bar{p}_{n-1}| \le$$

$$\le L\left(|y-\bar{y}| + |p_1-\bar{p}_1| + \ldots + |p_{n-1}-\bar{p}_{n-1}|\right). \ \diamond$$

Mavjudlik va yagonalik teoremasi **(MYaT).** Endi ushbu

$$\begin{cases} y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \\ y\big|_{x_0} = y_0,\ y'\big|_{x_0} = y'_0, \ldots, y^{(n-1)}\big|_{x_0} = y_0^{(n-1)}. \end{cases} \qquad \text{(II.2.3)}$$

Koshi masalasini qaraylik. Bu yerda $x_0, y_0, y'_0, \ldots, y_0^{(n-1)}$ – berilgan sonlar, $(x_0, y_0, y'_0, \ldots, y_0^{(n-1)}) \in D(f)$.

Teorema (MYaT). *Ushbu*

$$H = \{(x, y, p_1, \ldots, p_{n-1}) \in \mathbb{R}^{1+n} \mid |x-x_0| \le a,$$

$$|y-y_0| \le b, |p_1-y'_0| \le b, \ldots, |p_{n-1}-y_0^{(n-1)}| \le b\} \quad (a>0, b>0)$$

parallelipipedda $f(x, y, p_1, \ldots, p_{n-1})$ funksiya birato'la barcha o'zgaruvchilari bo'yicha uzluksiz hamda $(y, p_1, \ldots, p_{n-1})$ o'zgaruvchilarga nisbatan Lipshits shartini ham qanoatlantirsin.

U holda (II.2.3) Koshi masalasi biror $|x-x_0| \le h$ $(h>0)$ oraliqda $y=y(x)$ yagona yechimga ega bo'ladi.

Bu teoremani keyinroq isbotlaymiz, aniqrog'i, umumiyroq teoremadan keltirib chiqaramiz.

Agar $f(x, y, p_1, \ldots, p_{n-1})$ funksiya $G \subset \mathbb{R}^{1+n}$ sohada

C^n sinfga tegishli bo`lsa,

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

tenglamaning lokal umumiy yechimi $y = \varphi(x, c_1, c_2, \ldots, c_n)$ formula bilan beriladi, ya'ni u n dona ixtiyoriy o`zgarmasga bog`liq bo`ladi. Bu tasdiq MYaTdan $x_0$ ni tayinlab $y\big|_{x_0} = c_1,\ y'\big|_{x_0} = c_2, \ldots, y^{(n-1)}\big|_{x_0} = c_n$ qiymatlarni o`zgartirish natijasida hosil bo`ladi.

$n-$ tartibli **chiziqli oddiy differensial tenglama** deb

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = g(x)$$

ko`rinishdagi tenglamaga aytiladi. Bu tenglama uchun Koshi masalasi yechimining mavjudligi va yagonaligi haqidagi teorema quyida keltirilgan.

Teorema. *Aytaylik, chiziqli tenglama uchun quyidagi boshlang'ich masala qo'yilgan bo'lsin:*

$$\begin{cases} y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = g(x), \\ y\big|_{x_0} = y_0,\ y'\big|_{x_0} = y_0', \ldots, y^{(n-1)}\big|_{x_0} = y_0^{(n-1)}; \end{cases}$$

bu yerda $x_0 \in I;\ y_0, y_0', \ldots, y_0^{(n-1)} -$ *berilgan sonlar. Agar* $\{a_{n-1}(x), \ldots a_1(x), a_0(x), g(x)\} \subset C(I)$ *bo'lsa, u holda qo'yilgan bu masalaning birato'la* I *oraliqda aniqlangan yechimi mavjud va yagonadir.*

Bu teoremani ham keyinroq isbotlaymiz.

Masalalar

1. Faraz qilaylik, $T = \{(t, x) \in \mathbb{R}^2 \mid |t - t_0| \le a,\ |x - x_0| \le b\}$ $(a > 0, b > 0)$ va $f \in C(T)$ uchun

$$\begin{cases} x'' = f(t, x), \\ x(t_0) = x_0,\ x'(t_0) = y_0 \end{cases} \qquad (y_0 - \text{ixtiyoriy son})$$

Koshi masalasi berilgan bo'lsin. $x = \varphi(t)$ noma'lum funksiyaga nisbatan ushbu

$$\varphi(t) = x_0 + (t-t_0)y_0 + \int_{t_0}^{t} (t-s) f(s, \varphi(s))ds \qquad (*)$$

integral tenglamani tuzaylik.

1^0. Bu integral tenglamaning $x = \varphi(t) \in C(I)$, $t_0 \in I$, yechimi berilgan Koshi masalasining yechimi ekanligini isbotlang.

2^0. Quyidagi funksional ketma-ketlikni qarang:

$$x_0(t) = x_0,$$

$$x_k(t) = x_0 + (t-t_0)y_0 + \int_{t_0}^{t} (t-s) f(s, x_{k-1}(s))ds, \; k = 1, 2, \ldots .$$

$m = |y_0| + \frac{a}{2}\sup|f(t,x)|$, $h = \min\left\{a, \frac{b}{m}\right\}$ deylik. Barcha $x_k(t)$ lar $|t-t_0| \le h$ oraliqda aniqlangan va uzluksiz ekanligini ko`rsating.

3^0. Agar $x_k(t)$ yoki yning biror qismiy ketma-ketligi $\varphi(t)$ ga $[t_0-h, t_0+h]$ da tekis yaqinlashsa, $x = \varphi(t)$ funksiya (*) integral tenglamaning, demak, berilgan Koshi masalasining yechimi ekanligini isbotlang ($x_k(t)$ dan tekis yaqinlashuvchi qismiy ketma-ketlik ajratish mumkinligi Askoli-Arsela teoremasi yordamda isbotlanishi mumkin)

II.3. Yuqori tartibli tenglamaning tartibini pasaytirish va uni yechish usullari

1. Ushbu

$$y^{(n)} = f(x), \; f \in C(I), \qquad (II.3.1)$$

tenglamani n marta integrallash orqali yechish mumkin. Bunda yechim n dona ixtiyoriy o`zgarmasga bog`liq bo`lib chiqadi: $y = y(x, c_1, c_2, \ldots, c_n)$.

2. Tenglamada noma'lum funksiya $y = y(x)$ va uning $y', y'', \ldots, y^{(k-1)}$ hosilalari oshkor ko`rinishda qatnashmagan:

$$F(x, y^{(k)}, y^{(k+1)}, \ldots, y^{(n)}) = 0. \qquad (II.3.2)$$

Bu tenglamada

$$y^{(k)} = z \qquad (II.3.3)$$

deb, yangi $z = z(x)$ noma'lum funksiyani kiritamiz. U holda z ga nisbatan $(n-k)$ - tartibli ushbu

$$F(x, z, z', \ldots, z^{(n-k)}) = 0$$

tenglamani hosil qilamiz. Bu tenglamadan

$$z = z(x, c_1, c_2, \ldots, c_{n-k})$$

yechimni topamiz. Buni (II.3.3) belgilashga qo`yamiz va $k$ marta integrallashni bajarib, $y$ noma'lumni topamiz. Bunda yana $k$ ta ixtiyoriy o`zgarmas paydo bo`ladi:

$$y = y(x, c_1, c_2, \ldots, c_{n-k}, \ldots, c_n).$$

Misol 1. Ushbu $y'' = y'^2$ tenglamani yeching.

🗝 Tenglamada y oshkor ko`rinishda qatnashmagan. $y' = z$ yangi o'zgaruvchini kiritamiz. Demak, $z' = z^2$. Bu tenglamaning $z = 0$ yechimi mavjudligi ravshan. Qolgan yechimlarni o`zgaruvchilarni ajratish yordamida topamiz:

$$\frac{dz}{z^2} = dx, \qquad \int \frac{dz}{z^2} = \int dx, \quad -\frac{1}{z} = x + c_1, \quad z = -\frac{1}{x+c_1}, \quad c_1 \in \mathbb{R}.$$

Endi $y' = z$ belgilashga ko`ra dastlabki noma'lum y ga qaytamiz.

1) $y' = 0$ va 2) $y' = -\dfrac{1}{x+c_1}$ tenglamalardan

$$y = c_2 \text{ va } y = c_2 - \ln(x + c_1)$$

yechimlar majmuasini hosil qilamiz. ♣

3. Erkli o'zgaruvchi bevosita qatnashmagan tenglama

ushbu

$$F(y, y', y'', \ldots, y^{(n)}) = 0 \qquad \text{(II.3.4)}$$

ko`rinishda bo`ladi. U **avtonom tenglama** deb ham yuritiladi.

$y' = p$ yangi noma'lum funksiyani kiritamiz va erkli o`zgaruvchi sifatida y ni qabul qilamiz. $y'', y''', \ldots, y^{(n)}$ larni $p = p(y)$ va uning y bo'yicha hosilalari orqali ifodalab chiqamiz:

$$y''=\frac{dy'}{dx}=\frac{dp}{dy}\frac{dy}{dx}=\frac{dp}{dy}p$$

$$y'''=\frac{dy''}{dx}=\frac{d}{dx}(\frac{dp}{dy}\cdot p)=\frac{d}{dy}(\frac{dp}{dy}\cdot p)\cdot\frac{dy}{dx}=\left(\frac{d^2p}{dy^2}p+\left(\frac{dp}{dy}\right)^2\right)p$$

$$y^{(n)}=\omega\left(p,\frac{dp}{dy},...,\frac{d^{n-1}p}{dy^{n-1}}\right)$$

Shuning uchun (II.3.4) tenglama ushbu

$$F\left(y,p,\frac{dp}{dy}\cdot p,...,\omega(p,\frac{dp}{dy},...,\frac{d^{n-1}p}{dy^{n-1}})\right)=0 \qquad (II.3.5)$$

ko'rinishni oladi. Bu (II.3.5) tenglama esa $n-1$ tartiblidir. Agar (II.3.5) ni yechib

$$p=\varphi(y,c_1,...,c_{n-1})$$

yechimni topsak, dastlabki nomalum y ga qaytib, ushbu

$$y'=\varphi(y,c_1,...,c_{n-1})$$

birinchi tartibli (o`zgaruvchilari ajraladigan) tenglamani hosil qilamiz.

Biz yuqorida y noma'lum funksiyani erkli o`zgaruvchi sifatida qaradik, bunda $y=\text{const}$ yechimni yo'qotishimiz mumkin. Shuning uchun (II.3.4) tenglamaga $y=b$ qo`yib, hosil bo'lgan

$$F(b,0,...,0)=0$$

tenglamani yechib, $b=b_i$ ildizlarni topsak, u holda (II.3.4) tenglamaning $y=b_i$ ko'rinishdagi o'zgarmas yechimlariga ega bo'lamiz.

Misol 2. Ushbu

$$yy''+y'^2+1=0$$

tenglamada x erkli o`zgaruvchi oshkor ko`rinishda qatnashmagan. U avtonom tenglama. Uni yechish uchun $y'=p$, $p=p(y)$ noma'lum funksiyani kiritamiz. U holda $y''=\dfrac{dy'}{dx}=\dfrac{dp}{dy}\cdot p$ va berilgan tenglama $yp\frac{dp}{dy}+p^2+1=0$ ko'rinishga keladi. Bu

birinchi tartbli tenglama osongina yechiladi: $(p^2+1)y^2=c_1$.

Bundan $y'=p$ belgilashga ko'ra $y^2y'^2+y^2=c_1$ tenglamaga

kelamiz va uni yechib topamiz: $y^2+(x-c_2)^2=c_1$. Dastlabki tenglamaning, ravshanki, $y=const$ ko'rinishdagi yechimi yo'q.

4. Noma'lum funksiya va uning hosilalariga nisbatan bir jinsli bo'lgan tenglama. Agar ushbu

$$F(x,y,y'...,y^{(n)})=0 \tag{II.3.6}$$

tenglamaning chap tomonidagi funksiya $y,y',...,y^{(n)}$ o'zgaruvchilarni λ ga ko'paytirganda u λ ning biror $m-$ darajasiga ko'paysa, ya'ni

$$F(x,\lambda y,\lambda y',...,\lambda y^{(n)})=\lambda^m F(x,y,y',...,y^{(n)}) \tag{II.3.7}$$

bo'lsa, u holda (II.3.6) tenglama $y,y',...,y^{(n)}$ larga nisbatan bir **jinsli tenglama** deyiladi. Bunday tenglamani yechish uchun

$$y'=zy$$

deb, yangi $z=z(x)$ noma'lum funksiyani kiritamiz.

Quyidagilarga egamiz:

$$y'=yz,\quad y''=y'z+yz'=(yz)z+yz'=y(z^2+z'),$$

$$y'''=y(z^3+3z'+z''),\ ...,\ y^{(n)}=y\cdot\omega(z,z',...,z^{(n-1)}).$$

Endi bu munosabatlarni (II.3.6)ga qo'yib, (II.3.7) shartni e'tiborga olsak, quyidagi tenglikka kelamiz:

$$y^m F(x,1,z,z^2+z',...,\omega(z,z',...,z^{(n-1)}))=0$$

Oxirgi tenglikni. y^m ga qisqartirib, z ga nisbatan $(n-1)$ tartibli tenglamani hosil qilamiz. Agar hosil bo'lgan tenglamaning ushbu

$$z=\varphi(x,c_1,c_2,...,c_{n-1})$$

yechimini topa olsak, bu yerda z ni $\frac{y'}{y}$ bilan almashtirib,

$$\frac{y'}{y}=\varphi(x,c_1,c_2,\dots,c_{n-1})$$

tenglamaga ega bo'lamiz.

Nihoyat, oxirgi tenglamadan

$$y=c_n\exp\left(\int\varphi(x,c_1,c_2,\dots,c_{n-1})dx\right)$$

yechimni hosil qilamiz. Bu yechimlar oilasi ($c_n=0$ bo'lganda) $y=0$ yechimni o'z ichiga oladi, ya'ni yuqorida $m>0$ bo'lganda y^m ga qisqartirishda yechim yo'qolmaydi.

Misol 3. Ushbu

$$yy''-xy'^2+yy'=0$$

tenglamani yeching.

🗝 Berilgan tenglama uchun

$$F(x,y,y',y'')=yy''-xy'^2+yy' \text{ va}$$

$$F(x,\lambda y,\lambda y',\lambda y'')=\lambda^2(yy''-xy'^2+yy')=\lambda^2F(x,y,y',y'')$$

Demak, berilgan tenglama y,y',y'' larga nisbatan bir jinsli ($m=2$). Tenglama tartibini pasaytirish uchun unda $y'=zy$ va $y''=y(z'+z^2)$ almashtirish bajaramiz. U holda berilgan tenglama quyidagi ko'rinishni oladi:

$$y^2(z'+z^2)-xz^2y^2+y^2z=0 \Rightarrow z'+z^2-xz^2+z=0.$$

Oxirgi tenglamaning har ikkala tomonini z^2 ga bo'lamiz va $1/z=u$ deymiz. Natijada ushbu $u'-u=1-x$ tenglamani hosil qilamiz. Bu chiziqli tenglama osongina yechiladi: $u=x+c_1e^x$. Bu yerda $u=1/z$, $z=y'/y$ deb, dastlabki funksiyaga qaytamiz va

$$\frac{y'}{y}=\frac{1}{x+c_1e^x} \Rightarrow y=c_2\exp\left(\int\frac{dx}{x+c_1e^x}\right)$$

yechimni topamiz (yo'qolgan $y=0$ yechim $c_2=0$ da hosil bo'ladi). ☙

5. Umumlashgan bir jinsli tenglama.

Ushbu

$$F(x, y, y', ..., y^{(n)}) = 0 \qquad (II.3.8)$$

tenglama berilgan bo'lib,. $x, y, y', ..., y^{(n)}$ larni mos ravishda $\lambda, \lambda^k, \lambda^{(k-1)}, ..., \lambda^{k-n}$ larga ko'paytirilganda tenglamaning chap tomoni uchun ushbu

$$F(\lambda x, \lambda^k y, \lambda^{(k-1)} y', ..., \lambda^{k-n} y^{(n)}) = \lambda^m F(x, y, y', ..., y^{(n)}) \qquad (II.3.9)$$

shart bajarilsa, u holda bunday (II.3.8) tenglama **umumlashgan bir jinsli tenglama** deyiladi. Bu turdagi tenglamalarni yechish uchun x va y o'rniga

$$x = e^t, \quad y = ze^{kt} \quad (x > 0) \qquad (II.3.10_0)$$

deb yangi t va $z = z(t)$ o'zgaruvchilarni kiritamiz (bu almashtirish $x > 0$ bo'lganda qo'llaniladi, $x < 0$ bo'lganda esa $x = -e^t$ $y = ze^{kt}$ almashtirishdan foydalanish kerak). y ning x bo'yicha hosilalarini yangi nomalum funksiya z ning t argument bo'yicha hosilasi orqali ifodalaymiz. Ushbu

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} = \frac{dy}{dt} \cdot \frac{1}{e^t},$$

yoki

$$y' = \frac{dy}{dt} e^{-t} \qquad (II.3.11)$$

munosabatlar ravshandir. (II.3.10$_0$) tengliklardagi ikkinchi formulani t bo'yicha differensiallab,

$$\frac{dy}{dt} = \left(\frac{dz}{dt} + kz\right) e^{kt}$$

tenglikka ega bo'lamiz. Buni (II.3.11) ga qo'yib,

$$y' = \left(\frac{dz}{dt} + kz\right) e^{(k-1)t} \qquad (II.3.1?)$$

munosabatni hosil qilamiz. Biz y ning x bo'yicha hosilasini

z ning t bo'yicha hosilasi orqali ifodalovchi munosabatga ega bo'ldik.

Xuddi shu yo'sinda ishni davom ettirib, y ning x bo'yicha yuqori tartibli hosilalarini ham z ning t bo'yicha hosilalari orqali ifodalab chiqamiz:

$$y''=\frac{dy'}{dx}=\frac{dy'}{dt}e^{-t}=\left(\frac{d^2z}{dt^2}+(2k-1)\frac{dz}{dt}+k(k-1)z\right)e^{(k-2)t}, \quad (II.3.10_2)$$

$$y'''=\frac{dy''}{dx}=\frac{dy''}{dt}e^{-t}=$$

$$=\left(\frac{d^3z}{dt^3}+(3k-3)\frac{d^2z}{dt^2}+\left(k(k-1)+(k-2)(2k-1)\right)\frac{dz}{dt}+k(k-1)(k-2)z\right)e^{(k-3)t}$$

(II.3.10$_3$)

va, nihoyat,

$$y^{(n)}=\omega\left(z,\frac{dz}{dt},\dots,\frac{d^nz}{dt^n}\right)e^{(k-n)t}. \quad (II.3.10_n)$$

Endi (II.3.8) tenglamada (II.3.10$_0$), (II.3.10$_1$), (II.3.10$_2$), ..., (II.3.10$_n$) almashtirishlarni bajarib, ushbu

$$F\left(e^t, ze^{kt}, \left(\frac{dz}{dt}+kz\right)e^{(k-1)t},\dots,\omega\left(z,\frac{dz}{dt},\dots,\frac{d^nz}{dt^n}\right)e^{(k-n)t}\right)=0$$

tenglamani hosil qilamiz. (II.3.9) tenglikdagi t o'rniga e^t ni qo'yib, e^{mt} ni F funksiya ishorasi oldiga chiqarib, va unga qisqartirib,

$$F\left(1, z, \frac{dz}{dt}+kz,\dots,\omega\left(z,\frac{dz}{dt},\dots,\frac{d^nz}{dt^n}\right)\right)=0$$

n tartibli tenglamani hosil qilamiz. Bu tenglama erkli o'zgaruvchi t ni oshkor ko'rinishda o'z ichiga olmagan; shuning uchun 3. bandga ko'ra uning tartibi bittaga kamayadi.

Misol 4. Ushbu

$$xyy''-yy'-x^3=0$$

tenglamaning tartibini pasaytiring.

Ye— Qaralayotgan tenglama uchun

$F(x,y,y',y'')=xyy''-yy'-x^3x$. Demak,

$$F(\lambda x, \lambda^k y, \lambda^{k-1} y', \lambda^{k-2} y'') = \lambda^{1+k+k-2} xyy'' - \lambda^{k+k-1} yy' - \lambda^3 x^3 =$$

$$= \lambda^{2k-1} xyy'' - \lambda^{2k-1} yy' - \lambda^3 x^3 .$$

va ushbu

$$2k-1 = 2k-1 = 3$$

shartlar bajarilganda, ya'ni $k = 2$ bo'lganda (II.3.9) umumlashgan bir jinslilik sharti bajariladi ($m = 3$). Shuning uchun berilgan umumlashgan bir jinsli tenglamada (II.3.10$_0$) $x = e^t$, $y = ze^{2t}$ ($x > 0$) almashtirishni bajarib, uni erkli o'zgaruvchi bevosita qatnashmagan ko'rinishga olib kelish mumkin. Buning uchun y', y'' larni z yangi noma'lum funksiya, t erkli o'zgaruvchi va z ning t bo'yicha hosilalari orqali ifodalaymiz:

$$y' = \frac{dy}{dx} = \frac{dy}{dt} e^{-t} = \frac{d(ze^{2t})}{dt} e^{-t} = (z' + 2z)e^t ,$$

$$y'' = \frac{dy'}{dx} = \frac{dy'}{dt} e^{-t} = \frac{d((z' + 2z)e^t)}{dt} e^{-t} =$$

$$= ((z'' + 2z')e^t + (z' + 2z)e^t)e^{-t} = z'' + 3z' + 2z .$$

Endi berilgan tenglamada zarur almashtirishlarni bajaramiz:

$$e^t ze^{2t}(z'' + 3z' + 2z) - ze^{2t}(z' + 2z)e^t - e^{3t} = 0,$$

$$zz'' + 2zz' - 1 = 0 .$$

Oxirgi tenglamada erkli o'zgaruvchi t oshkor ko'rinishda qatnashmagan. Uning tartibini pasaytirish uchun yangi noma'lum funksiya sifatida $p = z'$ ni olib, z ni erkli o'zgaruvchi sifatida qaraymiz, $p = p(z)$. U holda $z'' = \frac{dz'}{dt} = \frac{dz'}{dz} \cdot \frac{dz}{dt} = \frac{dp}{dz} \cdot p$ bo'lgani uchun ushbu

$$z \frac{dp}{dz} \cdot p + 2zp - 1 = 0$$

birinchi tartibli tenglamaga kelamiz. ♁

6. Chap tomoni to'la hosiladan iborat bo'lgan tenglama.

Faraz qilaylik,

$$F(x, y, y', ..., y^{(n)}) = 0 \qquad \text{(II.3.17)}$$

tenglamaning **chap tomoni** $x, y, y', ..., y^{(n-1)}$ o'zgaruvchilarning funksiyasi bo'lmish biror $\Phi(x, y, y', ..., y^{(n-1)})$ ning x bo'yicha **to'la hosilasidan iborat** bo'lsin, ya'ni

$$F(x, y, y', ..., y^{(n)}) = \frac{d}{dx}\Phi(x, y, y', ..., y^{(n-1)}) \qquad \text{(II.3.18)}$$

yoki

$$F(x, y, y', ..., y^{(n)}) = \frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y}y' + \frac{\partial\Phi}{\partial y'}y'' + ... + \frac{\partial\Phi}{\partial y^{(n-1)}}y^{(n)} \quad \text{(II.3.19)}$$

tenglik $x, y, y', ..., y^{(n)}$ o'zgaruvchilarning barcha joiz qiymatlariga nisbatan aynan bajarilsin.

U holda (II.3.17) tenglamaning tartibi bittaga kamayadi va ushbu

$$\Phi(x, y, y', ..., y^{(n-1)}) = c_1$$

ko'rinishni oladi.

Agar (II.3.17) tenglamaning chap tomoni to'la hosiladan iborat bo'lmasa, ba'zi hollarda shunday $\mu = \mu(x, y, y', ..., y^{(n-1)})$ funksiyani topish mumkin bo'ladiki, (II.3.17) tenglamaning har ikkala tomonini shu μ ga ko'paytirib, to'la hosilali tenglama hosil qilinadi. Bu μ funksiya **integrallovchi ko'paytuvchi** deb ataladi.

Masalan, agar berilgan tenglama

$$y'' = \frac{\dfrac{\partial G(x,y)}{\partial x} + \dfrac{\partial G(x,y)}{\partial y}y'}{\dfrac{dF(y')}{dy'}}$$

($G(x, y)$, $F(y')$ – berilgan funksiyalar)

ko'rinishda bo'lsa, $\mu = \dfrac{dF(y')}{dy'}$ integrallovchi ko'paytuvchi yordamida u quyidagi birinchi tartibli differensial tenglamaga keltiriladi: $F(y') = G(x, y) + c_1$.

Misol 5. Ushbu
$$xy'' - y' - x^2yy' = 0$$
tenglamani yechaylik.

⊶ Berilgan tenglamaning har ikkala tomonini $\mu = \frac{1}{x^2}$ integrallovchi ko'paytuvchiga ko'paytirib, quyidagilarni hosil qilamiz:
$$\frac{xy'' - y'}{x^2} - yy' = 0\,,\quad \left(\frac{y'}{x} - \frac{y^2}{2}\right)' = 0\,,\quad \frac{y'}{x} - \frac{y^2}{2} = c_1\,.$$
Oxirgi tenglama o'zgaruvchilarini ajratib yechiladi. ✍

Masalalar

Tenglamalarni yeching:

1. $2yy'' - y'^2 - y'^3 = 0$. **2.** $x^2yy'' = (y + xy')^2$.

3. $y''(1 - y') = e^{y'}$. **4.** $y'' = xy' + y + 2x$.

II.4. n – tartibli chiziqli differensial tenglamaning umumiy xossalari

Biz bu yerda ushbu
$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = g(x) \qquad \text{(II.4.1)}$$
n – tartibli chiziqli differensial tenglama yechimlarining umumiy xossalarini o'rganamiz; bunda $y = y(x)$ – noma'lum funksiya, berilgan $a_{n-1}(x),...,a_1(x), a_0(x)$ koeffitsientlar va $g(x)$ ozod had (o'ng tomon) biror I oraliqda aniqlangan va uzluksiz deb hisoblanadi, ya'ni $\{a_{n-1}(x),...a_1(x), a_0(x), g(x)\} \subset C(I)$.

Quyidagi Koshi shartlarini (boshlang'ich shartlarni) qaraylik:
$$y\Big|_{x_0} = y_0,\ y'\Big|_{x_0} = y'_0, ..., y^{(n-1)}\Big|_{x_0} = y_0^{(n-1)}; \qquad \text{(II.4.2)}$$
bunda $x_0 \in I$ va $y_0, y'_0, ..., y_0^{(n-1)}$ berilgan ixtiyoriy sonlar.

Yuqorida e'tirof etilganidek, aytilgan shartlarda

(II.4.1),(II.4.2) Koshi masalasi birato'la I oraliqda aniqlangan yechimga ega va bunday yechim yagonadir.

(II.4.1) tenglamaning ozod hadi o'rniga 0 qo'yib, (II.4.1) ga mos bir jinsli tenglamani hosil qilamiz:

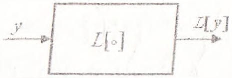
$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y = 0. \qquad \text{(II.4.1}_0\text{)}$$

Ravshanki, bu bir jinsli tenglama $y = 0$ trivial yechimga ega.

Qulaylik uchun ushbu

$$L[y] \stackrel{def}{=} y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y \qquad \text{(II.4.3)}$$

n – tartibli differensial operatorni kiritaylik. Biz bu operatorni I oraliqda n marta uzluksiz differensiallanuvchi funksiyalarga ta'sir ettiramiz, ya'ni (II.4.3) operatorni $y = y(x) \in C^n(I)$ funksiyalarda aniqlangan deb hisoblaymiz: $L[\circ]: C^n(I) \to C(I)$. $L[\circ]$ operator kirish funksiyasi y ga ko'ra chiqish funksiyasi $L[y]$ ni hisoblaydi (aniqlaydi):



Jumla. *(II.4.3) differensial operator chiziqlidir, ya'ni*

a) $\forall \{y_1, y_2\} \subset C^n(I) \quad L[y_1 + y_2] = L[y_1] + L[y_2]$ *(additivlik)*

b) $\forall \lambda \in \mathbb{R} \quad \forall y \in C^n(I) \quad L[\lambda \cdot y] = \lambda \cdot L[y]$ *(bir jinslilik)*

xossalari o'rinli.

Isboti hosilaning chiziqlilik xossasidan bevosita kelib chiqadi.

Natija. *Agar,* $\{y_1, y_2, ..., y_k\} \subset C^n(I)$ va $\{c_1, ..., c_k\} \subset \mathbb{R}$ *bo'lsa, u holda*

$$L\left[\sum_{j=1}^{k} c_j y_j\right] = \sum_{j=1}^{k} c_j L[y_j]$$

tenglik o'rinli.

$y_1, y_2, ..., y_k$ **funksiyalarning chiziqli kombinatsiyasi** deb ushbu $c_1y_1 + c_2y_2 + ... + c_ky_k$ yig'indiga aytiladi; bunda $\{c_1, ..., c_k\} \subset \mathbb{R}$.

Teorema. *(II.4.1₀) chiziqli bir jinsli differensial tenglama yechimlarining chiziqli kombinatsiyasi yana shu tenglamaning yechimidir.*

$y_1, y_2, ..., y_k$ – (II.4.1$_0$) ning yechimlari bo'lsin: $L[y_j] = 0$, $j = \overline{1,k}$. Biz ularning $\sum_{j=1}^{k} c_j y_j$ chiziqli kombinatsiya ham (II.4.1$_0$) tenglamani qanoatlantirishini tekshirishimiz kerak. Bu esa natija va berilganga ko'ra ravshan:

$$L\left[\sum_{j=1}^{k} c_j \cdot y_j\right] = \sum_{j=1}^{k} c_j \cdot L[y_j] = \sum_{j=1}^{k} c_j \cdot 0 = 0.$$

(II.4.1$_0$) tenglamaning barcha yechimlari to'plamini V_n bilan belgilaylik:

$$V_n = \{y \in C^n(I) \mid L[y] = 0\}. \qquad \text{(II.4.4)}$$

V_n da funksiyalarni qo'shish va funksiyani songa ko'paytirish amallarini odatdagicha (nuqtama-nuqta) kiritamiz.

Agar $\{y_1, y_2\} \subset V_n$ bo'lsa, $y_1 + y_2 \in V_n$, chunki (1_0) ning yechimlari yig'indisi yana yechim. Shunga o'xshash $y \in V_n \Rightarrow \lambda y \in V_n$ $(\lambda \in R)$, chunki (1_0) ning yechimini o'zgarmasga qo'paytirishdan yana yechim hosil bo'ladi. Bular V_n to'plam kiritilgan amallarga nisbatan yopiqligini anglatadi.

Endi ravshanki, (1_0) tenglamaning V_n yechimlari to'plami chiziqli (vektor) fazoni tashkil etadi. Bunda nol-vektor tenglamaning $y = 0$ trivial yechimidan iborat, $y \in V_n$ vektorga qarama-qarshi vektor $(-1)y = -y \in V_n$.

Masalalar

1. Ushbu $y = x^2$ funksiya biror $(-a, a)$ $(a > 0)$ oraliqda

$y'' + p(x)y' + q(x)y = 0, \{p,q\} \subset C((-a,a))$, ko`rinishdagi tenglamaning yechimi bo`lishi mumkinmi? $y = 1 - \cos x$ funksiyachi ?

2. $x_0 \in I$ nuqtani tayinlaylik.

$$V_n^0 = \{y \in C^n(I) \mid L[y] = 0, y(x_0) = 0\} = \{y \in V_n \mid y(x_0) = 0\} \subset V_n$$

to`plam V_n ning qismfazosimi?

3. $y'' + p(x)y' + q(x)y = 0$ tenglamaning p va q koeffitsientlariga qanday yetarli va zaruriy shart qo`yilsa u

$a)$ $y_1(x)$ va $xy_1(x)$,

$b)$ $y_1(x)$ va $1/y_1(x)$

yechimlarga ega bo`ladi?

II.5. Chiziqli erkli va chiziqli bog'langan funksiyalar

Biror I oraliqda aniqlangan $y_1 = y_1(x),\ y_2 = y_2(x),\ ...,\ y_n = y_n(x)$ funksiyalar berilgan bo'lsin. Bu funksialarning ushbu

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) + ... + \lambda_n y_n(x)$$

chiziqli kombinatsiyasini qaraylik; bu yerda $\lambda_1, \lambda_2, ..., \lambda_n$ – o'zgarmas sonlar, ular **chiziqli kombinatsiya koeffitsientlari** deb ataladi. Barcha koeffitsientlari nolga teng bo'lgan ($\lambda_1 = \lambda_2 = ... = \lambda_n = 0$) chiziqli kombinatsiya **trivial chiziqli kombinatsiya** deyiladi. Ravshanki, trivial chiziqli kombinatsiya I oraliqda aynan nolga teng. Agar berilgan funksiyalarning biror notrivial chiziqli kombinatsiyasi I oraliqda aynan nolga teng, ya'ni

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) + ... + \lambda_n y_n(x) = 0, x \in I,$$

$$\text{bunda } |\lambda_1| + |\lambda_2| + ... + |\lambda_n| \neq 0,$$

bo'lsa, u holda bu $y_1(x),\ y_2(x),\ ...,\ y_n(x)$ funksiyalar (funksiyalar sistemasi) I oraliqda **chiziqli bog'langan funksiyalar** deb ataladi. Aks holda esa, ya'ni $y_1(x),\ y_2(x),\ ...,\ y_n(x)$ funksiyalarning faqat trivial chiziqli kombinatsiyasigina I oraliqda aynan nolga teng bo'lsa, u holda bu funksiyalar I oraliqda **chiziqli erkli funksiyalar** deb yuritiladi. Shunday qilib, $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ funksiyalarning I oraliqda chiziqli erkliligi quyidagi implikatsiyaning rostligini anglatadi:

$$\sum_{i=1}^{n} \lambda_i y_i(x) = 0, x \in I \Rightarrow \lambda_1 = \lambda_2 = ... = \lambda_n = 0 .$$

Tushunarliki, $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ funksiyalarning chiziqli erkliligi bu funksiyalarning yozilish tartibiga bog'liq emas.

Misol 1. Ushbu $1, x, x^2, x^3$ funksiyalarni chiziqli erklilikka tekshiring.

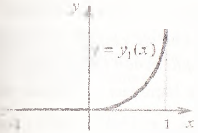
Faraz qilaylik, bu funksiyalar biror I oraliqda chiziqli bog'langan bo'lsin. U holda ularning biror notrivial chiziqli kombinatsiyasi I oraliqning har bir nuqtasida nolga aylanadi, ya'ni hammasi bir vaqtda nolga teng bo'lmagan biror λ_1, λ_2, λ_3, λ_4 ($|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda_4| \neq 0$) sonlar uchun $\lambda_1 \cdot 1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3 = 0$, $x \in I$, bo'ladi. Lekin bu holda $\lambda_1 \cdot 1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3$ ko'phad ko'pi bilan 3-darajali va, demak u ko'pi bilan 3ta nuqtada nolga aylanishi mumkin. Har qanday I oraliqda esa nuqtalar cheksiz ko'p. Shuning uchun ham $\lambda_1 \cdot 1 + \lambda_2 x + \lambda_3 x^2 + \lambda_4 x^3$ (bunda $|\lambda_1|+|\lambda_2|+|\lambda_3|+|\lambda_4| \neq 0$) ko'phad hech qanday oraliqda aynan nolga teng bo'la olmaydi. Hosil bo'lgan ziddiyat farazimizning noto'g'ri ekanligini ko'rsatadi. Demak, berilgan funksiyalar har qanday oraliqda chiziqli erkli.

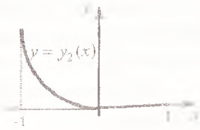
Misol 2. Ushbu

$$y_1(x) = \begin{cases} x^2, \text{ agar } x \geq 0 \text{ bo'lsa,} \\ 0, \text{ agar } x < 0 \text{ bo'lsa} \end{cases} \quad \text{va}$$

$$y_2(x) = \begin{cases} 0, \text{ agar } x \geq 0 \text{ bo'lsa,} \\ x^2, \text{ agar } x < 0 \text{ bo'lsa} \end{cases}$$

funksiyalar $I = [-1, 1]$ oraliqda chiziqli erkli (II.1- rasm.).





1.1- rasm

Haqiqatan ham,

$\lambda_1 y_1(x) + \lambda_2 y_2(x) = 0$, $x \in [-1,1]$, deylik. U holda oxirgi tenglikda $x = -1$ va $x = 1$ deb topamiz:

$$\lambda_1 y_1(-1) + \lambda_2 y_2(-1) = \lambda_1 \cdot 0 + \lambda_2 \cdot 1 = \lambda_2 = 0 \text{ va}$$

$$\lambda_1 y_1(1) + \lambda_2 y_2(1) = \lambda_1 = 0 \text{, ya'ni } \lambda_1 = \lambda_2 = 0 .$$

Demak, qaralayotgan funksiyalarning faqat trivial chiziqli kombinatsiyasigina $I = [-1,1]$ oraliqda nolga teng.

Misol 3. Ushbu $1, \sin^2 x, \cos^2 x$ funksiyalar $(-\infty,+\infty)$ oralig'ida chiziqli bog'langan, chunki ularning ushbu $(-1)\cdot 1 + 1\cdot \sin^2 x + 1\cdot \cos^2 x$ notrivial chiziqli kombinatsiyasi, ma'lumki, aynan nolga teng: $-1 + \sin^2 x + \cos^2 x = 0$.

Jumla 1. Berilgan funksiyalar chiziqli bog'langan bo'lishi uchun ularning birortasi qolganlarining chiziqli kombinatsiyasidan iborat bo'lishi yetarli va zarurdir.

Mustaqil isbotlang (chiziqli algebrani eslang).

Faraz qilaylik, berilgan ushbu $y_1 = y_1(x), y_2 = y_2(x), ..., y_n = y_n(x)$ funksiyalar I oraliqda $n-1$ marta differensiallanuvchi bo'lsin. Ularning **vronskiani (Vronskiy determinanti**) deb ushbu

$$W[y_1, y_2, ..., y_n] = \begin{vmatrix} y_1(x) & y_2(x) & ... & y_n(x) \\ y_1'(x) & y_2'(x) & ... & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & ... & y_n^{(n-1)}(x) \end{vmatrix}$$

determinantga aytiladi.

Teorema 1. *Agar* $y_1(x), y_2(x), ..., y_n(x)$ *funksiyalar* I *oraliqda* $n-1$ *marta differensiallanuvchi va chiziqli bog'langan bo'lsa, ularning vronskiani shu* I *oraliqda aynan nolga teng.*

Teoremaning shartiga ko'ra hammasi bir vaqtda nolga teng bo'lmagan $\lambda_1, \lambda_2, ..., \lambda_n$ $(|\lambda_1|+|\lambda_2|+...+|\lambda_n|\neq 0)$ sonlar uchun I oraliqda ushbu

$$\lambda_1 y_1(x)+\lambda_2 y_2(x)+...+\lambda_n y_n(x)=0,\ x\in I,$$

ayniyat o'rinli. Bu ayniyatni ketma-ket $n-1$ marta differensiallab, quyidagi ayniyatlar sistemasini hosil qilamiz:

$$\lambda_1 y_1(x)+\lambda_2 y_2(x)+...+\lambda_n y_n(x)=0$$

$$\lambda_1 y_1'(x)+\lambda_2 y_2'(x)+...+\lambda_n y_n'(x)=0$$

....

$$\lambda_1 y_1^{(n-1)}(x)+\lambda_2 y_2^{(n-1)}(x)+...+\lambda_n y_n^{(n-1)}(x)=0$$

yoki vektor ko'rinishda

$$\lambda_1\begin{pmatrix} y_1(x)\\ y_1'(x)\\ \vdots\\ y_1^{(n-1)}(x)\end{pmatrix}+\lambda_2\begin{pmatrix} y_2(x)\\ y_2'(x)\\ \vdots\\ y_2^{(n-1)}(x)\end{pmatrix}+...+\lambda_n\begin{pmatrix} y_n(x)\\ y_n'(x)\\ \vdots\\ y_n^{(n-1)}(x)\end{pmatrix}=\begin{pmatrix}0\\0\\ \vdots\\0\end{pmatrix}$$

Oxirgi vektor tenglik berilgan funksiyalar Vronskiy determinantining ustunlari orasida ixtiyoriy $x\in I$ nuqtada chiziqli bog'lanish mavjudligini anglatadi $(|\lambda_1|+|\lambda_2|+...+|\lambda_n|\neq 0)$. Demak, algebradan ma'lum teoremaga ko'ra berilgan funksiyalarning vronskiani har bir $x\in I$ nuqtada nolga teng.

Natija. *Agar* I *oraliqda* $n-1$ *marta differensiallanuvchi* $y_1(x), y_2(x), ..., y_n(x)$ *funksiyalarning* $W(x)=W[y_1, y_2, ..., y_n]$ *vronskiani* I *ning biror nuqtasida noldan farqli bo'lsa, u holda bu funksiyalar* I *oraliqda chiziqli erkli bo'ladi.*

Bu yerda shuni e'tirof etaylikki, teorema 1 va uning natijasining teskarisi o'rinli emas, ya'ni I oraliqda

$W(x)=W[y_1,y_2,\dots,y_n]=0$ ekanligidan $y_1,y_2,\dots,y_n$ funksiyalarning I oraliqda chiziqli bog'langanligi kelib chiqmaydi. Bu tasdiqni quyidagi misol asoslaydi.

Misol. Ushbu

$$y_1(x)=\begin{cases} x^2, \text{ agar } x\ge 0 \text{ bo'lsa}, \\ 0, \text{ agar } x<0 \text{ bo'lsa} \end{cases} \qquad \text{va}$$

$$y_2(x)=\begin{cases} 0, \text{ agar } x\ge 0 \text{ bo'lsa}, \\ x^2, \text{ agar } x<0 \text{ bo'lsa} \end{cases}$$

funksiyalarni qaraylik. Ravshanki, ular $C^1([-1,1])$ sinfga tegishli. Osongina tekshirib ko'rish mumkinki, $[-1,1]$ oraliqda $W(x)=W[y_1,y_2]=0$. Lekin biz bu funksiyalarning $[-1,1]$ oraliqda chiziqli erkli ekanligini yuqorida ko'rsatgan edik.

Agar qaralayotgan funksiyalar biror chiziqli (uzluksiz koeffitsientli) bir jinsli differensial tenglamaning yechimlari bo'lsa, u holda bu funksiyalar vronskianining nolga tengligidan ularning chiziqli bog'langan ekanligi kelib chiqadi. Bu tasdiq quyidagi teoremaning bir qismidir.

Teorema 2. *Aytaylik, $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ funksiyalar n-tartibli chiziqli (I oraliqda uzluksiz koeffitsientli) bir jinsli differensial tenglama $L[y]=0$ ning yechimlari bo'lsin. U holda bu $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ yechimlarning (I da) chiziqli bog'langan bo'lishi uchun $W(x)=W[y_1,y_2,\dots,y_n]$ vronskianning biror nuqtada nolga teng bo'lishi yetarli va zarurdir.*

◄ **Yetarliligi.** Faraz qilaylik, $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ yechimlarning vronskiani $x_0\in I$ nuqtada nolga teng bo'lsin, $W(x_0)=0$. Demak, algebradan ma'lum teoremaga ko'ra $\lambda_1, \lambda_2, \dots, \lambda_n$ noma'lumlarga nisbatan

$$\lambda_1 y_1(x_0)+\lambda_2 y_2(x_0)+\dots+\lambda_n y_n(x_0)=0$$
$$\lambda_1 y_1'(x_0)+\lambda_2 y_2'(x_0)+\dots+\lambda_n y_n'(x_0)=0$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\lambda_1 y_1^{(n-1)}(x_0)+\lambda_2 y_2^{(n-1)}(x_0)+...+\lambda_n y_n^{(n-1)}(x_0)=0$$

chiziqli bir jinsli algebraik sistema $\lambda_1, \lambda_2, ..., \lambda_n$ $(|\lambda_1|+|\lambda_2|+...+|\lambda_n|\neq 0)$ notrivial yechimga ega. Endi ana shu notrivial yechimlarning birortasi $\lambda_1, \lambda_2, ..., \lambda_n$ ni olib, bu sonlarga ko'ra $y(x)=\lambda_1 y_1(x)+\lambda_2 y_2(x)+...+\lambda_n y_n(x)$ funksiyani tuzaylik. Bu funksiya yechimlarning chiziqli kombinatsiyasi sifatida $L[y]=0$ tenglamaning yechimi va $\lambda_1, \lambda_2, ..., \lambda_n$ larning tanlab olinishiga ko'ra $y(x_0)=0, y'(x_0)=0, ..., y^{(n-1)}(x_0)=0$. Yechimning yagonalik xossasiga ko'ra $y(x)\equiv 0$, ya'ni tanlangan $\lambda_1, \lambda_2, ..., \lambda_n$ $(|\lambda_1|+|\lambda_2|+...+|\lambda_n|\neq 0)$ lar uchun I oraliqda $\lambda_1 y_1(x)+\lambda_2 y_2(x)+...+\lambda_n y_n(x)=0$. Bu $y_1(x), y_2(x), ..., y_n(x)$ yechimlarning chiziqli bog'langan ekanligini anglatadi.

Zarurligi. $y_1(x), y_2(x), ..., y_n(x)$ yechimlar chiziqli bog'langan bo'lsin. Bu holda ularning vronskiani teorema 1ga ko'ra barcha nuqtalarda nolga teng. ♨

Isbotlangan bu teoremadan quyidagi bevosita kelib chiqadi.

Teorema 3. *Aytaylik, $y_1(x), y_2(x), ..., y_n(x)$ funksiyalar $n-$tartibli chiziqli bir jinsli $L[y]=0$ differensial tenglamaning I oraliqda aniqlangan yechimlari bo'lsin. U holda $y_1(x), y_2(x), ..., y_n(x)$ yechimlarning vronskiani yo I oraliqda nolga aylanmaydi va bu yechimlar chiziqli erkli, yoki yechimlarning vronskiani I oraliqda aynan nolga teng va bu yechimlar chiziqli bog'langan bo'ladi.*

⊶ Faraz qilaylik, $y_1(x), y_2(x), ..., y_n(x)$ yechimlarning $W(x)=W[y_1, y_2, ..., y_n]$ vronskiani I oraliqda nolga aylanmasin. U holda bu yechimlar chiziqli erkli bo'ladi, chunki, agar ular chiziqli bog'langan bo'lganda edi, teorema 2 ga ko'ra $W(x)$ vronskian farazimizga zid ravishda aynan nolga teng bo'lardi.

Endi teskarisini faraz qilaylik, ya'ni

$y_1(x), y_2(x), ..., y_n(x)$ yechimlarning $W(x)$ vronskiani I oraliqning biror nuqtasida nolga aylangan bo'lsin. U holda yana teorema 2 ga ko'ra bu yechimlar chizqli bog'langan va, demak, $W(x)$ vronskian I oraliqda aynan nolga teng.

Masalalar

1. $y_1 \equiv 0, y_2(x), ..., y_n(x), x \in I$, funksiyalar chiziqli erkli bo'lishi mumkunmi?
2. Ikkita funksiya biri ikkinchisiga proporsional bo'lgan taqdirdagina chiziqli bog'liq bo'ladi. Shuni isbotlang.
3. Jumla 1 ni isbotlang.

II.6. Chiziqli bir jinsli tenglama umumiy yechimining tuzilishi

Bu paragragda quyidagi n – tartibli chiziqli uzluksiz koeffitsientli bir jinsli differensial tenglama yechimining tuzilishini o'rganamiz:

$$L[y] = 0, \qquad \text{(II.6.1)}$$

bunda $L[y] = y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_1(x)y' + a_0(x)y$ va

$$\{a_{n-1}(x), ..., a_1(x), a_0(x)\} \subset C(I).$$

n – tartibli chiziqli bir jinsli differensial tenglama (II.6.1) ning n dona chiziqli erkli yechimlari bu **tenglamaning bazis yechimlari** yoki **yechimlarning fundamental sistemasi** deb ataladi. Yuqoridagi II.5 paragrafda asoslanganiga ko'ra (II.6.1) tenglamaning n dona yechimlari bazis yechimlarni tashkil etishini (etmasligini) bu yechimlarning vronskiani orqali aniqlash mumkin: agar bu yechimlarning vronskiani nolga aylanmasa, ular bazis yechimlarni tashkil etadi. Quyidagi teorema (II.6.1) tenglama umumiy yechimining tuzilishini ochadi.

Teorema. *Ixtiyoriy chiziqli bir jinsli (II.6.1) differensial tenglamaning bazis yechimlari mavjud va uning har qanday yechimi biror bazis yechimlarining chiziqli kombinatsiyasidan iborat, ya'ni* $\dim V_n = n$.

Quyidagi n guruh boshlang'ich shartlarni qarayluk

$(x_0 \in I)$:

$$y(x_0)=1,\ y'(x_0)=0,\ y''(x_0)=0, \ldots, y^{(n-2)}(x_0)=0,\ y^{(n-1)}(x_0)=0; \tag{II.6.2$_1$}$$

$$y(x_0)=0,\ y'(x_0)=1,\ y''(x_0)=0, \ldots, y^{(n-2)}(x_0)=0,\ y^{(n-1)}(x_0)=0; \tag{II.6.2$_2$}$$

....

$$y(x_0)=0,\ y'(x_0)=0,\ y''(x_0)=0, \ldots, y^{(n-2)}(x_0)=0,\ y^{(n-1)}(x_0)=1. \tag{II.6.2$_n$}$$

Ushbu (II.6.1),(II.6.2$_1$); (II.6.1),(II.6.2$_2$); ... ; (II.6.1),(II.6.2$_n$) n dona Koshi masalasining har biri I oraliqda aniqlangan yagona yechimga ega. Bu yechimlarni mos ravishda $y=\varphi_1(x), y=\varphi_2(x), \ldots, y=\varphi_n(x)$ bilan belgilaylik. Shunday qilib,

$$L[\varphi_i(x)]=0,\ \ \varphi_i^{(i-1)}(x_0)=1, \varphi_i^{(j)}(x_0)=0;\ i,j=\overline{1,n},\ j\neq i-1$$

$$(\text{ya'ni } \varphi_i^{(j)}(x_0)=\delta_{i,j+1}). \tag{II.6.3}$$

Bu $y=\varphi_1(x), y=\varphi_2(x), \ldots, y=\varphi_n(x)$ yechimlar chiziqli erkli, chunki ularning vronskiani $x=x_0$ nuqtada, (II.6.3) ga ko'ra, birga teng. Demak, ular bazis yechimlarni tashkil etadi.

Endi ixtiyoriy $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$ bazis yechimlar berilgan bo'lsin. Ular yuqorida qurilgan bazis yechimlardan farqli bo'lishi ham mumkin. Ixtiyoriy $y=y(x)$ yechim shu bazis yechimlarning chiziqli kombinatsiyasidan iborat ekanligini ko'rsatishimiz kerak. $\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)$ chiziqli erkli yechimlar bo'lgani uchun ularning vronskiani ixtiyoriy $x_0 \in I$ nuqtada noldan farqli. Demak, ushbu

$$c_1\varphi_1(x_0)+c_2\varphi_2(x_0)+\ldots+c_n\varphi_n(x_0)=y(x_0),$$
$$c_1\varphi_1'(x_0)+c_2\varphi_2'(x_0)+\ldots+c_n\varphi_n'(x_0)=y'(x_0),$$
....
$$c_1\varphi_1^{(n-1)}(x_0)+c_2\varphi_2^{(n-1)}(x_0)+\ldots+c_n\varphi_n^{(n-1)}(x_0)=y^{(n-1)}(x_0)$$

algebraik sistemani qanoatlantiruvchi yagona $c_1, c_2, \ldots, c_n$ yechim mavjud. Ana shu $c_1, c_2, \ldots, c_n$ larga ko'ra $\tilde{y}(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + \ldots + c_n\varphi_n(x)$ funksiyani tuzaylik. Yechimlarning chiziqli kombinatsiyasi sifatida $\tilde{y}(x)$ ham yechim. Bundan tashqari, $c_1, c_2, \ldots, c_n$ larning tanlanishiga ko'ra $\tilde{y}(x_0) = y(x_0), \tilde{y}'(x_0) = y'(x_0), \ldots, \tilde{y}^{(n-1)}(x_0) = y^{(n-1)}(x_0)$. Demak, $\tilde{y}(x)$ va $y(x)$ yechimlar bir xil boshlang'ich shartlarni qanoatlantiradi. Yechimning yagonalik xossasiga ko'ra $\tilde{y}(x) \equiv y(x)$, ya'ni $y(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + \ldots + c_n\varphi_n(x)$. ■

Isbotlangan teoremada $L[\circ]$ operatorning koeffitsientlari uzluksiz ekanligi ahamiyatli. Agar $L[\circ]$ operatorning koeffitsientlari uzluksiz bo'lmasa, teoremaning xulosasi noto'g'ri bo'lishi mumkin. Buni quyidagi misol asoslaydi.

Misol. Ushbu

$$y'' - a(x)y' = 0, \quad a(x) = \begin{cases} \dfrac{2}{x}, & \text{agar } x \neq 0 \text{ bo'lsa}, \\ 0, & \text{agar } x = 0 \text{ bo'lsa}, \end{cases}$$

ikkinchi tartibli (uzluksiz bo'lmagan koeffitsientli) chiziqli differensial tenglamani qaraylik. Bu tenglama $(-\infty; +\infty)$ oraliqda aniqlangan $y_1 = 1, y_2 = x^3, y_3 = |x|^3$ yechimlarga ega (tekshirib ko'ring). Bu yechimlar $\mathbb{R}$ da chiziqli erkli. Haqiqatan ham, faraz qilaylik,

$$\lambda_1 + \lambda_2 x^3 + \lambda_3 |x|^3 = 0, \quad x \in \mathbb{R},$$

bo'lsin. U holda

$$\lambda_1 + \lambda_2 x^3 + \lambda_3 x^3 = 0, \quad x > 0,$$

$$\lambda_1 + \lambda_2 x^3 - \lambda_3 x^3 = 0, \quad x \le 0,$$

bo'ladi. Bu tengliklardan $\lambda_1 = 0, \lambda_2 + \lambda_3 = 0, \lambda_2 - \lambda_3 = 0$, ya'ni $\lambda_1 = \lambda_2 = \lambda_3 = 0$ ekanligini topamiz. Demak, qaralayotgan $y_1 = 1, y_2 = x^3, y_3 = |x|^3$ funksiyalarning faqat trivial chiziqli kombinatsiyasigina aynan nolga teng, ya'ni ular chiziqli erkli.

Shunday qilib, berilgan ikkinchi tartibli chiziqli differensial tenglama uch dona chiziqli erkli yechimga ega ekan. Agar $a(x)$ koeffitsient $(-\infty;+\infty)$ oraliqda uzluksiz bo'lganda edi, bu differensial tenglama ikkita chiziqli erkli yechimga ega bo'lib, ixtiyoriy uchinchi yechim shu ikki yechimning chiziqli kombinatsiyasidan iborat bo'lar edi. ♣

Masalalar

1. $y''+\omega^2 y=0$, $\omega=\text{const}>0$, tenglamaning umumiy yechimi $y=c_1\cos\omega x+c_2\sin\cos\omega x$ formula bilan berilishini ko'rsating.

2. Ushbu $x^2y''+pxy'+qy=0$, $p,q-\text{const}$, $x>0$, Eyler tenglamasini qaraylik. $\lambda^2+(p-1)\lambda+q=0$ kvadrat tenglamaning diskriminantini $D=(p-1)^2-4q$, ildizlarini esa λ_1,λ_2 bilan belgilaylik. Eyler tenglamasining umumiy yechimi

$$y=c_1x^{\lambda_1}+c_2x^{\lambda_2},\ \text{agar } D>0 \text{ bo'lsa;}$$

$$y=(c_1+c_2\ln x)x^{\lambda_1},\ \text{agar } D=0 \text{ bo'lsa;}$$

$$y=(c_1\cos(\operatorname{Im}\lambda_1\ln x)+c_2\sin(\operatorname{Im}\lambda_1\ln x))x^{\operatorname{Re}\lambda_1},\ \text{agar } D<0 \text{ bo'lsa,}$$

ko'rinishda tasvirlanadi. Shuni isbotlang.

II.7. Bazis yechimlariga ko'ra chiziqli bir jinsli differensial tenglamani tiklash. Ostrogradskiy-Liuvill formulasi

Bazis yechimlariga ko'ra mos chiziqli bir jinsli differensial tenglamani tiklash. Biz yuqorida $n-$tartibli chiziqli bir jinsli differensial tenglama $L[y]=0$ ning bazis yechimlari mavjud ekanligini asoslagan edik. Endi teskari masala bilan shug'ullanamiz, ya'ni bazis yechimlariga ko'ra mos differensial tenglamani tiklash masalasini o'rganamiz.

Teorema. *Aytaylik,* $\{\varphi_1(x),\ \varphi_2(x),\ \ldots,\ \varphi_n(x)\}\subset C^n(I)$ *funksiyalarning* $W(x)=W[\varphi_1(x),\ \varphi_2(x),\ \ldots,\ \varphi_n(x)]$ *vronskiani*

I oraliqda nolga aylanmasin. U holda bazis yechimlari shu funksiyalardan iborat bo'lgan $L[y]=0$ ko'rinishdagi n- tartibli chiziqli uzluksiz koeffitsientli bir jinsli differensial tenglama mavjud va yagonadir.

$y=y(x)$ noma'lum funksiyaga nisbatan usbu

$$\frac{1}{W(x)}\cdot\begin{vmatrix}\varphi_1(x) & \varphi_2(x) & \dots & \varphi_n(x) & y\\ \varphi_1'(x) & \varphi_2'(x) & \dots & \varphi_n'(x) & y'\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \varphi_1^{(n-1)}(x) & \varphi_2^{(n-1)}(x) & \cdots & \varphi_n^{(n-1)}(x) & y^{(n-1)}\\ \varphi_1^{(n)}(x) & \varphi_2^{(n)}(x) & \cdots & \varphi_n^{(n)}(x) & y^{(n)}\end{vmatrix}=0 \qquad (II.7.1)$$

differensial tenglamani tuzaylik. Ravshanki, bu tenglama n dona $y=\varphi_1(x), y=\varphi_2(x), \dots, y=\varphi_n(x)$ yechimlarga ega (ikkita ustuni bir xil bo'lgan determinantning qiymati nolga teng). (II.7.1) dagi determinantni oxirgi ustuni bo'ylab Laplas formuasiga ko'ra yoyib, tuzilgan tenglamani quyidagi ko'rinishda yozamiz:

$$y^{(n)}+\bar{a}_{n-1}(x)y^{(n-1)}+\dots+\bar{a}_1(x)y'+\bar{a}_0(x)y=0;$$

bu yerda $\bar{a}_{n-1}(x),\dots,\bar{a}_1(x),\bar{a}_0(x)$ koeffitsientlar berilgan funksiyalar va ularning n- tartibligacha hosilalari orqali ko'paytirish, qo'shish va $W(x)\neq 0$ vronsianga bo'lish amallari yordamida ifodalanadi va shuning uchun ular I oraliqda uzluksiz. Tushunarliki, berilgan funksiyalar qurilgan shu n- tartibli chiziqli bir jinsli differensial tenglamaning bazis yechimlaridir ($W(x)\neq 0$). Endi bunday ko'rinishdagi tenglamaning yagonaligini ko'rsatamiz.

Faraz qilaylik, berilgan $y=\varphi_1(x), y=\varphi_2(x), \dots, y=\varphi_n(x)$ funksiyalar ushbu $y^{(n)}+a_{n-1}(x)y^{(n-1)}+\dots+a_1(x)y'+a_0(x)y=0$ uzluksiz koeffitsientli tenglamaning yechimlari bo'lsin, ya'ni I oraliqda

$$a_0(x)\varphi_1(x)+a_0(x)\varphi_1'(x)+\dots+a_{n-1}(x)\varphi_1^{(n-1)}(x)+\varphi_1^{(n)}(x)\equiv 0,$$

$$a_0(x)\varphi_2(x)+a_0(x)\varphi_2'(x)+\dots+a_{n-1}(x)\varphi_2^{(n-1)}(x)+\varphi_2^{(n)}(x)\equiv 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_0(x)\varphi_n(x)+a_0(x)\varphi_n'(x)+\dots+a_{n-1}(x)\varphi_n^{(n-1)}(x)+\varphi_n^{(n)}(x)\equiv 0$$

Bu ayniyatlarni $a_0(x), a_1(x), \dots, a_{n-1}(x)$ noma'lumlarga nisbatan chiziqli algebraik tenglamalar sistemasi deb qaraylik. Sistemaning determinanti

$$\begin{vmatrix} \varphi_1(x) & \varphi_1'(x) & \dots & \varphi_1^{(n-1)}(x) \\ \varphi_2(x) & \varphi_2'(x) & \dots & \varphi_2^{(n-1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_n(x) & \varphi_n'(x) & \dots & \varphi_n^{(n-1)}(x) \end{vmatrix} = W(x) \neq 0, x \in I.$$

Demak, $a_0(x), a_1(x), \dots, a_{n-1}(x)$ lar bu sistemadan Kramer formulalari yordamida $\varphi_i^{(j)}(x)$ $\left(i, j = \overline{1,n}\right)$ uzluksiz funksiyalar orqali bir qiymatli topiladi. ♁

Ostrogradskiy-Liuvill formulasi. $L[y] = 0$ chiziqli tenglama yechimlarining vronskiani uchun Ostrogradskiy-Liuvill formulasi deb ataluvchi formulani hosil qilamiz.

Dastlab determinantni differensiallash qoidasini keltiramiz.

Lemma. *Differensiallanuvchi $d_{ij} = d_{ij}(x)$ funksiyalardan tuzilgan n - tartibli*

$$\Delta = \begin{vmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{vmatrix}$$

determinantning hosilasi uchun quyidagi formula o'rinli:

$$\Delta' = \Delta_1 + \Delta_2 + \dots + \Delta_n, \qquad (II.7.2)$$

bunda Δ_i determinant Δ ning i - satridagi elementlarning o'rniga ularning hosilasini yozishdan hosil bo'lgan.

🗝 Determinant ta'rifiga ko'ra

$$\Delta = \sum (-1)^{\sigma(j_1, j_2, \dots, j_n)} d_{1j_1} d_{2j_2} \dots d_{nj_n}, \qquad (II.7.3)$$

bunda yig'indi $1, 2, \dots, n$ sonlarining barcha $j_1, j_2, \dots, j_n$ o'rinalmashtirishlari bo'yicha hisoblangan, $\sigma(j_1, j_2, \dots, j_n)$ bilan

$j_1, j_2, \dots, j_n$ o'rinalmashtirishning juftliligi belgilangan.

Quyidagilarga egamiz:

$$\Delta' = \sum (-1)^{\sigma(j_1,j_2,\dots,j_n)} (d_{1j_1} d_{2j_2} \dots d_{nj_n})' =$$

$$= \sum (-1)^{\sigma(j_1,j_2,\dots,j_n)} \left(d'_{1j_1} d_{2j_2} \dots d_{nj_n} + d_{1j_1} d'_{2j_2} \dots d_{nj_n} + \dots + d_{1j_1} d_{2j_2} \dots d'_{nj_n}\right) =$$

$$= \sum (-1)^{\sigma(j_1,j_2,\dots,j_n)} d'_{1j_1} d_{2j_2} \dots d_{nj_n} + \sum (-1)^{\sigma(j_1,j_2,\dots,j_n)} d_{1j_1} d'_{2j_2} \dots d_{nj_n} +$$

$$+ \dots + \sum (-1)^{\sigma(j_1,j_2,\dots,j_n)} d_{1j_1} d_{2j_2} \dots d'_{nj_n} = \Delta_1 + \Delta_2 + \dots + \Delta_n .$$

bu yerda

$$\Delta_i = \sum (-1)^{\sigma(j_1,j_2,\dots,j_n)} d_{1j_1} d_{2j_2} \dots d'_{ij_i} \dots d_{nj_n} \quad (i = \overline{1,n})$$

determinant Δ determinantdagi i- satrni uning hosilasi bilan almashtirishdan hosil bo'lgan.

Teorema (Ostrogradskiy-Liuvill formulasi). *n- tartibli chiziqli bir jinsli differensial tenglama* $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = 0$ *ning n dona* $y = y_1(x),\ y = y_2(x),\ \dots,\ y = y_n(x)$ *yechimlarining* $W(x)$ *vronskiani uchun ushbu*

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} a_{n-1}(s)ds\right) \qquad \text{(II.7.4)}$$

***Ostrogradskiy-Liuvill formulasi** deb ataluvchi formula o'rinli.*

Vronskian ta'rifiga ko'ra

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y'_1(x) & y'_2(x) & \dots & y'_n(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix}.$$

Bu determinantni (II.7.2) formuladan foydalanib differensiallaymiz. Bunda hosil bo'luvchi determinantlarning dastlabki $n-1$ tasi nolga teng bo'ladi, chunki ularning ikkita satri bir xil. Natijada

$$W'(x)=\begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \dots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & y_2^{(n)}(x) & \cdots & y_n^{(n)}(x) \end{vmatrix} \qquad (II.7.5)$$

tenglik hosil bo'ladi. Endi $y_i(x)$ larning yechim ekanligidan foydalanamiz:

$$y_i^{(n)}(x)+a_{n-1}(x)y_i^{(n-1)}(x)+\ldots+a_1(x)y_i'(x)+a_0(x)y_i(x)=0,$$

ya'ni

$$y_i^{(n)}(x)=-a_{n-1}(x)y_i^{(n-1)}(x)-\ldots-a_1(x)y_i'(x)-a_0(x)y_i(x)$$

$$(i=\overline{1,n}). \qquad (II.7.6)$$

(II.7.5) da o'ng tomondagi determinantning birinchi satrini $a_0(x)$ ga, ikkinchi satrini $a_1(x)$ ga va h.k. $n-1$ satrini $a_{n-2}(x)$ ga ko'paytirib, ularni oxirgi n - satrga qo'shamiz. (II.7.6) tengliklarga ko'ra oxirgi satrda $-a_{n-1}(x)$ umumiy ko'paytuvchi hosil bo'ladi. Uni determinant oldiga chiqarib, ushbu

$$W'(x)=-a_{n-1}(x)W(x)$$

tenglikni topamiz. Bu tenglikdan endi (II.7.4) formula ravshan.

Chiziqli tenglama tartibini pasaytirish. Agar $L[y]=0$ chiziqli tenglamaning $y=\varphi_1(x)$, $\varphi_1(x)\neq 0$, yechimi ma'lum bo'lsa ($L[\varphi_1(x)]=0$), bu tenglamaning tartibini bittaga kamaytirish mumkin. Buning yechun tenglamada $y=\varphi_1(x)u$ almashtirishni bajaramiz; bunda $u=u(x)$ yangi noma'lum funksiya. Kerakli hosilalarni hisoblaymiz:

$$y'=\varphi_1(x)u'+\varphi_1'(x)u,$$

$$y''=\varphi_1(x)u''+2\varphi_1'(x)u'+\varphi_1''(x)u,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$y^{(n)}=\varphi_1(x)u^{(n)}+n\varphi_1'(x)u^{(n-1)}+\frac{n(n-1)}{2}\varphi_1''(x)u^{(n-2)}+\cdots+\varphi_1^{(n)}(x)u$$

Bularni $L[y]=0$ tenglamaga qo'yib, quyidagi tenglamani hosil qilamiz:

$$L[\varphi_1(x)u] \equiv \varphi_1(x)u^{(n)} + \tilde{a}_{n-1}(x)u^{(n-1)} + \cdots + \tilde{a}_1(x)u' + \tilde{a}_0(x)u = 0 .$$

Ravshanki, $u=1$ bu tenglamaning yechimi, chunki $L[\varphi_1(x)]=0$. Demak, $\tilde{a}_0(x) \equiv 0$ va $u=u(x)$ noma'lum funksiya uchun ushbu

$$\varphi_1(x)u^{(n)} + \tilde{a}_{n-1}(x)u^{(n-1)} + \cdots + \tilde{a}_1(x)u' = 0$$

tenglama hosil bo'ladi. Bu tenglamada $u=u(x)$ noma'lum oshkor ko'rinishda qatnashmaganligi uchun $v=u'$ deb, ($\varphi_1(x)$ nolga aylanmagan oraliqda) v ga nisbatan $(n-1)$- tartibli differensial tenglamaga kelamiz.

Agar ikkinchi tartibli chiziqli bir jinsli tenglama $y''+a_1(x)y'+a_0(x)y=0$ ning biror $y=\varphi_1(x)$, $\varphi_1(x)\neq 0$, yechimi ma'lum bo'lsa, tenglamaning bu yechimga chiziqli bog'liq bo'lmagan ikkinchi $y=y(x)$ yechimini Ostrogradskiy-Liuvill formulasidan foydalanib topish ham mumkin:

$$\begin{vmatrix} \varphi_1(x) & y \\ \varphi_1'(x) & y' \end{vmatrix} = c_1 \exp\left(-\int_{x_0}^{x} a_1(s)ds\right),$$

$$\varphi_1(x)y' - \varphi_1'(x)y = c_1 \exp\left(-\int_{x_0}^{x} a_1(s)ds\right).$$

Oxirgi chiziqli birinchi tartibli tenglama standart usullar yordamida yechiladi.

Masalalar

1. $\varphi(x)$ funksiya $C^2(I)$ sinfga tegishli va I oraliqda nolga aylanmasin. Yechimlari

   a) $\varphi(x)$ va $x\varphi(x)$;

   b) $\varphi(x)$ va $1/\varphi(x)$ ($\varphi'(x)\neq 0$)

   bo'lgan chiziqli ikkinchi tartibli differensial tenglama tuzing.

2. Agar ikkinchi tartibli chiziqli bir jinsli tenglama $y''+a_1(x)y'+a_0(x)y=0$ ning biror $y=\varphi_1(x)$, $\varphi_1(x)\neq 0$, yechimi ma'lum bo'lsa, tenglamaning bu yechimga chiziqli bog'liq bo'lmagan

ikkinchi yechimini toping.

3. Agar $y''+a_1(x)y'+a_0(x)y=0$, $\{a_1(x),a_0(x)\}\subset C([a,+\infty))$ tenglamaning har qanday yechimi o`zining birinchi tartibli hosilasi bilan birgalikda $x\to+\infty$ da nolga intilsa, $a_1(x)$ funksiya to`g`risida nima deyish mumkin?

II.8. n-tartibli chiziqli bir jinsli bo`lmagan tenglamani yechish

Bir jinsli bo'lmagan $L[y]=g(x)$ tenglamani qaraylik. Bu tenglama umumiy yechimining tuzilishini quyidagi teorema ochadi.

Теорема. *Aytaylik, $y=\psi(x)$ funksiya bir jinsli bo'lmagan $L[y]=g(x)$ tenglamaning biror xususiy yechimi ($L[\psi]=g(x)$), $\varphi_1,\dots,\varphi_n$ funksiyalar esa mos bir jinsli $L[y]=0$ tenglamaning bazis yechimlari bo'lsin. U holda $L[y]=g(x)$ tenglamaning umumiy yechimi $y=\psi+\sum_{i=1}^{n}c_i\varphi_i$ $(c_1,\dots,c_n-const)$ formula bilan beriladi, ya'ni bir jinsli bo'lmagan tenglamaning umumiy yechimi shu tenglamaning biror xususiy yechimiga mos bir jinsli tenglamaning umumiy yechimini qo'shishdan hosil bo'ladi.*

8→ Ravshanki, $y=\sum_{i=1}^{n}c_i\varphi_i+\psi$ funksiya $c_1,\dots,c_n$ larning ixtiyoriy tayinlangan qiymatlarida bir jinsli bo'lmagan $L[y]=g(x)$ tenglamaning yechimi:

$$L[y]=L\left[\sum_{i=1}^{n}c_i\varphi_i+\psi\right]=\sum_{i=1}^{n}c_iL[\varphi_i]+L[\psi]=\sum_{i=1}^{n}c_i\cdot 0+L[\psi]=g(x).$$

Endi $L[y]=g(x)$ tenglamaning ixtiyoriy y yechimini qaraylik. $L[y]=g(x)$, $L[\psi]=g(x)$ ham bo'lgani uchun $L[\circ]$ operatorning chiziqliligidan $L[y-\psi]=0$ ekanligi kelib chiqadi, ya'ni $y-\psi$ funksiya mos bir jinsli tenglamaning yechimi. U fundamental sistemaning biror chiziqli kombinatsiyasidan iborat bo'lishi kerak, ya'ni biror $c_1,\dots,c_n$ larda $y-\psi=\sum_{i=1}^{n} c_i\varphi_i$ bo'ladi.

Demak, $y=\psi+\sum_{i=1}^{n} c_i\varphi_i$. ♘

Bir jinsli bo'lmagan tenglamaning xususiy yechimini topishda quyidagi superpozitsiya prinsipidan foydalanish mumkin.

Jumla (superpozitsiya prinsipi). *Agar* $y=\psi_1(x)$ *funksiya* $L[y]=g_1(x)$ *tenglamaning* $y=\psi_2(x)$ *funksiya esa* $L[y]=g_2(x)$ *tenglamaning yechimi bo'lsa, u holda* $y=\psi_1(x)+\psi_2(x)$ *funksiya ushbu* $L[y]=g_1(x)+g_2(x)$ *tenglamaning yechimi bo'ladi.*

Isboti ravshan.

Misol 1. Ushbu

$$y''-3y'+2y=2+4e^{3x}$$

tenglamaning xususiy yechimini toping.

⊶ $y''-3y'+2y=2$ va $y''-3y'+2y=4e^{3x}$

tenglamalarni qaraylik. Birinchi tenglama, ravshanki, $y=1$ yechimga ega. Ikkinchi tenglamaning yechimini $y=ke^{3x}$ ko'rinishda izlab ko'raylik. Uni tenglamaga qo'yib, noma'lum k soni uchun $2k=4$ munosabatni hosil qilamiz. Demak, $k=2$, ya'ni $y=2e^{3x}$ funksiya $y''-3y'+2y=4e^{3x}$ tenglamaning xususiy yechimi. Superpozitsiya prinsipiga ko'ra $y=1+2e^{3x}$ funksiya berilgan $y''-3y'+2y=2+4e^{3x}$ tenglamaning (xususiy) yechimidir. ♘

Lagranjning ixtiyoriy o'zgarmaslarni variatsiyalash usuli. Bir jinsli tenglama $L[y]=0$ ning $\varphi_1(x),\varphi_2(x),\dots,\varphi_n(x)$ bazis yechimlari ma'lum bo'lsin. U holda bu tenglamaning umumiy yechimi, ma'lumki, $y=\sum_{i=1}^{n} c_i\varphi_i(x)$ ko'rinishda ifodalanadi; bu yerda $c_1,c_2,\dots,c_n$ – ixtiyoriy o'zgarmaslar. Lagranjning ixtiyoriy o'zgarmaslarni variatsiyalash usuliga ko'ra bir jinsli bo'lmagan $L[y]=g(x)$ tenglamaning yechimi ushbu

$$y=\sum_{i=1}^{n}c_i(x)\varphi_i(x) \qquad (II.8.1_0)$$

ko'rinishda izlanadi; bu yerda $c_1(x), c_2(x), \dots, c_n(x)$ – hozircha noma'lum uzluksiz differensiallanuvchi funksiyalar. Ularning soni n ta. Bu noma'lum funksiyalarni (II.8.1₀) ifoda $L[y]=g(x)$ tenglamaning yechimi bo'lishi kerakligidan aniqlaymiz; bu bitta shart. Umuman olganda, biz yana $n-1$ ta shartni o'zimizdan qo'yishimiz mumkin. Hosilani hisoblaymiz:

$$y'=\sum_{i=1}^{n}c_i'(x)\varphi_i(x)+\sum_{i=1}^{n}c_i(x)\varphi_i'(x).$$

Hosilaning ko'rinishi sodda bo'lishi uchun noma'lum funksiyalarga nisbatan ushbu

$$\sum_{i=1}^{n}c_i'(x)\varphi_i(x)=0 \qquad (II.8.2_0)$$

shartni qo'yib,

$$y'=\sum_{i=1}^{n}c_i(x)\varphi_i'(x) \qquad (II.8.1_1)$$

tenglikni topamiz. Endi y'' ni hisoblaymiz:

$$y''=\sum_{i=1}^{n}c_i'(x)\varphi_i'(x)+\sum_{i=1}^{n}c_i(x)\varphi_i''(x).$$

Ushbu

$$\sum_{i=1}^{n}c_i'(x)\varphi_i'(x)=0 \qquad (II.8.2_1)$$

shartni qo'yib, ikkinchi tartibli hosila uchun

$$y''=\sum_{i=1}^{n}c_i(x)\varphi_i''(x) \qquad (II.8.1_2)$$

sodda ko'rinishli formulani hosil qilamiz. Shunga o'xshash fikr yuritib, $c_i'(x)$ larga nisbatan mos shartlarni qo'yib, topamiz:

. . . .

$$\sum_{i=1}^{n} c_i'(x)\varphi_i^{(n-2)}(x) = 0, \qquad (II.8.2_{n-2})$$

$$y^{(n-1)} = \sum_{i=1}^{n} c_i(x)\varphi_i^{(n-1)}(x), \qquad (II.8.1_{n-1})$$

Endi

$$y^{(n)} = \sum_{i=1}^{n} c_i'(x)\varphi_i^{(n-1)}(x) + \sum_{i=1}^{n} c_i(x)\varphi_i^{(n)}(x) \qquad (II.8.1_n)$$

hosilani hisoblab va $y, y', y'', ..., y^{(n)}$ larning qiymatlarini (II.8.1_1)-(II.8.1_n) dan yechilayotgan $L[y] = g(x)$ tenglamaga qo'yib,

$$L[y] = \sum_{i=1}^{n} c_i'(x)\varphi_i^{(n-1)}(x) + \sum_{i=1}^{n} c_i(x)L[\varphi_i] = g(x),$$

ya'ni

$$\sum_{i=1}^{n} c_i'(x)\varphi_i^{(n-1)}(x) = g(x) \qquad (II.8.2_{n-1})$$

tenglikni topamiz. Shunday qilib, $c_1'(x), c_2'(x), ..., c_n'(x)$ larga nisbatan quyidagi chiziqli algebraik sistemani (ya'ni (II.8.2_0)-(II.8.2_{n-1}) shartlarni) hosil qildik:

$$\sum_{i=1}^{n} c_i'(x)\varphi_i(x) = 0$$

$$\sum_{i=1}^{n} c_i'(x)\varphi_i'(x) = 0$$

. . . .

$$\sum_{i=1}^{n} c_i'(x)\varphi_i^{(n-2)}(x) = 0$$

$$\sum_{i=1}^{n} c_i'(x)\varphi_i^{(n-1)}(x) = g(x).$$

Bu sistema $c_1'(x), c_2'(x), ..., c_n'(x)$ larga nisbatan yagona yechimga ega, chunki sistemaning determinanti $\varphi_1(x),\varphi_2(x),\dots,\varphi_n(x)$ bazis yechimlarning $W(x)$ vronskianidan iborat bo'lgani uchun u I oraliqda nolga aylanmaydi. Kramer formulalariga ko'ra

$$c_1'(x)=\frac{1}{W(x)}\begin{vmatrix} 0 & \varphi_2(x) & \dots & \varphi_n(x)\\ \vdots & \vdots & \ddots & \vdots\\ 0 & \varphi_2^{(n-2)}(x) & \dots & \varphi_n^{(n-2)}(x)\\ g(x) & \varphi_2^{(n-1)}(x) & \dots & \varphi_n^{(n-1)}(x)\end{vmatrix}=g(x)\omega_1(x),$$

$$c_2'(x)=\frac{1}{W(x)}\begin{vmatrix} \varphi_1(x) & 0 & \varphi_3(x) & \dots\\ \vdots & \vdots & \vdots & \vdots\\ \varphi_1^{(n-2)}(x) & 0 & \varphi_3^{(n-2)}(x) & \dots\\ \varphi_1^{(n-1)}(x) & g(x) & \varphi_3^{(n-1)}(x) & \dots\end{vmatrix}=g(x)\omega_2(x),$$

....

$$c_n'(x)=\frac{1}{W(x)}\begin{vmatrix} \varphi_1(x) & \dots & \varphi_{n-1}(x) & 0\\ \vdots & \vdots & \vdots & \vdots\\ \varphi_1^{(n-2)}(x) & \dots & \varphi_{n-1}^{(n-2)}(x) & 0\\ \varphi_1^{(n-1)}(x) & \dots & \varphi_{n-1}^{(n-1)}(x) & g(x)\end{vmatrix}=g(x)\omega_n(x);$$

bu yerdagi $\omega_1(x),\omega_2(x),\dots,\omega_n(x)$ funksiyalar I oraliqda C^1 sinfga tegishli.

Endi $c_1(x), c_2(x), \dots, c_n(x)$ noma'lum funksiyalarni ushbu

$$c_i(x)=\int_{x_0}^{x} g(s)\omega_i(s)\,ds,\ i=\overline{1,n}\ \ (x_0\in I-\text{tayinlangan nuqta})$$

ko'rinishda tanlaymiz va (II.8.1₀) formulaga ko'ra $L[y]=g(x)$ tenglamaning

$$y=\sum_{i=1}^{n}c_i(x)\varphi_i(x)=\sum_{i=1}^{n}\varphi_i(x)\int_{x_0}^{x} g(s)\omega_i(s)\,ds$$

yoki

$$y=\int_{x_0}^{x}K(x,s)g(s)ds \quad \text{(bunda} \quad K(x,s)=\sum_{i=1}^{n}\varphi_i(x)\omega_i(s) \text{)} \quad \text{(II.8.4)}$$

yechimini topamiz. Bu (II.8.4) formula **Koshi formulasi**, $K(x,s)$ funksiya esa **Koshi funksiyasi** deyiladi.

Bu Koshi formulasini Dyuamel prinsipidan foydalaninib keltirib chiqarish ham mumkin.

Jumla (Dyuamel prinsipi). *Tayinlangan ixtiyoriy* $s \in I$ *nuqta uchun* $\Gamma = \Gamma(x,s)$ *bilan ushbu*

$$\begin{cases} L[\Gamma]=0, \\ \Gamma|_{x=s}=0,\ \Gamma'|_{x=s}=0,\ \dots,\ \Gamma^{(n-2)}|_{x=s}=0,\ \Gamma^{(n-1)}|_{x=s}=g(s). \end{cases} \quad \text{(II.8.5)}$$

Koshi masalasining yechimini belgilaylik. U holda

$$y=\int_{x_0}^{x}\Gamma(x,s)ds \quad \text{(II.8.6)}$$

funksiya

$$\begin{cases} L[y]=g(x), \\ y|_{x=x_0}=0,\ y'|_{x=x_0}=0,\ \dots,\ y^{(n-1)}|_{x=x_0}=0, \end{cases} \quad \text{(II.8.7)}$$

Koshi masalasining yechimi bo'ladi.

Isboti. $\Gamma=\Gamma(x,s)$ yechimni $L[y]=0$ tenglamaning $\varphi_1(x),\varphi_2(x),\dots,\varphi_n(x)$ bazis yechimlari orqali qurish oson, chunki

$$\Gamma=\sum_{i=1}^{n}c_i\varphi_i(x)$$

va quyidagi boshlang'ich shartlar qanoatlanishi kerak:

$$\Gamma|_{x=s}=0: \quad \sum_{i=1}^{n}c_i\varphi_i(s)=0,$$

$$\Gamma'|_{x=s}=0: \quad \sum_{i=1}^{n}c_i\varphi_i'(s)=0,$$

$$\dots\dots\dots\dots\dots\dots$$

$$\Gamma^{(n-2)}|_{x=s}=0: \quad \sum_{i=1}^{n}c_i\varphi_i^{(n-2)}(s)=0,$$

$$\Gamma^{(n-1)}|_{s=x}= g(s): \quad \sum_{i=1}^{n} c_i\varphi_i^{(n-1)}(s) = g(s).$$

Oxirgi sistemadan yuqoridagi belgilashlardan foyalaninb, $c_1, c_2, ..., c_n$ noma'lumlarni topamiz:

$$c_1 = g(s)\omega_1(s),\ c_2 = g(s)\omega_2(s),\ ...,\ c_n = g(s)\omega_n(s).$$

Demak, $\Gamma = \sum_{i=1}^{n} c_i\varphi_i(x) = \sum_{i=1}^{n} \varphi_i(x)\omega_i(s)g(s)$. Bu formuladan ravshanki, Γ funksiya s bo'yicha uzluksiz, x bo'yicha esa C^n sinfga tegishli. Endi $y = \int_{x_0}^{x} \Gamma(x,s)\,ds$ formulani Leybnits qoidasiga ko'ra differensiallaymiz va Γ funksiya uchun boshlang'ich shartlardan foydalanib, quyidagilarni topamiz:

$$y' = \int_{x_0}^{x} \frac{\partial\Gamma}{\partial x}\,ds + \Gamma|_{s=x} = \int_{x_0}^{x} \frac{\partial\Gamma}{\partial x}\,ds,$$

$$y'' = \int_{x_0}^{x} \frac{\partial^2\Gamma}{\partial x^2}\,ds + \left.\frac{\partial\Gamma}{\partial x}\right|_{s=x} = \int_{x_0}^{x} \frac{\partial^2\Gamma}{\partial x^2}\,ds,$$

....

$$y^{(n-1)} = \int_{x_0}^{x} \frac{\partial^{n-1}\Gamma}{\partial x^{n-1}}\,ds + \left.\frac{\partial^{n-2}\Gamma}{\partial x^{n-2}}\right|_{s=x} = \int_{x_0}^{x} \frac{\partial^{n-1}\Gamma}{\partial x^{n-1}}\,ds,$$

$$y^{(n)} = \int_{x_0}^{x} \frac{\partial^{n}\Gamma}{\partial x^{n}}\,ds + \left.\frac{\partial^{n-1}\Gamma}{\partial x^{n-1}}\right|_{s=x} = \int_{x_0}^{x} \frac{\partial^{n}\Gamma}{\partial x^{n}}\,ds + g(x).$$

Demak,

$$L[y] = \int_{x_0}^{x} L[\Gamma]\,ds + g(x) = g(x),$$

ya'ni $L[y] = g(x)$ tenglama qanoatlandi. $y|_{x=x_0} = 0,\ y'|_{x=x_0} = 0, ..., y^{(n-1)}|_{x=x_0} = 0$ boshlang'ich shartlarning bajarilishi ravshan. ◊

Izoh. Jumlaning isboti jarayonidan tushunarliki, Lagranjning ixtiyoriy o'zgarmaslarni variatsiyalash metodi Dyuamel prinsipiga mohiyat (mazmun) jihatidan teng kuchli.

$K(x,s)g(s)=\Gamma(x,s)$ va $K(x,s)$ funksiya $g(x)\equiv 1$ ga ko'ra qurilgan mos Γ funksiyaga teng. Demak, agar $K(x,s)$ Koshi funksiyasini ushbu

$$\begin{cases} L[K]=0, \\ K|_{x=s}=0,\ K'|_{x=s}=0,\ \dots,\ K^{(n-2)}|_{x=s}=0,\ K^{(n-1)}|_{x=s}=1. \end{cases} \quad \text{(II.8.8)}$$

Koshi masalasining yechimi sifatida topsak (bu yerda hosilalar x o'zgaruvchi bo'yicha hisoblanadi, s – parametr), u holda

$$y=\int_{x_0}^{x}K(x,s)g(s)ds \quad \text{(II.8.9)}$$

funksiya $L[y]=g(x)$ tenglamaning

$y|_{x=x_0}=0$, $y'|_{x=x_0}=0$, ..., $y^{(n-1)}|_{x=x_0}=0$, shartlarni qanoatlantiruvchi xususiy yechimi bo'ladi.

Misol 2. Ushbu

$$y''+\omega^2 y=f(x) \quad (\omega>0,\ f\in C(I))$$

tenglamani yeching (garmonik ossilyatorning majburiy harakati, x – vaqt). Mos bir jinsli tenglama $y''+\omega^2 y=0$ ning bazis yechimlari $y_1=\cos\omega x$, $y_2=\sin\omega x$ umumiy yechimi $y=c_1\cos\omega x+c_2\sin\omega x$ (bu garmonik tebranishlarni ifodalaydi). Berilgan bir jinsli bo'lmagan tenglamaning xususiy yechimini Lagranj usulidan foydalanib topamiz, ya'ni bu tenglamaning yechimini $y=c_1(x)\cos\omega x+c_2(x)\sin\omega x$ ko'rinishda izlaymiz. Ushbu

$$c_1'(x)\cos\omega x+c_2'(x)\sin\omega x=0$$

shartni qo'yib, $y'=-c_1(x)\omega\sin\omega x+c_2(x)\omega\cos\omega x$ formulani hosil qilamiz. Bundan y'' ni hisoblab, berilgan tenglamadan

$$-c_1'(x)\omega\sin\omega x+c_2'(x)\omega\cos\omega x=f(x)$$

shartni hosil qilamiz. Shunday qilib, $c_1(x),c_2(x)$ noma'lum funksiyalar uchun

$$\begin{cases} c_1'(x)\cos\omega x + c_2'(x)\sin\omega x = 0 \\ -c_1'(x)\omega\sin\omega x + c_2'(x)\omega\cos\omega x = f(x) \end{cases}$$

sistemani topdik. Bu sistemadan $c_1'(x), c_2'(x)$ lar bir qiymatli aniqlanadi:

$$c_1'(x) = -\frac{1}{\omega}\sin\omega x \cdot f(x), \quad c_2'(x) = \frac{1}{\omega}\cos\omega x \cdot f(x).$$

Bundan

$$c_1(x) = -\frac{1}{\omega}\int \sin\omega x \cdot f(x)\,dx, \quad c_2(x) = \frac{1}{\omega}\int \cos\omega x \cdot f(x)\,dx.$$

Demak, berilgan bir jinsli bo'lmagan tenglamaning

$$y = c_1(x)\cos\omega x + c_2(x)\sin\omega x =$$

$$= -\frac{1}{\omega}\cos\omega x \int \sin\omega x \cdot f(x)\,dx + \frac{1}{\omega}\sin\omega x \int \cos\omega x \cdot f(x)\,dx.$$

xususiy yechimini topdik. Bu yechimni aniq integral yordamida quyidagicha yozish mumkin:

$$y = -\frac{1}{\omega}\cos\omega x \int_{x_0}^{x} \sin\omega s \cdot f(s)\,ds + \frac{1}{\omega}\sin\omega x \int_{x_0}^{x} \cos\omega s \cdot f(s)\,ds$$

yoki

$$y = \frac{1}{\omega}\int_{x_0}^{x} \sin\omega(x-s) \cdot f(s)\,ds \quad .$$

Berilgan bir jinsli bo'lmagan tenglamaning umumiy yechimi

$$y = \frac{1}{\omega}\int_{x_0}^{x} \sin\omega(x-s) \cdot f(s)\,ds + c_1\cos\omega x + c_2\sin\omega x$$

ko'rinishda ifodalanadi.

Endi ushbu

$$y'' + \omega^2 y = F\cos(\Omega x + \varphi_0)$$

($\omega > 0$, $F > 0$, $\Omega > 0$, φ_0 berilgan sonlar)

tenglamani qaraylik. Agar x vaqtni belgilasa, bu tenglama garmonik ossilyatorning $F\cos(\Omega x+\varphi_0)$ tashqi kuch ta'siridagi majburiy tebranishlarini tavsiflaydi. Tenglamaning umumiy yechimi

$$y=\frac{F}{\omega}\int_{x_0}^{x}\sin\omega(x-s)\cdot\cos(\Omega s+\varphi_0)ds\ +c_1\cos\omega x+c_2\sin\omega x$$

formula bilan beriladi. Bu yerdagi integralni hisoblash uchun ikki holni qaraymiz.

$\omega\neq\Omega$ bo'lsin. Bu holda kerakli hisoblashlarni va ixchamlashlarni bajarib, topamiz:

$$y=\frac{F}{\omega^2-\Omega^2}\cos(\Omega x+\varphi_0)+c_1\cos\omega x+c_2\sin\omega x.$$

Yechim garmonik tebranishlar yig'indisidan iborat bo'lib, u chegaralangan.

$\omega=\Omega$ bo'lganda yechim quyidagi ko'rinishda ifodalanadi

$$y=\frac{F}{2\omega}x\sin(\omega x+\varphi_0)+c_1\cos\omega x+c_2\sin\omega x.$$

Bu formuladan qaralayotgan holda tebranishlar amplitudasi vaqt o'tishi bilan $\frac{F}{2\omega}x$ ko'paytuvchi hisobiga keskin ortishini ko'ramiz. Bu $\omega=\Omega$ hol **rezonans holi** deb ataladi.

Misol 3. Ushbu

$$y'''-y''-y'+y=g(x)\quad(g\in C(I))$$

tenglamaga mos bir jinsli tenglamaning e^{-x},e^{x},xe^{x} bazis yechimlarini bilgan holda uning xususiy yechimini toping.

Koshi formulasidan foydalanamiz. Buning uchun $K(x,s)$ Koshi funksiyasini yuqorida keltirilgan izohga ko'ra ushbu

$$\begin{cases}K'''-K''-K'+K=0\\ K|_{x=s}=0,\ K'|_{x=s}=0,\ K''|_{x=s}=1\end{cases}$$

boshlang'ich masalaning yechimi sifatida topamiz. $y''' - y'' - y' + y = 0$ tenglamaning bazis yechimlari e^{-x}, e^{x}, xe^{x} bo'lgani uchun

$$K = c_1 e^{-x} + c_2 e^{x} + c_3 x e^{x}$$

bo'lishi kerak. Bu yerdagi c_1, c_2, c_3 noma'lumlar boshlang'ich shartlarning qanoatlanishidan topiladi:

$$K|_{x=s} = 0: \quad c_1 e^{-s} + c_2 e^{s} + c_3 s e^{s} = 0,$$

$$K'|_{x=s} = 0: \quad -c_1 e^{-s} + c_2 e^{s} + c_3 (1+s) e^{s} = 0,$$

$$K''|_{x=s} = 1: \quad c_1 e^{-s} + c_2 e^{s} + c_3 (2+s) e^{s} = 1.$$

Bu sistemani yechib, c_1, c_2, c_3 noma'lumlarni topamiz:

$$c_1 = \frac{e^{s}}{4}, \; c_2 = -\frac{(1+2s)e^{-s}}{4}, \; c_3 = \frac{e^{-s}}{2}.$$

Demak,

$$K = K(x,s) = c_1 e^{-x} + c_2 e^{x} + c_3 x e^{x} =$$

$$= \frac{e^{s}}{4} e^{-x} - \frac{(1+2s)e^{-s}}{4} e^{x} + \frac{e^{-s}}{2} x e^{x} =$$

$$= \frac{1}{4}\left((-1 + 2(x-s)) e^{x-s} + e^{-(x-s)} \right).$$

Endi Koshi formulasiga ko'ra izlangan xususiy yechimni yozamiz:

$$y = \int_{x_0}^{x} K(x,s) g(s) ds =$$

$$= \frac{1}{4} \int_{x_0}^{x} \left((-1 + 2(x-s)) e^{x-s} + e^{-(x-s)} \right) g(s) ds.$$

Bu yerda $x_0 \in \mathbb{R}$ – tayinlangan nuqta, $x \in \mathbb{R}$ – erkli o'zgaruvchi. Berilgan tenglamaning qurilgan xususiy yechimi $y(x_0) = y'(x_0) = y''(x_0) = 0$ boshlang'ich shartlarni qanoatlantiradi.

Masalalar

1. Ushbu $x^2y''+3xy'+y=1+\cos x$, $x>0$, tenglamaning umumiy yechimini quring (mos bir jinsli tenglamaning yechimlari $y_1=\frac{1}{x}$, $y_2=\frac{\ln x}{x}$).

2. Ushbu

$$x''+(1+\varphi(t))x=0,\ |\varphi(t)|\le\frac{c}{t^2},\ t\ge t_0>0\ (c=\mathrm{const}>0)$$

ikkinchi tartibli chiziqli differensial tenglamaning har qanday yechimi $t\to+\infty$ da chegaralangan bo'lishini isbotlang. $\varphi(t)\in C((0,+\infty))$ deb hisoblanadi

II.9. Tenglamani komplekslashtirish

Biz yuqorida $L[y]=g(x)$ chiziqli tenglamaning $y=\varphi(x)$, $\varphi:I\to\mathbb{R}$, haqiqiy yechimlarini o'rgandik (tenglamada qatnashgan funksiyalar haqiqiy edi). Ba'zan bu tenglamaning kompleks yechimlarini topishga to'g'ri keladi.

Dastlab $I\subset\mathbb{R}$ oraliqda aniqlangan kompleks funksiya, uning uzluksizligi, hosilasi va integrali tushunchalarini kiritaylik.

Kompleks sonlar maydonini odatdagidek $\mathbb{C}$ bilan belgilaymiz. $w:I\to\mathbb{C}$ akslantirish I oraliqda aniqlangan **kompleks funksiya** ($x\in I$ haqiqiy o'zgaruvchining kompleks funksiyasi) deyiladi. U har qanday $x\in I$ haqiqiy songa $w(x)\in\mathbb{C}$ kompleks sonni mos keltiradi. Bu $w(x)$ kompleks sonning haqiqiy va mavhum qismlarini ajratib, uni $w(x)=u(x)+iv(x)$ ko'rinishda yozish mumkin; bu yerda i – mavhum birlik ($i^2=-1$). $u:I\to\mathbb{R}$, $u(x)=\mathrm{Re}\,w(x)$ – haqiqiy qism, $v:I\to\mathbb{R}$, $v(x)=\mathrm{Im}\,w(x)$ – mavhum qism. Demak, bitta kompleks funksiyani berish ikkita haqiqiy funksiyani (haqiqiy va mavhum qismlarni) berish demakdir. $w(x)=u(x)+iv(x)$ kompleks funksiyani $(u(x),v(x))$ vektor-funksiya kabi tushunish ham mumkin. Vektor-funksiya uchun koordinatalar bo'ylab kiritilgan analiz tushunchalari (limit, uzluksizlik, hosila, integral, ...) bevosita kompleks funksiya holiga ko'chiriladi. Masalan, $x_0 \in I$ nuqta uchun

$$\lim_{x\to x_0} w(x) = \lim_{x\to x_0}(u(x)+iv(x)) = \lim_{x\to x_0} u(x) + i\lim_{x\to x_0} v(x) \qquad \text{deb}$$

hisoblanadi. Agar berilgan $x_0 \in I$ nuqtada $u(x)$ va $v(x)$ haqiqiy funksiyalar uzluksiz bo'lsa, u holda $w(x) = u(x) + iv(x)$ kompleks funksiya shu x_0 **nuqtada uzluksiz** deb qabul qilinadi. Xuddi shunga o'xshash hosila va integral tushunchalari kiritiladi:

$$w'(x) = \left(u(x)+iv(x)\right)' = u'(x)+iv'(x),$$

$$\int_a^b w(x)dx = \int_a^b u(x)dx + i\int_a^b v(x)dx, \ \{a,b\} \subset I.$$

Osongina ko'rsatish mumkinki,

$$w'(x) = \lim_{h\to 0}\frac{w(x+h)-w(x)}{h}$$

formula ham o'rinli bo'ladi. Haqiqiy sohada hosila hisoblash uchun asosiy qoidalar kompleks funksiyalar uchun ham saqlanadi: z_1, z_2 kompleks sonlar va differensiallanuvchi (ya'ni hosilaga ega bo'lgan) $w_1(x), w_2(x)$ kompleks funksiyalar uchun

$$\left(z_1w_1(x)+z_2w_2(x)\right)' = z_1w_1'(x)+z_2w_2'(x),$$

$$\left(w_1(x)\cdot w_2(x)\right)' = w_1'(x)\cdot w_2(x)+w_1(x)\cdot w_2'(x),$$

$$\left(\frac{w_1(x)}{w_2(x)}\right)' = \frac{w_1'(x)\cdot w_2(x)-w_1(x)\cdot w_2'(x)}{w_2^2(x)} \quad (w_2(x)\neq 0) \qquad \text{(II.9.0)}$$

formulalar o'rinli bo'ladi. Agar $w(x) = u(x)+iv(x)$ kompleks funksiyaning $u(x)$ haqiqiy va $v(x)$ mavhum qismlari I oraliqda uzluksiz, ya'ni $\{u(x), v(x)\} \subset C(I,\mathbb{R})$ bo'lsa, u holda $w(x)$ kompleks funksiya I **oraliqda uzluksiz** deyiladi va bu $w(x) \in C(I,\mathbb{C})$ kabi belgilanadi. Agar $\{u(x), v(x)\} \subset C^1(I,\mathbb{R})$ bo'lsa, u holda $w(x) = u(x)+iv(x)$ kompleks funksiya I oraliqda uzluksiz differensiallanuvchi deyiladi va bu $w(x)\in C^1(I,\mathbb{C})$ kabi belgilanadi. I oraliqda n - tartibli hosilasi bilan birgalikda uzluksiz (n marta uzluksiz differensiallanuvchi) bo'lgan kompleks funksiyalar sinfi ham shunga o'xshash kiritiladi va bu sinf $C^n(I,\mathbb{C})$ bilan belgilanadi.

z_1, z_2 kompleks sonlar va $w_1(x), w_2(x)$ integrallanuvchi kompleks funksiyalar uchun

$$\int_a^b (z_1 w_1(x) + z_2 w_2(x))\,dx = z_1 \int_a^b w_1(x)dx + z_2 \int_a^b w_2(x)dx,$$

$$\left|\int_a^b w_1(x)dx\right| \le \int_a^b |w_1(x)|\,dx \quad (a<b)$$

munosabatlar o'rinlidir. Agar $w(x)\in C^1(I,\mathbb{C})$ bo'lsa, u holda har qanday $\{a,b\}\subset I$ uchun ushbu

$$\int_a^b w'(x)dx = w(b) - w(a)$$

Nyuton-Leybnits formulasi o'rinli bo'ladi.

Ikki t va x haqiqiy o'zgaruvchilarning $w(t,x)$ kompleks funksiyasi yuqoridagiga o'xshash kiritiladi va o'rganiladi. Bizga quyidagi tasdiq kerak bo'ladi. Agar $\dfrac{\partial^2 w}{\partial t\partial x}$, $\dfrac{\partial^2 w}{\partial x\partial t}$ aralash xususiy hosilalarning biri uzluksiz bo'lsa, ular teng bo'ladi: $\dfrac{\partial^2 w}{\partial x\partial t} = \dfrac{\partial^2 w}{\partial x\partial t}$.

Bu tasdiq haqiqiy funksiyalar uchun mos teoremadan bevosita kelib chiqadi. Demak, haqiqiy funksiyalar holidagidek silliq kompleks funksiyaning yuqori tartibli xususiy hosilasi hosilani hisoblash tartibiga bog'liq bo'lmaydi.

Endi ixtiyoriy $z=\alpha+i\beta\in\mathbb{C}$ ($\{\alpha,\beta\}\subset\mathbb{R}$) kompleks son uchun

$$e^z \overset{def}{=} e^\alpha(\cos\beta + i\sin\beta) \qquad \text{(II.9.1)}$$

kompleks sonni aniqlaylik. Agar bu formulada $z = ix$, $x \in \mathbb{R}$, desak, u holda

$$e^{ix} = \cos x + i\sin x \qquad \text{(II.9.2)}$$

tenglik hosil bo'ladi. Ravshanki, ushbu

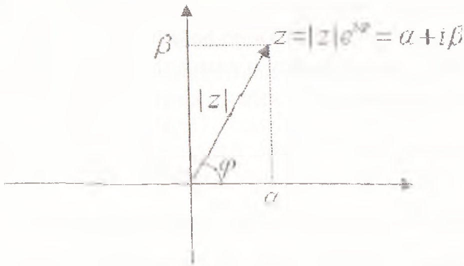
$$e^{-ix} = \cos x - i\sin x, \qquad \text{(II.9.3)}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \text{(II.9.4)}$$

formulalar ham o'rinlidir. Bu (II.9.2)- (II.9.4) formulalar **Eyler formulalari** deb ataladi. Agar $z = \alpha + i\beta \neq 0$ kompleks sonning modulini (absolyut qiymatini) $|z|$ $\left(|z| = \sqrt{\alpha^2 + \beta^2}\right)$ bilan, argumentini $\arg z = \varphi$ $(|z|\cos\varphi = \alpha, |z|\sin\varphi = \beta;\ 0 \leq \varphi < 2\pi)$ bilan belgilasak, u holda, ravshanki,

$$z = |z|e^{i\varphi} \quad \left(z = |z|(\cos\varphi + i\sin\varphi)\right)$$

bo'ladi (II.2 - rasm).



II.2 - rasm. $z = \alpha + i\beta = |z|e^{i\varphi}$ kompleks sonning moduli $|z|$ va argumenti φ

Osongina tekshirib ko'rish mumkinki, ixtiyoriy z_1 va z_2 kompleks sonlar uchun

$$e^{z_1 + z_2} = e^{z_1}e^{z_2}$$

bo'ladi. Ixtiyoriy $a > 0$ haqiqiy va ixtiyoriy z kompleks son uchun $a^z = e^{z\ln a}$ deymiz.

Berilgan $z = \alpha + i\beta \in \mathbb{C}$ ($\{\alpha, \beta\} \subset \mathbb{R}$) kompleks son uchun ushbu

$$e^w = z$$

tenglamaning yechim(lar)i z **sonning logarifmi** deyiladi va $w = \mathrm{Ln}\, z$ bilan belgilanadi; shunday qilib, agar $\mathrm{Ln}\, z$ mavjud bo'lsa, $e^{\mathrm{Ln}\, z} = z$ tenglik o'rinlidir. Berilgan $z \neq 0$ ga ko'ra $e^w = z$ tenglamadan $w = u + iv \in \mathbb{C}$ ($\{u, v\} \subset \mathbb{R}$) noma'lumni topish mumkin:

$$e^w = z \Leftrightarrow e^u(\cos v + i\sin v) = \alpha + i\beta \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} e^u \cos v = \alpha \\ e^u \sin v = \beta \end{cases} \Leftrightarrow \begin{cases} e^{2u} = \alpha^2 + \beta^2 \\ \sqrt{\alpha^2 + \beta^2}\cos v = \alpha,\ \sqrt{\alpha^2 + \beta^2}\sin v = \beta \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} u = \ln|z| \\ v = \arg z + 2\pi k, k \in \mathbb{Z}\ (z \neq 0) \end{cases}$$

Demak,

$$\mathrm{Ln}\, z = \ln|z| + i\arg z + i2\pi k,\ k \in \mathbb{Z},\ z \neq 0.$$

Masalan, $\mathrm{Ln}(-1) = \ln 1 + i\pi + i2\pi k = i\pi(1 + 2k)$, $k \in \mathbb{Z}$.

Shunday qilib, noldan farqli har qanday $z \neq 0$ kompleks sonning logarifmi cheksiz ko'p. Bu qiymatlar farqi $i2\pi$ ga karrali. **Logarifmning bosh qiymati** deb $\ln z = \ln|z| + i\arg z$, $z \neq 0$, songa aytiladi. Tushunarliki, $\mathrm{Ln}\, z = \ln z + i2\pi k$, $k \in \mathbb{Z}$, $z \neq 0$.

Endi ixtiyoriy $z = \alpha + i\beta$ ($\{\alpha, \beta\} \subset \mathbb{R}$) kompleks sonni tayinlab, $x \in \mathbb{R}$ haqiqiy o'zgaruvchining ushbu

$$w(x) = e^{zx} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$

kompleks funksiyasini kiritamiz. Uning hosilasini hisoblaylik. Ravshanki,

$$(e^{zx})' = (e^{\alpha x})'(\cos\beta x + i\sin\beta x) + e^{\alpha x}(\cos\beta x + i\sin\beta x)' =$$

$$= \alpha e^{\alpha x}(\cos\beta x + i\sin\beta x) + e^{\alpha x}(-\beta\sin\beta x + i\beta\cos\beta x) =$$

$$= \alpha e^{\alpha x}(\cos\beta x + i\sin\beta x) + e^{\alpha x} i\beta(i\sin\beta x + \cos\beta x) =$$
$$= (\alpha + i\beta)e^{\alpha x}(\cos\beta x + i\sin\beta x) =$$
$$= z e^{zx},$$

ya'ni

$$(e^{zx})' = z e^{zx}. \qquad (II.9.5)$$

Demak, haqiqiy z uchun bizga ma'lum bo'lgan bu formula kompleks z uchun ham o'z kuchini saqlaydi.

Endi noma'lumi haqiqiy o'zgaruvchining kompleks funksiyasidan iborat bo'lgan differensial tenglamalarni o'rganish mumkin. (II.9.5) formuladan ravshanki, $w' = zw$ ($z \in \mathbb{C}$ berilgan o'zgarmas kompleks son) tenglama $w(x) = e^{zx}$ yechimga ega. Bu tenglamaning barcha yechimlari $w = ce^{zx}$ formula bilan berilishini ko'rsatish mumkin; bu yerda c – ixtiyoriy o'zgarmas kompleks son.

Faraz qilaylik,

$$L[y] = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = g(x) \qquad (II.9.6)$$

tenglamadagi berilgan funksiyalar I oraliqda uzluksiz kompleks funksiyalar bo'lsin, ya'ni $\{a_{n-1}(x), \ldots, a_1(x), a_0(x), g(x)\} \subset C(I,\mathbb{C})$. U holda bu tenglamaning berilgan

$$y\big|_{x_0} = y_0,\ y'\big|_{x_0} = y_0', \ldots, y^{(n-1)}\big|_{x_0} = y_0^{(n-1)} \qquad (II.9.7)$$

boshlang'ich shartlarni $(\{y_0, y_0', \ldots, y_0^{(n-1)}\} \subset \mathbb{C})$ qanoatlantiruvchi $y = y(x)$ kompleks yechimi bira to'la I oraliqda aniqlangan, $C^n(I,\mathbb{C})$ sinfga tegishli va yagona bo'ladi (bu tasdiqni keyinroq isbotlaymiz).

Berilgan $y_1 = y_1(x),\ y_2 = y_2(x),\ \ldots,\ y_n = y_n(x)$ kompleks funksiyalarning chiziqli kombinatsiyasi ushbu $\lambda_1 y_1(x) + \lambda_2 y_2(x) + \ldots + \lambda_n y_n(x)$ yig'indidan iborat; bu yerda endi $\lambda_1, \lambda_2, \ldots, \lambda_n$ koeffitsientlar kompleks sonlardir. Kompleks funksiyalarning chiziqli erkliligi va chiziqli bog'langanligi tushunchalari haqiqiy funksiyalar holidagidek kiritiladi. Agar $2n$ ta

$u_1(x), v_1(x), u_2(x), v_2(x), \ldots, u_n(x), v_n(x)$ haqiqiy funksiyalar chiziqli erkli (chiziqli bog'langan) bo'lsa, u holda

$$y_1(x) = u_1(x) + iv_1(x),\ y_2(x) =$$
$$= u_2(x) + iv_2(x),\ \ldots, y_n(x) = u_n(x) + iv_n(x)$$

$(u_j(x) = \operatorname{Re} y_j(x),\ v_j(x) = \operatorname{Im} y_j(x);\ j = \overline{1, n})$ kompleks funksiyalar ham chiziqli erkli (mos ravishda chiziqli bog'langan) bo'ladi. Haqiqiy funksiyalar holida isbotlangan ... teoremalar kompleks funksiyalar uchun ham saqlanadi. Faqat endi kompleks koeffitsientli $L[y] = 0$ tenglamaning (kompleks) yechimlari ($\mathbb{C}$ maydon ustida qurilgan) n o'lchamli kompleks chiziqli fazoni tashkil etadi.

Bu yerda biz funksiyalar bir muhim sinfining chiziqli erkliligini isbotlaymiz. Buning uchun dastlab yordamchi lemmani keltiraylik.

Lemma. *Agar biror kompleks koeffitsientli* $P(x), Q(x)$ *ko'phadlar va* $\sigma \neq \lambda$ *kompleks sonlar uchun biror* I *oraliqda*

$$P(x)e^{\lambda x} + Q(x)e^{\sigma x} = 0,\ x \in I, \tag{II.9.8}$$

ayniyat o'rinli bo'lsa, u holda $P(x) = Q(x) = 0$ *bo'ladi.*

⊶ Tushunarliki. $Q(x) = 0$ ekanligini isbotlash kifoya. Berilgan ayniyatning har ikkala tomonini $e^{-\lambda x}$ ga ko'paytiramiz:

$$P(x) + Q(x)e^{\theta x} = 0, x \in \mathbb{R},\ \theta \stackrel{\text{def}}{=} \sigma - \lambda \neq 0. \tag{II.9.9}$$

Bu ayniyatni differensiallab topamiz:

$$P'(x) + \left(\theta Q(x) + Q'(x)\right)e^{\theta x} = 0,\ x \in I. \tag{II.9.10}$$

Agar $Q(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ bo'lsa,

$$\theta Q(x) + Q'(x) = \theta a_k x^k + (\theta a_{k-1} + k a_k)x^{k-1} + \cdots + (\theta a_0 + a_1)$$

bo'ladi. Bundan ravshanki, agar $Q(x) = 0$, ya'ni $a_k = a_{k-1} = \cdots = a_1 = a_0 = 0$ bo'lsa, u holda $\theta Q(x) + Q'(x)$ ko'phad ham nolga teng va aksincha: agar $\theta Q(x) + Q'(x) = 0$, ya'ni $\theta a_k = \theta a_{k-1} + k a_k = \cdots = \theta a_0 + a_1 = 0$ bo'lsa, u holda $\theta \neq 0$

bo'lganligi uchun $a_k = a_{k-1} = \dots = a_1 = a_0 = 0$ ham bo'ladi. Demak, $\theta Q(x) + Q'(x)$ va $Q(x)$ ko'phadlarning koeffitsientlari bir vaqtda nolga teng (yoki tengmas): $\theta Q(x) + Q'(x) = 0 \Leftrightarrow Q(x) = 0$. Endi (II.9.10) ayniyatni yana ketma-ket differensiallab, $R(x)e^{\theta x} = 0$, $x \in I$, ayniyatni hosil qilamiz. Bunda $R(x) = 0 \Leftarrow Q(x) = 0$ ham bo'ladi. Oxirgi ayniyatdan $R(x) = 0$, $x \in I$, ekanligi kelib chiqadi. Bundan ravshanki, $R(x)$ ko'phadning barcha koeffitsientlari nolga teng, chunki aks holda bu ko'phad ko'pi bilan $\deg R(x)$ dona nuqtada nolga aylanardi, I oraliqda nuqtalar cheksiz ko'p bo'lgani uchun u I da aynan nolga teng bo'lolmasdi. Demak, $Q(x) = R(x) = 0$. ✍

Teorema. *Berilgan* $\lambda_1, \lambda_2, \dots, \lambda_s$ *turli kompleks sonlar va* $k_1, k_2, \dots, k_s$ *nomanfiy butun sonlar uchun ushbu*

$$e^{\lambda_1 x}, xe^{\lambda_1 x}, \dots, x^{k_1}e^{\lambda_1 x}, e^{\lambda_2 x}, xe^{\lambda_2 x}, \dots, x^{k_2}e^{\lambda_2 x}, \dots, e^{\lambda_s x}, xe^{\lambda_s x}, \dots, x^{k_s}e^{\lambda_s x}$$

(kompleks) funksiyalar ixtiyoriy I *oraliqda chiziqli erkli.*

✍ s bo'yicha matematik induksiya metodini qo'llaymiz. $s = 1$ holida teorema ravshan. Teoremani s o'rnida $s-1$ bo'lganda o'rinli deb faraz qilamiz va bundan teoremani keltirib chiqaramiz. Berilgan funksiyalarning biror chiziqli kombinatsiyasi nolga teng bo'lsin:

$$c_0e^{\lambda_1 x} + c_1xe^{\lambda_1 x} + \dots + c_{k_1}x^{k_1}e^{\lambda_1 x} + d_0e^{\lambda_2 x} + d_1xe^{\lambda_2 x} + \cdots d_{k_2}x^{k_2}e^{\lambda_2 x} +$$

$$+ \dots + u_0e^{\lambda_s x} + u_1xe^{\lambda_s x} + \dots + u_{k_s}x^{k_s}e^{\lambda_s x} = 0,\ x \in I;$$

bu yerdagi koeffitsientlar – kompleks sonlar. Bu ayniyatni qisqaroq ko'rinishda yozaylik:

$$P_1(x)e^{\lambda_1 x} + P_2(x)e^{\lambda_2 x} + \dots + P_s(x)e^{\lambda_s x} = 0, x \in I; \qquad \text{(II.9.11)}$$

bu yerda $P_j(x)$ – ko'phadlar ($\deg P_j(x) \le k_j$, $j = \overline{1,n}$). biz ularning nolga tengligini ko'rsatishimiz kerak. Lemmaning isbotiga o'xshash fikr yuritamiz. Oxirgi ayniyatning har ikkala tomonini $e^{-\lambda_1 x}$ ga ko'paytiramiz va differensiallaymiz:

$$
\begin{aligned}
&P_1'(x)+\left((\lambda_2-\lambda_1)P_2(x)+P_2'(x)\right)e^{(\lambda_2-\lambda_1)x}+\\
&+\cdots+\left((\lambda_s-\lambda_1)P_s(x)+P_s'(x)\right)e^{(\lambda_s-\lambda_1)x}=0,\ x\in I
\end{aligned}
\quad ; \quad \text{(II.9.12)}
$$

bunda $\lambda_2-\lambda_1\neq 0,\ \ldots,\ \lambda_s-\lambda_1\neq 0$ va

$$(\lambda_2-\lambda_1)P_2(x)+P_2'(x)=0 \Leftrightarrow P_2(x)=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$(\lambda_s-\lambda_1)P_s(x)+P_s'(x)=0 \Leftrightarrow P_s(x)=0$$

ekvivalentliklar o'rinli. Endi (II.9.12) ayniyatni ketma-ket differensiallab topamiz:

$$\tilde{P}_2(x)e^{(\lambda_2-\lambda_1)x}+\cdots+\tilde{P}_s(x)e^{(\lambda_s-\lambda_1)x}=0,\ x\in\mathbb{R}\ ; \qquad \text{(II.9.13)}$$

bunda $\tilde{P}_2(x)=0 \Leftrightarrow P_2(x)=0,\ \ldots,\ \tilde{P}_s(x)=0 \Leftrightarrow P_s(x)=0$ bo'ladi. Oxirgi (II.9.13) ayniyatdagi qo'shiluvchilar soni (II.9.11) ayniyatdagidan bitta kam. Induksiya faraziga ko'ra $\tilde{P}_2(x)=\ldots=\tilde{P}_{s-1}(x)=0$ bo'lishi kerak. Demak, $P_2(x)=\ldots=P_{s-1}(x)=0$. Bu tengliklarni (II.9.11) ga qo'yib, undan $P_1(x)=0$ ekanligini ham topamiz. ♘

Endi

$$L[y]=y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y=0$$

tenglamaning koeffitsientlari yana haqiqiy bo'lsin. Haqiqiy funksiyalar kompleks funksiyalarning xususiy holi bo'lgani uchun biz bu tenglamani kompleks sohada o'rganishimiz, ya'ni uning kompleks yechimlarini topishimiz mumkin. Bu – berilgan haqiqiy koeffitsientli **tenglamaning komplekslashtirilishi** deyiladi.

Jumla. *Faraz qilaylik, $L[y]=0$ haqiqiy koeffitsientli tenglamaning $y=u(x)+iv(x)$ kompleks yechimi ma'lum bo'lsin. U holda bu yechimning $u(x)$ haqiqiy va $v(x)$ mavhum qismlari berilgan shu $L[y]=0$ tenglamaning yechimlari bo'ladi.*

⊶ Haqiqatan ham, $L[\circ]$ operatorning chiziqlilik xossasi va koeffitsientlarining haqiqiyligiga ko'ra

$$L[u(x)+iv(x)]=L[u(x)]+iL[v(x)]=0 \Rightarrow L[u(x)]=L[v(x)]=0 .$$

Masalalar

1. Yuqorida keltirilgan (II.9.0) munosabatlarni isbotlang.
2. $w(x)=(x+z)^k$ $(z\in\mathbb{C},\ k\in\mathbb{N})$ funksiyaning hosilasi $w'(x)=k(x+z)^{k-1}$ ga teng bo'lishini isbotlang. ($k\in\mathbb{N}$ bo'yicha induksiya qo'llang)
3. Ixtiyoriy $z\in\mathbb{C}$ uchun $e^z\neq 0$ ekanligini ko'rsating.
4. $\mathrm{Ln}(z_1 z_2)=\mathrm{Ln}\,z_1+\mathrm{Ln}\,z_2$ tenglikni isbotlang.
5. $z\neq 0$ kompleks son berilgan bo'lsin. Ushbu $w(x)=\ln(zx)$, $x\in(-\infty,0)\bigcup(0,+\infty)$, kompleks funksiyani qaraylik. Bu funksiyaning hosilasi uchun

$$w'(x)=\frac{d}{dx}\ln(zx)=\frac{1}{x},\ x\in(-\infty,0)\bigcup(0,+\infty),$$

formulani isbotlang.
6. Agar $I\subset\mathbb{R}$ oraliqda aniqlangan $y(x)$ kompleks funksiya uchun $y'(x)=0,\ x\in I$, bo'lsa, u holda bu funksiya I oraliqda o'zgarmas ekanligini ko'rsating.
7. Ushbu $y'(x)=zy(x)$ $(z\in\mathbb{C})$ tenglamaning barcha kompleks yechimlari $y=ce^{zx}$ formula bilan berilishini isbotlang; bunda c – ixtiyoriy o'zgarmas kompleks son.

II.10. n - tartibli chiziqli o'zgarmas koeffitsientli bir jinsli differensial tenglamalar

Quyidagi n- tartibli chiziqli o'zgarmas koeffitsientli bir jinsli differensial tenglamani qaraylik:

$$L[y]\overset{def}{=}y^{(n)}+a_{n-1}y^{(n-1)}+\dots+a_1y'+a_0y=0. \qquad (II.10.1)$$

Bu yerdagi koeffitsientlar kompleks bo'lishi mumkin. Qaralayotgan (II.10.1) tenglamaning umumiy yechimini topish uchun uning n dona chiziqli erkli yechimlarini, ya'ni bazis yechimlarni topish kerak. Umumiy yechim bazis yechimlarning ixtiyoriy chiziqli kombinatsiyasi ko'rinishida ifodalanadi.

Berilgan (II.10.1) differensial tenglamaning bazis yechimlarini Eyler usuli bilan topamiz. Bu usulga ko'ra tenglamaning yechimi

$$y = e^{\lambda x} \qquad \text{(II.10.2)}$$

ko'rinishda izlanadi; bu yerda λ – hozircha noma'lum son. Ravshanki,

$$(e^{\lambda x})' = \lambda e^{\lambda x},\ (e^{\lambda x})'' = \lambda^2 e^{\lambda x},\ \ldots,\ (e^{\lambda x})^{(n)} = \lambda^n e^{\lambda x},$$

$$L[e^{\lambda x}] = L(\lambda)e^{\lambda x}; \qquad \text{(II.10.3)}$$

bu yerda

$$L(\lambda) \stackrel{def}{=} \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \qquad \text{(II.10.4)}$$

Demak, (II.10.2) funksiya (II.10.1) tenglamaning yechimi bo'lishi uchun λ soni ushbu

$$L(\lambda) = 0 \text{, ya'ni}$$

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0 \qquad \text{(II.10.5)}$$

n- darajali algebraik tenglamaning ildizi bo'lishi kerak. (II.10.4) ko'phad va (II.10.5) algebraik tenglama (II.10.1) differensial tenglamaning mos ravishda **xarakteristik ko'phadi** va **xarakteristik tenglamasi** deb ataladi. Xarakteristik tenglamaning ildizlari mos **differensial tenglamaning xarakteristik sonlari** deyiladi.

Masalan, $y'' - 3y' + 2y = 0$ differensial tenglamaning xarakteristik ko'phadi $\lambda^2 - 3\lambda + 2$, xarakteristik tenglamasi $\lambda^2 - 3\lambda + 2 = 0$, xarakteristik sonlari esa $\lambda_1 = 1$ va $\lambda_2 = 2$ dan iborat bo'ladi.

Algebradan ma'lumki, (II.10.5) n- darajali algebraik tenglama (xarakteristik tenglama) kompleks sohada karraliligi bilan birgalikda hisoblanganda n ta ildizga ega. Aniqrog'i, k_1 ($k_1 \geq 1$) karrali λ_1, k_2 ($k_2 \geq 1$) karrali λ_2, . . . , k_s ($k_s \geq 1$) karrali λ_s har xil ildizlar mavjud va $k_1 + k_2 + \cdots + k_s = n$ ($s \leq n$) bo'ladi, ya'ni

$$L(\lambda)=\lambda^n+a_{n-1}\lambda^{n-1}+\cdots+a_1\lambda+a_0=(\lambda-\lambda_1)^{k_1}(\lambda-\lambda_2)^{k_2}\cdots(\lambda-\lambda_s)^{k_s}. \quad \text{(II.10.6)}$$

Dastlab bir foydali jumlani isbotlaylik.

Jumla 1. *Agar* $\lambda=\mu+i\nu$, $\{\mu,\nu\}\subset\mathbb{R}$, *bo'lsa, u holda ixtiyoriy natural* m *uchun*

$$L[x^m e^{\lambda x}]=\big(L(\mu+i\nu)x^m+C_m^1 L'(\mu+i\nu)x^{m-1}+$$
$$+C_m^2 L''(\mu+i\nu)x^{m-2}+\cdots+L^{(m)}(\mu+i\nu)\big)e^{\lambda x}=$$
$$=\sum_{j=0}^{m} C_m^j L^{(j)}(\mu+i\nu)x^{m-j}e^{\lambda x}, \quad \text{(II.10.7)}$$

formula o'rinli bo'ladi; bu yerdagi barcha hosilalar μ *haqiqiy o'zgaruvchi bo'yicha hisoblangan,* $C_m^j=\dfrac{m!}{j!(m-j)!}$ *– binomial koeffitsientlar.*

⊸ (II.10.3) formulaga ko'ra $\lambda=\mu+i\nu$, $\{\mu,\nu\}\subset\mathbb{R}$, uchun

$$L[e^{\mu x}e^{i\nu x}]=L(\mu+i\nu)e^{(\mu+i\nu)x}.$$

Bu ayniyatni μ haqiqiy o'zgaruvchi bo'yicha m marta differensiallaymiz. Chap tomonda x va μ bo'yicha differensiallash tartibini almashtirib, topamiz:

$$\frac{\partial}{\partial\mu}L[e^{\mu x}e^{i\nu x}]=L\Big[\frac{\partial}{\partial\mu}e^{\mu x}e^{i\nu x}\Big]=L[xe^{\mu x}e^{i\nu x}]=L[xe^{\lambda x}]=$$
$$=\frac{\partial}{\partial\mu}\big(L(\mu+i\nu)e^{(\mu+i\nu)x}\big).$$

Shu ishni takrorlab,

$$L[x^m e^{\lambda x}]=\frac{\partial^m}{\partial\mu^m}\big(L(\mu+i\nu)e^{(\mu+i\nu)x}\big)$$

formulaga ega bo'lamiz. Endi quyidagi hisoblashlarni, ko'paytmaning hosilasi uchun Leybnits formulasidan foydalanib, bajaramiz:

$$L[x^m e^{\lambda x}] = \frac{\partial^m}{\partial \mu^m}\left(L(\mu+i\nu)e^{(\mu+i\nu)x}\right) =$$

$$= \frac{\partial^m}{\partial \mu^m}\left(L(\mu+i\nu)e^{\mu x}\right)e^{i\nu x} =$$

$$= \sum_{j=0}^{m} C_m^j L^{(j)}(\mu+i\nu)(e^{\mu x})^{(m-j)}e^{i\nu x} =$$

$$= \sum_{j=0}^{m} C_m^j L^{(j)}(\mu+i\nu)x^{m-j}e^{\mu x}e^{i\nu x} =$$

$$= \sum_{j=0}^{m} C_m^j L^{(j)}(\mu+i\nu)x^{m-j}e^{\lambda x}.$$

Jumla 2. *Agar* λ_j *xarakteristik son* k_j *karrali bo'lsa, u holda (II.10.1) differensial tenglama* k_j *dona*

$$y_1 = e^{\lambda_j x},\ y_2 = xe^{\lambda_j x},\ \ldots,\ y_{k_j} = x^{k_j-1}e^{\lambda_j x} \qquad \text{(II.10.8)}$$

yechimlarga ega.

λ_j ning k_j karrali xarakteristik son ekanligi ushbu

$$L(\lambda) = (\lambda-\lambda_j)^{k_j} M(\lambda)$$

tenglikning o'rinliligini anglatadi; bu yerda $M(\lambda)$ ko'phadning darajasi $(n-k_j)$ ga teng va $M(\lambda_j)\neq 0$. Demak, $\lambda = \mu + i\nu$, $\{\mu,\nu\}\subset\mathbb{R}$, uchun

$$L(\mu+i\nu) = (\mu+i\nu-\lambda_j)^{k_j} M(\mu+i\nu) \text{ va}$$

$$L(\lambda_j) = L'(\lambda_j) = \ldots = L^{(k_j-1)}(\lambda_j) = 0,\ L^{(k_j)}(\lambda_j) = k_j!\cdot M(\lambda_j)\neq 0. \quad \text{(II.10.9)}$$

Yuqoridagi (II.10.3) ayniyatdan $L[e^{\lambda_j x}] = 0$, ya'ni $y_1 = e^{\lambda_j x}$ funksiya (II.10.1) tenglamaning yechimi ekanligi ravshan. Endi (II.10.8) dagi qolgan funksiyalarning ham yechim bo'lishini ko'rsatamiz. Buning uchun (II.10.7) formuladan $m = 1, 2, \ldots, k_j - 1$ uchun $\lambda = \mu + i\nu = \lambda_j$ deb va (II.10.9) ni hisobga olib quyidagilarni topamiz:

$$L[xe^{\lambda_j x}] = \left(L(\lambda_j)x + L'(\lambda_j) \right)e^{\lambda_j x} = 0,$$

$$L[x^2e^{\lambda_j x}] = \left(L(\lambda_j)x^2 + 2L'(\lambda_j)x + L''(\lambda_j) \right)e^{\lambda_j x} = 0,$$

. . . .

$$L[x^{k_j-1}e^{\lambda_j x}] = \Big(L(\lambda_j)x^{k_j-1} + C_{k_j-1}^{1}L'(\lambda_j)x^{k_j-2} +$$

$$+ C_{k_j-1}^{2}L''(\lambda_j)x^{k_j-3} + \cdots + L^{(k_j-1)}(\lambda_j)\Big)e^{\lambda_j x} = 0.$$

Shunday qilib. (II.10.8) da keltirilgan barcha funksiyalar (II.10.1) tenglamaning yechimi bo'lishi isbotlandı.

Izoh. Biz kompleks funksiyadan kompleks o'zgaruvchi bo'yicha hosila tushunchası va unıng xossalari bilan tanish bo'lmaganligimiz sababli $\lambda \in \mathbb{C}$ ning haqiqiy μ va mavhum ν qismlarini ajratib, μ haqiqiy o'zgaruvchi bo'yicha differensiallashdan foydalandik. Kompleks o'zgaruvchiga nisbatan hosila tushunchası bilan tanish o'quvchilar to'g'ridan-to'g'ri $\lambda \in \mathbb{C}$ o'zgaruvchı bo'yıcha hosıladan foydalanishi mumkın.

Isbotlangan jumla 2 dan foydalanib, (II.10.1) differensial tenglamaning k_1 karrali λ_1, k_2 karrali λ_2, . . . , k_s karrali λ_s turli xarakteristik sonlariga ko'ra uning $k_1 + k_2 + \cdots + k_s = n$ dona yechimini tuzamiz:

$$y_1 = e^{\lambda_1 x},\ y_2 = xe^{\lambda_1 x},\ \ldots,\ y_{k_1} = x^{k_1-1}e^{\lambda_1 x} \quad (\,k_1 \text{ ta})$$

$$y_{k_1+1} = e^{\lambda_2 x},\ y_{k_1+2} = xe^{\lambda_2 x},\ \ldots,\ y_{k_1+k_2} = x^{k_2-1}e^{\lambda_2 x} \quad (\,k_2 \text{ ta})$$

. . . .

$$y_{k_1+k_2+\cdots+k_{s-1}+1} = e^{\lambda_s x},\ y_{k_1+k_2+\cdots+k_{s-1}+2} = xe^{\lambda_s x},$$

$$\ldots,\ y_{k_1+k_2+\cdots+k_s} = x^{k_s-1}e^{\lambda_s x} \quad (\,k_s \text{ ta})$$

Bu yechımlarning chiziqlı erklıligi … da isbotlangan edi. Demak, biz (II.10.1) tenglamaning bazis yechimlarini topdik.

Misol 1. Ushbu

$$y''' - 4y'' + 5y' - 2y = 0$$

diefferensial tenglamani yeching.

8⟶ Mos xarakteristik tenglamani tuzamiz:

$$\lambda^3-4\lambda^2+5\lambda-2=0.$$

Xarakteristik tenglamaning ildizlarini topamiz:

$$\lambda^3-4\lambda^2+5\lambda-2=(\lambda^3-4\lambda^2+4\lambda)+\lambda-2=$$
$$=\lambda(\lambda-2)^2+\lambda-2=(\lambda-1)^2(\lambda-2)=0$$

$\lambda=1-$ ikki karrali ildiz, $\lambda=2-$ oddiy (bir karrali) ildiz.
Bu xarakteristik sonlarga ko'ra qaralayotgan differensial tenglamaning bazis yechimlarini tuzamiz:

$$e^x,\ xe^x,\ e^{2x}.$$

Berilgan differensial tenglamaning umumiy yechimi $y=c_1e^x+c_2xe^x+c_3e^{2x}$ formula bilan ifodalanadi. Bu yerdagi c_1,c_2,c_3 lar ixtiyoriy kompleks qiymatlar qabul qilsa, kompleks sohadagi umumiy yechim, ular ixtiyoriy haqiqiy qiymatlar qabul qilganda esa, haqiqiy sohadagi umumiy yechim hosil bo'ladi. ♨

Misol 2. Ushbu

$$y''-2y'+2y=0$$

diefferensial tenglamaning umumiy yechimini toping.

8⟶ Bu differensial tenglamaning xarakteristik sonlari kompleks: $\lambda^2-2\lambda+2=0\Rightarrow\lambda_1=1+i$ va $\lambda_2=1-i$. Demak, (kompleks) bazis yechimlar $e^{(1+i)x}, e^{(1-i)x}$; umumiy yechim esa kompleks sohada $y=c_1e^{(1+i)x}+c_2e^{(1-i)x}$ ko'rinishga ega (c_1,c_2- ixtiyoroy kompleks o'zgarmaslar). Biz yechimni kompleks sohada topdik. Differensial tenglama esa haqiqiy sohada berilgan edi. Bunday paytlarda odarda haqiqiy yechimlarni topish talab etiladi. Haqiqiy bazis yechimlarni qurish uchun kompleks yechimning haqiqiy va mavhum qismilarini ajratamiz yoki Eyler formulalaridan foydalanamiz:

$$\cos x=\frac{e^{(1+i)x}+e^{(1-i)x}}{2},\quad \sin x=\frac{e^{(1+i)x}-e^{(1-i)x}}{2i}.$$

Endi tushunarliki, $e^x\cos x, e^x\sin x$ $-$haqiqiy bazis yechimlar, differensial tenglamaning haqiqiy sohadagi umumiy yechimi $y = c_1 e^x \cos x + c_2 e^x \sin x$ (c_1, c_2 – ixtiyoroy haqiqiy o'zgarmaslar). ♘

Endi (II.10.1) tenglamaning koeffitsientlari haqiqiy bo'lganda uning haqiqiy yechimlarini qurish usulida to'xtalamiz. Xarakteristik tenglama (II.10.5) ning koeffitsientlari haqiqiy bo'lgani uchun u k karrali $\lambda = \mu + i\nu$ $(\nu \neq 0)$ kompleks ildiz bilan birgalikda unga qo'shma bo'lgan k karrali $\overline{\lambda} = \mu - i\nu$ $(\nu \neq 0)$ kompleks ildizga ham ega. Eyler formulalariga ko'ra $2k$ ta $e^{(\mu+i\nu)x}, xe^{(\mu+i\nu)x}, \ldots, x^{k-1}e^{(\mu+i\nu)x}, e^{(\mu-i\nu)x}, xe^{(\mu-i\nu)x}, \ldots, x^{k-1}e^{(\mu-i\nu)x}$ kompleks yechimlardan yana $2k$ ta ushbu

$$e^{\mu x}\cos\nu x = \frac{1}{2}(e^{(\mu+i\nu)x} + e^{(\mu-i\nu)x}), e^{\mu x}\sin\nu x = \frac{1}{2i}(e^{(\mu+i\nu)x} - e^{(\mu-i\nu)x}),$$

$$xe^{\mu x}\cos\nu x = \frac{1}{2}(xe^{(\mu+i\nu)x} + xe^{(\mu-i\nu)x}),$$

$$xe^{\mu x}\sin\nu x = \frac{1}{2i}(xe^{(\mu+i\nu)x} - xe^{(\mu-i\nu)x}),$$

....

$$x^{k-1}e^{\mu x}\cos\nu x = \frac{1}{2}(x^{k-1}e^{(\mu+i\nu)x} + x^{k-1}e^{(\mu-i\nu)x}),$$

$$x^{k-1}e^{\mu x}\sin\nu x = \frac{1}{2i}(x^{k-1}e^{(\mu+i\nu)x} - x^{k-1}e^{(\mu-i\nu)x})$$

haqiqiy yechimlarni quramiz. Boshqa kompleks yechimlarga ko'ra ham mos haqiqiy yechimlarni tuzamiz va berilgan haqiqiy koeffitsientli differensial tenglamaning haqiqiy bazis yechimlarini topamiz.

Misol 3. Ushbu

$$y^{V} - y^{IV} + 18y''' - 18y'' + 81y' - 81y = 0$$

differensial tenglamani yeching.

⊶ Mos xarakteristik tenglamani tuzamiz va uni yechamiz:

$$\lambda^5 - \lambda^4 + 18\lambda^3 - 18\lambda^2 + 81\lambda - 81 = 0,$$

$(\lambda-1)(\lambda^2+9)^2=0$.

Demak, xarakteristik sonlar : $\lambda=1$ (oddiy), $\lambda=\pm 3i$ (ikki karrali).

Haqiqiy bazis yechimlar: e^x, $\cos 3x$, $\sin 3x$, $x\cos 3x$, $x\sin 3x$.

Umumiy yechim (haqiqiy sohada):

$y=c_1e^x+c_2\cos 3x+c_3\sin 3x+c_4x\cos 3x+c_5x\sin 3x$. ♧

Masalalar

Tenglamalarni yeching :

1. $y'''-3y''+3y'-y=0$. 2. $y^{IV}-y=0$.
3. $y^{IV}-4y'''+8y''-8y'+4y=0$.
4. $y^{IV}-5y'''+9y''-7y'+2y=0$.

II.11. Bir jinsli bo'lmagan chiziqli o'zgarmas koeffitsientli tenglama

Chiziqli o'zgarmas koeffitsientli differensial operator $L[\circ]$ (II.10.1) formula bilan aniqlangan bo'lsin. Bir jinsli bo'lmagan

$$L[y]=g(x) \qquad \text{(II.11.1)}$$

tenglamaning umumiy yechimini topish uchun bu tenglamaning xususiy yechimiga mos bir jinsli $L[y]=0$ tenglamaning umumiy yechimini qo'shish kerak (II.8 paragrafga qarang). $L[y]=0$ tenglamaning bazis yechimlarini qurishni o'rgandik. Bu yechimlardan foydalanib $L[y]=g(x)$ tenglamaning xususiy yechimini Lagranjning ixtiyoriy o'zgarmaslarni variatsiyalash usuli bilan yoki (II.8.4) Koshi formulasiga ko'ra qurish mumkin. Bunda integrallash amali ishlatiladi. Lekin o'ng tomondagi $g(x)$ funksiya $g(x)=P(x)e^{\sigma x}$ ($P(x)-$ko'phad) maxsus ko'rinishga ega bo'lganda xususiy yechimni quyida bayon etilgan **noma'lum koeffitsientlar metodi** deb ataluvchi metod yordamida integrallash amalini ishlatmasdan turib topish mumkin.

Faraz qilaylik, $P(x)$ ko'phad

$$P(x)=b_mx^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0,\ b_m\neq 0, \qquad \text{(II.11.2)}$$

ko'rinishda bo'lsin. Ushbu

$$L[y]=P(x)e^{\sigma x} \qquad \text{(II.11.3)}$$

tenglamaning xususiy yechimini topish uchun (II.10.3) va (II.10.7) formulalardan kelib chiqib, quyidagicha ish tutish mumkin. Bunda σ son $L[y]=0$ tenglamaning xarakteristik soni bo'lmagan va bo'lgan hollarni alohida-alohida qaraymiz.

Teorema 1. *Agar (II.11.3) tenglamadagi* σ *son uchun* $L(\sigma)\neq 0$ *bo'lsa, u holda*

$$L[y]=(b_m x^m + b_{m-1}x^{m-1}+\cdots+b_1x+b_0)e^{\sigma x}$$

(II.11.3) tenglama

$$y=(r_m x^m + r_{m-1}x^{m-1}+\cdots+r_1x+r_0)e^{\sigma x} \qquad \text{(II.11.4)}$$

ko'rinishdagi xususiy yechimga ega.

$P(x)$ ko'phadning darajasi m bo'yicha matematik induksiya metodini qo'llaymiz. Dastlab $\deg P(x)=0$ deymiz va

$$L[y]=b_0e^{\sigma x}$$

tenglama $y=r_0e^{\sigma x}$ ko'rinishdagi xususiy yechimga egaligini ko'rsatamiz. (II.11.4) formulaga ko'ra

$$L[r_0e^{\sigma x}]=r_0e^{\sigma x}L(\sigma)=b_0e^{\sigma x}.$$

Bundan $L[y]=b_0e^{\sigma x}$ tenglama qanoatlanishi uchun $r_0=\dfrac{b_0}{L(\sigma)}$ deyish kifoya ekanligi kelib chiqadi. Demak, $m=0$ holida teorema isbotlandi. Endi induksiya farazini qilib, ya'ni $L[y]=\tilde{P}(x)e^{\sigma x}$, $\deg\tilde{P}(x)\le m-1$, tenglamaning $y=\tilde{Q}(x)e^{\sigma x}$, $\deg\tilde{Q}(x)\le m-1$ ko'rinishdagi yechimga ega ekanligini ma'lum deb, $L[y]=P(x)e^{\sigma x}$, $\deg P(x)=m$, tenglamaning $y=Q(x)e^{\sigma x}$, $\deg Q(x)=m$, ko'rinishdagi yechimga egaligini isbotlashimiz kerak. Berilgan

$$L[y]=P(x)e^{\sigma x},\ P(x)=(b_m x^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0)e^{\sigma x},$$

tenglamaning yechimini

$$y=r_m x^m e^{\sigma x}+u \qquad \text{(II.11.5)}$$

ko'rinishda izlaymiz; bu yerda r_m – hozircha noma'lum son, u – yangi noma'lum funksiya. Demak,

$L[r_m x^m e^{\sigma x} + u] = P(x)e^{\sigma x}$, ya'ni

$$r_m L[x^m e^{\sigma x}] + L[u] = (b_m x^m + b_{m-1}x^{m-1} + \dots + b_1 x + b_0)e^{\sigma x} \quad \text{(II.11.6)}$$

tenglik qanoatlanishi kerak. Endi r_m ni tanlash evaziga (II.11.6) tenglamadan x^m qatnashgan hadni yo'qotishga harakat qilamiz. (II.10.3) ayniyatga ko'ra

$$\begin{aligned} L[x^m e^{\sigma x}] &= \big(L(\sigma)x^m + C_m^1 L'(\sigma)x^{m-1} + C_m^2 L''(\sigma)x^{m-2} + \\ &+ \dots + L^{(m)}(\sigma)\big)e^{\sigma x} = L(\sigma)x^m e^{\sigma x} + R(x)e^{\sigma x} \end{aligned} \quad \text{(II.11.7)}$$

bu yerdagi $R(x) = C_m^1 L'(\sigma)x^{m-1} + C_m^2 L''(\sigma)x^{m-2} + \dots + L^{(m)}(\sigma)$ ko'phadning darajasi $\deg R(x) \le m-1$. Endi (II.11.7) ni (II.11.6) ga qo'yamiz, $r_m = \dfrac{b_m}{L(\sigma)}$ deb tanlaymiz va u noma'lum funksiya uchun ushbu

$$L[u] = \tilde{P}(x)e^{\sigma x},$$

$$\tilde{P}(x) \stackrel{def}{=} b_{m-1}x^{m-1} + \dots + b_1 x + b_0 - R(x), \ \deg \tilde{P}(x) \le m-1,$$

x^m qatnashmagan tenglamani hosil qilamiz.

Induksiya faraziga ko'ra bu tenglama $u = \tilde{Q}(x)e^{\sigma x}$, $\deg \tilde{Q}(x) \le m-1$, ko'rinishdagi yechimga ega. Demak, (II.11.5) formulaga ko'ra $L[y] = P(x)e^{\sigma x}$, $\deg P(x) = m$, tenglama $y = r_m x^m e^{\sigma x} + \tilde{Q}(x)e^{\sigma x}$, ya'ni

$y = Q(x)e^{\sigma x}$, $Q(x) \stackrel{def}{=} r_m x^m + \tilde{Q}(x)$, $\deg Q(x) = m$, ko'rinishdagi yechimga ega. 👍

Teorema 2. *Agar (II.11.3) tenglamadagi σ son $L(\lambda) = 0$ xarakteristik tenglamaning k karrali ildizi bo'lsa, u holda*

$$L[y] = (b_m x^m + b_{m-1}x^{m-1} + \dots + b_1 x + b_0)e^{\sigma x}$$

(II.11.3) tenglama

$$y = x^k(r_m x^m + r_{m-1}x^{m-1} + \cdots + r_1 x + r_0)e^{\sigma x} \qquad (II.11.8)$$

ko'rinishdagi yechimga ega.

Yuqoridagi teoremaning isbotiga o'xshash fikr yuritamiz. $\deg P(x) = m$ bo'yicha matematik induksiya metodini qo'llaymiz, faqat endi sal nozikroq fikrlash kerak bo'ladi. Dastlab teoremaning shartiga ko'ra $L(\lambda) = (\lambda - \sigma)^k M(\lambda)$, $M(\sigma) \neq 0$ $(k \in \mathbb{N})$, ekanligini e'tirof etaylik. Demak,

$$L(\sigma) = L'(\sigma) = \ldots = L^{(k-1)}(\sigma) = 0,\ L^{(k)}(\sigma) = k!M(\sigma) \neq 0. \qquad (II.11.9)$$

$m = 0$ bo'lsin. Ushbu

$$L[y] = b_0 e^{\sigma x} \qquad (II.11.10)$$

tenglama $y = r_0 x^k e^{\sigma x}$ ko'rinishdagi yechimga ega bo'lishini ko'rsatishimiz kerak. Buning uchun $L[r_0 x^k e^{\sigma x}]$ ni hisoblaymiz. (II.10.7) formuladan (II.11.9) ga ko'ra quyidagini topamiz :

$$L[r_0 x^k e^{\sigma x}] = r_0\big(L(\sigma)x^k + C_k^0 L'(\sigma)x^{k-1} +$$
$$+ C_k^1 L''(\sigma)x^{k-2} + \cdots + L^{(k)}(\sigma)\big)e^{\sigma x} = r_0 k! \cdot M(\sigma)e^{\sigma x}.$$

Demak, agar $r_0 = \dfrac{b_0}{k! \cdot M(\sigma)}$ desak, u holda $y = r_0 x^k e^{\sigma x}$ funksiya (II.11.10) tenglamaning yechimi bo'ladi. Teorema $m = 0$ holida isbot bo'ldi.

Endi induksiya farazini qilamiz, ya'ni $L[y] = \tilde{P}(x)e^{\sigma x}$, $\deg \tilde{P}(x) \leq m-1$, tenglama $y = x^k \tilde{Q}(x)e^{\sigma x}$, $\deg \tilde{Q}(x) \leq m-1$, ko'rinishdagi yechimga ega ekanligini ma'lum deymiz va $L[y] = P(x)e^{\sigma x}$, $\deg P(x) = m$, tenglamaning $y = x^k Q(x)e^{\sigma x}$, $\deg Q(x) = m$, ko'rinishdagi yechimga egaligini isbotlaymiz. Berilgan

$$L[y] = P(x)e^{\sigma x},\ P(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

tenglamaning yechimini

$$y = r_m x^{k+m} e^{\sigma x} + u \qquad (II.11.11)$$

ko'rinishda izlaymiz: bu yerda $u = u(x)$ – yangi noma'lum funksiya, r_m – noma'lum son. Biz ularni berilgan tenglamaning qanoatlanishi, ya'ni

$$L[r_m x^{k+m} e^{\sigma x} + u] = P(x)e^{\sigma x}$$

shartidan topishimiz kerak. Bu tenglamani

$$r_m L[x^{k+m} e^{\sigma x}] + L[u] = (b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0)e^{\sigma x} \quad \text{(II.11.12)}$$

ko'rinishda yozib olamiz va r_m ni shunday tanlaymizki, natijada (II.11.12) tenglamada x^m qatnashmasin. Buning uchun $L[x^{k+m} e^{\sigma x}]$ ni hisoblaymiz. (II.10.7) formuladan (II.11.9) munosabatlarni hisobga olib topamiz:

$$L[x^{k+m} e^{\sigma x}] = \Big(L(\sigma)x^{k+m} + C^1_{k+m} L'(\sigma)x^{k+m-1} + \cdots + C^{k-1}_{k+m} L^{(k-1)}(\sigma)x^{m+1} +$$

$$+ C^k_{k+m} L^{(k)}(\sigma)x^m + C^{k+1}_{k+m} L^{(k+1)}(\sigma)x^{m-1} + \cdots + L^{(k+m)}(\sigma) \Big) e^{\sigma x} =$$

$$= \Big(C^k_{k+m} L^{(k)}(\sigma)x^m + C^{k+1}_{k+m} L^{(k+1)}(\sigma)x^{m-1} + \cdots + L^{(k+m)}(\sigma) \Big) e^{\sigma x} =$$

$$= \Big(C^k_{k+m} k! \cdot M(\sigma)x^m + S(x) \Big) e^{\sigma x}, \quad \text{(II.11.13)}$$

bunda $S(x) = C^{k+1}_{k+m} L^{(k+1)}(\sigma)x^{m-1} + \cdots + L^{(k+m)}(\sigma)$ ko'phadning darajasi $\deg S(x) \le m-1$. Endi (II.11.13) ni (II.11.12) ga qo'yib, $r_m = \dfrac{b_m}{C^k_{k+m} k! \cdot M(\sigma)}$ deb tanlab, u noma'lum funksiya uchun ushbu

$$L[u] = \tilde{P}(x)e^{\sigma x},$$

$$\tilde{P}(x) \overset{def}{=} (b_{m-1}x^{m-1} + \cdots + b_1 x + b_0 - S(x)), \; \deg \tilde{P}(x) \le m-1, \quad \text{(II.11.14)}$$

tenglamani hosil qilamiz. Induksiya faraziga ko'ra (II.11.14) tenglama $u = x^k \tilde{Q}(x)e^{\sigma x}$, $\deg \tilde{Q}(x) \le m-1$, ko'rinishdagi yechimga ega. (II.11.11) formulaga ko'ra $L[y] = P(x)e^{\sigma x}$, $\deg P(x) = m$, tenglama esa

$$y = r_m x^{k+m} e^{\sigma x} + u = r_m x^{k+m} e^{\sigma x} + x^k \bar{Q}(x) e^{\sigma x} =$$

$$= x^k \left(r_m x^m + \bar{Q}(x) \right) e^{\sigma x} = x^k P(x) e^{\sigma x}$$

ko'rinishdagi yechimga ega bo'ladi; bunda $P(x) = r_m x^m + \bar{Q}(x)$, $\deg P(x) = m$.

Isbotlangan teoremalardan

$L[y] = (b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0) e^{\sigma x}$ tenglamaning xususiy yechimini topish uchun quyidagi **noma'lum koeffitsientlar metodidan** foydalanish mumkinligi kelib chiqadi:

1^0. $L(\sigma)$ ni hisoblaymiz. Agar $L(\sigma) \neq 0$ bo'lsa, $k = 0$ deymiz, aks holda esa, ya'ni $L(\sigma) = 0$ bo'lganda k bilan σ ildizning karralilik darajasini belgilaymiz.

2^0. Tenglamaning yechimini

$$y = x^k (r_m x^m + r_{m-1} x^{m-1} + \cdots + r_1 x + r_0) e^{\sigma x}$$

ko'rinishda izlaymiz; bunda $r_m, r_{m-1}, \ldots, r_1, r_0 -$ hozircha noma'lum koeffitsientlar.

3^0. Bu y ni berilgan tenglamaga qo'yib, chap tomonni soddalashtirib, tenglamani $e^{\sigma x}$ ga qisqartirib, ko'phadlarning tengligini hosil qilamiz.

4^0. Chap va o'ng tomondagi kophadlardagi x ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib, $r_m, r_{m-1}, \ldots, r_1, r_0 -$ noma'lum koeffitsientlarga nisbatan chiziqli algebraik tenglamalar sistemasini topamiz.

5^0. Bu sistemadan noma'lum koeffitsientlarni aniqlaymiz (sistemaning yechimga egaligini teoremalar ta'minlaydi).

6^0. Koeffitsientlarning topilgan qiymatlarini $y = x^k (r_m x^m + r_{m-1} x^{m-1} + \cdots + r_1 x + r_0) e^{\sigma x}$ ga qo'yib, izlangan xususiy yechimni hosil qilamiz.

Misol 4. Tenglamani yeching

$$y'' - (1+2i)y' + (-1+i)y = 2xe^{ix}.$$

🗝 Berilgan tenglamaning umumiy yechimi mos bir jinsli tenglamaning umumiy yechimiga berilgan bir jinsli bo'lmagan tenglamaning xususiy yechimini qo'shishdan hosil bo'ladi. Bir jinsli tenglamaning xarakteristik ko'phadi: $L(\lambda)=\lambda^2-(1+2i)\lambda-1+i$. Uning ildizlari $\lambda_1=i, \lambda_2=1+i$. Bir jinsli tenglamaning umumiy yechimi

$$y=c_1e^{ix}+c_2e^{(1+i)x} \quad (\{c_1,c_2\}\subset\mathbb{C}).$$

Endi berilgan bir jinsli bo'lmagan tenglamaning xususiy yechimini topamiz. Tenglamaning o'ng tomoni $P(x)e^{\sigma x}$ uchun $\sigma=i,\ P(x)=2x,\ \deg P(x)=1$. Ravshanki i soni $L(\lambda)=0$ tenglamaning $k=1$ karrali (oddiy) ildizi. Demak, xususiy yechimni $y=x(r_1x+r_0)e^{ix}$ ko'rinishda izlash mumkin. Bu y ni va uning $y'=\left(ir_1x^2+(2r_1+ir_0)x+r_0\right)e^{ix}$,

$y''=\left(-r_1x^2+(-r_0+i4r_1)x+2r_1+r_0\right)e^{ix}$ hosilalarini berilgan tenglamaga qo'yib, e^{ix} ga qisqartirib, chap tomondagi o'xshash hadlarni ixchamlab, topamiz:

$$-2r_1x+2r_1-r_0=2x.$$

Bundan $r_1=-1,\ r_0=2r_1=-2$ bo'lishi kelib chiqadi. Bu qiymatlarni $y=x(r_1x+r_0)e^{ix}$ ga qo'yib, berilgan tenglamaning $y=-x(x+2)e^{ix}$ xususiy yechimini hosil qilamiz. Uning umumiy yechimi esa

$$y=c_1e^{ix}+c_2e^{(1+i)x}-x(x+2)e^{ix} \quad (\{c_1,c_2\}\subset\mathbb{C})$$

formula bilan beriladi. 👍

Endi haqiqiy sohada berilgan va o'ng tomoni maxsus ko'rinishga ega bo'lgan ushbu

$$L[y]=e^{\alpha x}(P(x)\cos\beta x+Q(x)\sin\beta x) \qquad (II.11.15)$$

tenglamaning xususiy yechimini topishda to'xtalamiz. Bunda $L[\circ]$ operatorning koeffitsientlari va α,β – haqiqiy sonlar; $P(x), Q(x)$ – haqiqiy koeffitsientli ko'phadlar va

$\deg P(x)=m_1$, $\deg Q(x)=m_2$. Kompleks sohaga o'tib Eyler formulalariga ko'ra quyidagini yozamiz:

$$e^{\alpha x}(P(x)\cos\beta x+Q(x)\sin\beta x)=e^{\alpha x}(P(x)\operatorname{Re}e^{i\beta x}+$$
$$+Q(x)\operatorname{Re}(-ie^{i\beta x}))=\operatorname{Re}\left((P(x)-iQ(x))e^{(\alpha+i\beta)x}\right).$$

Demak,

$$L[y]=(P(x)-iQ(x))e^{(\alpha+i\beta)x}$$

tenglamaning $y=x^k(r_mx^m+r_{m-1}x^{m-1}+\cdots+r_1x+r_0)e^{(\alpha+i\beta)x}$ (bunda $m=\max\{m_1,m_2\}$, agar $L(\alpha+i\beta)\neq 0$ bo'lsa, $k=0$, $L(\alpha+i\beta)=0$ bo'lganda esa k soni $\sigma=\alpha+i\beta$ ildizning karralilik darajasi) ko'rinishdagi kompleks yechimini topib, uning haqiqiy qismini hisoblasak, berilgan (II.11.15) tenglamaning xususiy yechimini topgan bo'lamiz.

Misol 5. Ushbu

$$y''-4y'+5y=e^x(3\cos x+\sin x) \qquad \text{(II.11.16)}$$

tenglamaning xususiy yechimini toping.

⊶ Tenglamaning o'ng tomonini kompleks funksiyaning haqiqiy qismi ko'rinishida ifodalaymiz:

$$e^x(3\cos x+\sin x)=e^x\left(3\operatorname{Re}e^{ix}-\operatorname{Re}(ie^{ix})\right)=\operatorname{Re}\left((3-i)e^{(1+i)x}\right).$$

Endi ushbu

$$y''-4y'+5y=(3-i)e^{(1+i)x} \qquad \text{(II.11.17)}$$

tenglamaning xususiy yechimini topamiz. Bu tenglamaga mos bir jinsli tenglamaning xarakteristik ko'phadi $L(\lambda)=\lambda^2-4\lambda+5$ bo'lgani uchun $\sigma=1+i$ uning ildizi emas. Demak, (II.11.17) differensial tenglamaning xususiy yechimini $y=re^{(1+i)x}$ ko'rinishda izlash mumkin. Bu $y=re^{(1+i)x}$ ni tenglamaga qo'yib, $e^{(1+i)x}$ ga qisqartirib, topamiz:

$$(1+i)^2r-4(1+i)r+5r=3-i.$$

Bundan $r=1+i$. Demak, $y=(1+i)e^{(1+i)x}$ funksiya (II.11.17) tenglamaning xususiy yechimi. Endi oxirgi funksiyaning haqiqiy qismini topamiz:

$$\mathrm{Re}\left((1+i)e^{(1+i)x}\right)=e^x\,\mathrm{Re}\left((1+i)(\cos x+i\sin x)\right)=e^x(\cos x-\sin x).$$

Topilgan $y=e^x(\cos x-\sin x)$ haqiqiy funksiya berilgan (II.11.16) tenglamaning xususiy yechimidir. ♘

Biz yuqorida haqiqiy sohada berilgan (II.11.15) tenglamaning xususiy yechimini kompleks sohaga o'tib topdik. Xususiy yechimni kompleks sohaga chiqmasdan, ya'ni haqiqiy sohada turib ham topish mumkin. Buning uchun (II.11.15) ning yechimini

$$y=x^k e^{\alpha x}\left(M(x)\cos\beta x+N(x)\sin\beta x\right) \qquad \text{(II.11.18)}$$

ko'rinishda izlash kerak; bunda $k-$ yuqorida aytilgan ma'noga ega, $M(x)$ va $N(x)$ ko'phadlarlarning darajalari esa $\le m=\max\{m_1,m_2\}$. Ularning noma'lum koeffitsientlarini topish uchun (II.11.18) funksiyani (II.11.15) tenglamaga qo'yib, chap va o'ng tomondagi o'xshash hadlardagi koefftsientlarni tenglashtirib, hosil bo'lgan chiziqli algebraik tenglamalar sistemasini yechish kerak.

Misol 6. Ushbu

$$y''-4y'+5y=4e^{2x}(x\cos x+\sin x)$$

tenglamaning xususiy yechimini toping.

⊶ Ravshanki, $\sigma=2+i$ son $\lambda^2-4\lambda+5=0$ xarakteristik tenglamaning $k=1$ karrali ildizi,

$$m_1=1,\ m_2=0,$$

$$m=\max\{m_1,m_2\}=1$$

Demak, berilgan tenglamaning yechimini $y=xe^{2x}\left((ax+b)\cos x+(cx+d)\sin x\right)$ ko'rinishda izlash mumkin; bunda $a,b,c,d-$hozircha noma'lum sonlar. Bu $y=e^{2x}\left((ax^2+bx)\cos x+(cx^2+dx)\sin x\right)$ ni tenglamaga qo'yib, o'xshash hadlarni ixchamlab va $2e^{2x}$ ga qisqartirib, topamiz:

$$2cx\cos x-2ax\sin x+(a+d)\cos x+(c-b)\sin x=2x\cos x+2\sin x.$$

Bundan, chap va o'ng tomondagi o'xshash hadlardagi koeffitsientlarni tenglashtirib, $2c=2,\ -2a=0,\ a+d=0,\ c-b=2$ shartlarga ega bo'lamiz. Oxirgi sistemadan $a=0,\ b=-1,\ c=1,\ d=0$ qiymatlarni topamiz. Bularni $y = xe^{2x}\left((ax+b)\cos x + (cx+d)\sin x\right)$ ga qo'yib, berilgan tenglamaning $y = xe^{2x}\left(x\sin x - \cos x\right)$ xususiy yechimini topamiz. ♘

(Bir jinsli) **Eyler tenglamasi** deb ushbu

$$x^n y^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + xa_1 y' + a_0 y = 0 \qquad \text{(II.11.19)}$$

ko'rinishdagi chiziqli differensial tenglamaga aytiladi; bunda $a_{n-1}, \ldots, a_1, a_0$ – o'zgarmas sonlar. Bu tenglama erkli o'zgaruvchini $x>0$ bo'lganda $x=e^t$ ($x<0$ bo'lganda esa $x=-e^t$) almashtirish yordamida o'zgarmas koeffitsientli chiziqli tenglamaga keltiriladi. Bunga ishonch hosil qilish uchun y noma'lum funksiyaning x bo'yicha hosilalarini y ning t bo'yicha hosilalari orqali ifodalash kerak. Quyidagi hisoblashlarni bajaramiz ($x>0$, $x=e^t$):

$$x'_t = e^t = x,\ t'_x = x^{-1} = e^{-t},$$

$$y'_x = y'_t \cdot t'_x = y'_t e^{-t} \ (= x^{-1} y'_t),$$

$$y''_{xx} = (y'_t e^{-t})'_t \cdot t'_x = (y''_t - y'_t)e^{-2t} \left(= x^{-2}(y''_t - y'_t)\right),$$

Endi ixtiyoriy $k \in \mathbb{N}$ uchun

$$y^{(k)}_x = (b_{k,k} y^{(k)}_t + b_{k,k-1} y^{(k-1)}_t + \cdots + b_{k,1} y'_t)e^{-kt},$$

bunda $b_{k,j}$ lar – o'zgarmas sonlar, bo'lishini ko'rish qiyin emas. Buni matematik induksiya metodi yordamida qat'iy isbotlashni o'quvchiga havola etamiz. Hosilalar uchun tayyorlangan ifodalarni (II.11.19) tenglamaga qo'yib, o'zgarmas koeffitsientli chiziqli tenglamaga kelamiz. Uning xarakteristik tenglamasini tuzish uchun tenglamaga $y = e^{\lambda t}$ qo'yib, $e^{\lambda t}$ ga qisqartirish kerak. $x = e^t$ bo'lgani uchun buning o'rniga $y = x^{\lambda}$ funksiyani to'g'ridan-to'g'ri

(II.11.19) tenglamaga qo'yib, x^{λ} ga qisqartirish mumkin. Natijada λ noma'lumga nisbatan ushbu

$$\begin{aligned}&\lambda(\lambda-1)\ldots(\lambda-n+1)+a_{n-1}\lambda(\lambda-1)\ldots(\lambda-n+2)+\\&+\cdots+a_1\lambda+a_0=0\end{aligned}\qquad\text{(II.11.20)}$$

n - darajali algebraik tenglama hosil bo'ladi. Bu (II.11.20) tenglama va uning ildizlari (II.11.19) ning mos ravishda **xarakteristik tenglamasi** va **xarakteristik sonlari** deb ataladi. (II.11.19) Eyler tenglamasining umumiy yechimi uning xarakteristik sonlari bilan aniqlanadi. Har bir k karrali λ xarakteristik songa (II.11.19) tenglamaning k dona ushbu

$$x^{\lambda},\ x^{\lambda}\ln x,\ \ldots,\ x^{\lambda}(\ln x)^{k-1}$$

chiziqli erkli yechimlari mos keladi. Agar (II.11.19) tenglamaning barcha koeffitsientlari haqiqiy bo'lsa, (II.11.20) xarakteristik tenglama k karrali $\lambda=\mu+i\nu$ $(\mu\neq 0)$ xarakteristik son bilan birgalikda k karrali $\bar{\lambda}=\mu-i\nu$ xarakteristik songa ham ega bo'ladi. Bu xarakteristik sonlarga $2k$ dona ushbu

$$\begin{aligned}&x^{\mu}\cos(\nu\ln x),\ x^{\mu}\sin(\nu\ln x),\ x^{\mu}\ln x\cos(\nu\ln x),\\&x^{\mu}\ln x\sin(\nu\ln x),\ \ldots,\ x^{\mu}(\ln x)^{k-1}\cos(\nu\ln x),\ x^{\mu}(\ln x)^{k-1}\sin(\nu\ln x)\end{aligned}$$

chiziqli erkli haqiqiy yechimlar mos keladi. Barcha xarakteristik sonlarga mos keluvchi yechimlarni to'plab, Eyler tenglamasining bazis yechimlarini hosil qilamiz.

Misol 7. Tenglamaning umumiy yechimini toping

$$x^3y'''+2x^2y''-xy'+y=0\,.$$

Xarakteristik tenglamani tuzamiz:

$$\lambda(\lambda-1)(\lambda-2)+2\lambda(\lambda-1)-\lambda+1=0\,.$$

Xarakteristik sonlar: $\lambda_1=-1$, $\lambda_2=\lambda_3=1$.

Oddiy xarakteristik son ($\lambda_1=-1$) ga mos keluvchi yechim:

$$y_1=\frac{1}{x}.$$

Ikki karrali xarakteristik son ($\lambda_2=\lambda_3=1$) ga mos keluvchi yechimlar: $y_2 = x$, $y_3 = x\ln x$ $(x > 0)$. Demak, $x > 0$ sohadagi

umumiy yechim $y = \frac{c_1}{x} + c_2 x + c_3 x \ln x$.

$x < 0$ sohadagi umumiy yechim esa $y = \frac{c_1}{x} + c_2 x + c_3 x \ln(-x)$

formula bilan beriladi.

Masalalar

1. Tenglamalarning umumiy yechimini toping:

$a)$ $y''' - 4y'' + 9y' - 10y = 5e^{2x}$;

$b)$ $x^3 y''' - x^2 y'' + 2xy' - 2y = 0$ $(x > 0)$.

2. Ushbu $L[y] = (b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0) e^{\alpha x}$

tenglamada $y = e^{\alpha x} u$ almashtirishni bajaring ($u = u(x)$ – yangi noma'lum funksiya). Bunda hosil bo'luvchi

$$\tilde{a}_n u^{(n)} + \tilde{a}_{n-1} u^{(n-1)} + \cdots + \tilde{a}_1 u' + \tilde{a}_0 u = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$$

tenglamaning koeffitsientlarini hisoblang.

3. Agar uzluksiz koeffitsientli $a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y = 0$

tenglama barcha $\bar{x} = x + \lambda$ almashtirishlarga nisbatan invariant (ko'rinishi o'zgarmas) bo'lsa, tenglamaning koeffitsientlari o'zgarmas bo'lishini isbotlang.

4. Agar uzluksiz koeffitsientli $a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y = 0$ tenglama barcha $\bar{x} = \lambda x$ almashtirishlarga nisbatan invariant bo'lsa, $a_j(x) = a_j x^j$ (a_j – const, $j = \overline{1, n}$) ekanligini, ya'ni bu tenglama Eyler tenglamasidan iborat bo'lishini isbotlang.

5. Operatorli metod. D differensiallash operatorini kiritaylik; bu operator $y = y(x)$ $(x \in I)$ silliq funksiyaga ushbu $Dy = \frac{dy}{dx} = y'$ formulaga ko'ra ta'sir etadi. E bilan birlik operatorini belgilaymiz: $Ey = y$. Ravshanki, D *va* E *operatorlarlar chiziqli.* A chiziqli operatorning λ songa ko'paytmasi λA operator ushbu $(\lambda A)y = \lambda(Ay)$ formula bilan kiritiladi. A bilan birgalikda λA *operator ham chiziqlidir.* A va B chiziqli operatorlarning yig'indisi $A+B$ ushbu $(A+B)y = Ay + By$ formula bilan, ularning AB ko'paytmasi (ketma-ket bajarilishi) esa $(AB)y = A(By)$ formula bilan aniqlanadi. *A va B operatorlar bilan birgalikda $A+B$ va AB operatorlar ham chiziqlidir.* A operatorning darajalari $A^2 = AA$, $A^3 = AA^2$, ..., $A^n = AA^{n-1}$ formulalar bilan aniqlanadi. Masalan, $D^2 = DD$ ko'paytma $D^2y = D(Dy) = \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2} = y''$ ikki marta differensiallash operatorini anglatadi. Demak, ixtiyoriy $a_{n-1}, \ldots, a_1, a_0$ sonlar uchun ushbu

$$L(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 E$$

n- tartibli differensial operator (differensial ko'phad) ma'noga ega. Kiritilgan ta'riflarga ko'ra bu operator $y = y(x)$ silliq funksiyaga ushbu

$$L(D)y = (D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 E)y =$$

$$= D^n y + a_{n-1}D^{n-1}y + \cdots + a_1 Dy + a_0 Ey = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y$$

ya'ni $L(D)y = L[y]$ formula asosida ta'sir etadi.

1°. Yuqorida kursiv bilan ajratilgan tasdiqlarni tekshiring.

2°. Ixtiyoriy $R(D), S(D), T(D)$ differensial ko'phadlar uchun quyidagi xossalarni isbotlang:

$$R(D) + \big(S(D) + T(D)\big) = \big(R(D) + S(D)\big) + T(D),$$

$$S(D) + T(D) = T(D) + S(D),$$

$$R(D)E = ER(D) = R(D),$$

$$R(D)\big(S(D)T(D)\big) = \big(R(D)S(D)\big)T(D),$$

$$S(D)T(D) = S(D)T(D),$$

$$\big(R(D) + S(D)\big)T(D) = R(D)T(D) + S(D)T(D),$$

agar $S(D)y_1 = 0$, $T(D)y_2 = 0$ bo'lsa, ixtiyoriy o'zgarmas c_1, c_2 sonlar uchun $T(D)S(D)(c_1 y_1 + c_2 y_2) = 0$ bo'ladi.

Oxirgi xossadan foydalanib, $L(D)y=0$ tenglmaning yechimlarini topish mumkin. Buning uchun $L(D)$ differensial ko'phadni ko'paytuvchilarga ajratish kerak. Masalan, $L(D)=D^2+E=(D-iE)(D+iE)$, $(D-iE)e^{ix}=0$ va $(D+iE)e^{-ix}=0$ bo'lgani uchun $L(D)y=(D^2+E)y=y''+y=0$ tenglamaning $y=c_1e^{ix}+c_2e^{-ix}$ yechimini topamiz.

$L(D)$ operatorni $e^{\lambda x}$ funksiyaga ta'sir ettirib, quyidagi formulani hosil qilamiz:

$$L(D)e^{\lambda x}=e^{\lambda x}L(\lambda).$$

Endi $L(D)$ operatorning $e^{\lambda x}u$ funksiyaga ta'sirini o'rganaylik; bunda $u=u(x)$ – silliq funksiya. Ravshanki, ko'paytmani differensiallash qoidasiga ko'ra

$$De^{\lambda x}u=e^{\lambda x}u'+\lambda e^{\lambda x}u=e^{\lambda x}(D+\lambda E)u,$$

ya'ni $De^{\lambda x}u$ ifodada $e^{\lambda x}$ oldinga o'tkazilganda D siljib, $D+\lambda E$ ga aylanadi. Bu munosabatni umumlashtirib,

3. quyidagi formulani isbotlang:

$$L(D)e^{\lambda x}u=e^{\lambda x}L(D+\lambda E)u$$

Bu formula siljish formulasi deb ataladi: $e^{\lambda x}$ oldinga chiqarilganda (o'tkazilganda) $L(D)$ siljib $L(D+\lambda E)$ ga aylanadi.

4. Agar $k\in\mathbb{N}$ va $M(\lambda)$ ko'phad uchun $L(\lambda)=(\lambda-\sigma)^k M(\lambda)$, $M(\sigma)\neq 0$ bo'lsa,

$L(D)(x^k e^{\sigma x})$ ni hisoblang.

5. Ushbu $L(D)=(D-\lambda_0 E)^{k_0}$ $(k_0\in\mathbb{N})$ differensial operator uchun

$$(D-\lambda_0 E)^{k_0}e^{\lambda_0 x}=0,\ (D-\lambda_0 E)^{k_0}(xe^{\lambda_0 x})=0,$$
$$\ldots,\ (D-\lambda_0 E)^{k_0}(x^{k_0-1}e^{\lambda_0 x})=0$$

tengliklarning o'rinliligini ko'rsating.

6. Agar $L(D)=(D-\lambda_1 E)^{k_1}(D-\lambda_2 E)^{k_2}\ldots(D-\lambda_s E)^{k_s}$ bo'lsa, $L(D)y=0$ differensial tenglamaning umumiy yechimini quring.

Endi bir jinsli bo'lmagan $L(D)y=\left(b_m x^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0\right)e^{\sigma x}$ tenglama bilan shug'ullanamiz. Bu tengalamning xususiy yechimini quyidagi usul bilan topish mumkin. Ravshanki, bu tenglamaning o'ng tomoni ushbu $(D-\sigma E)^{m+1}y=0$ bir jinsli tenglamaning yechimi, ya'ni $(D-\sigma E)^{m+1}\left(\left(b_m x^m+b_{m-1}x^{m-1}+\cdots+b_1x+b_0\right)e^{\sigma x}\right)=0$. Berilgan bir jinsli bo'lmagan tenglamaning har ikkala tomoniga $(D-\sigma E)^{m+1}$ operatorni ta'sir ettirib, ushbu $(D-\sigma E)^{m+1}L(D)y=0$ bir jinsli (yuqoriroq tartibli) tenglamani hosil qilamiz. Bu tenglamani yechib, undan $L(D)y=0$ tenglamaning yechimini tushirib qoldirib, berilgan bir jinsli bo'lmagan tenglamaning yechimi ko'rinishini topamiz.

7°. σ ning xususiyatidan kelib chiqib, berilgan bir jinsli bo'lmagan tenglamaning yechimi ko'rinishini yozing.

8°. Yuqoridagi fikrlashlarni

$$L(D)y=\sum_{j=1}^{s}\left(b_{j,m_j}x^{m_j}+b_{j,m_j-1}x^{m_j-1}+\cdots+b_{j,1}x+b_{j,0}\right)e^{\sigma_j x}$$

tenglamaga umumlashtiring.

II.12. Tenglamalarni darajali qatorlar yordamida yechish

Koeffitsientlar boshlang'ich nuqtada analitik.

Differensial tenglamalarning yechimlari har doim ham kvadraturalarda ifodalanavermaydi. Tenglamada qatnashgan funksiyalarning barchasi analitik bo'lganda, yechimlarni darajali qator yig'indisi sifatida topish mumkin bo'ladi. Shu munosabat bilan dastlab analiz kursidan ma'lum bo'lgan analitik funksiya ta'rifi va uning ba'zi xossalarini eslaylik.

Agar bir o'zgaruvchining $y=f(x)$ funksiyasi x_0 nuqtaning biror atrofida biror $\sum_{n=0}^{+\infty}a_n(x-x_0)^n$ (x_0 markazli) darajali qatorning yig'inidisi sifatida tasvirlansa, ya'ni

$$f(x)=\sum_{n=0}^{+\infty}a_n(x-x_0)^n,\ |x-x_0|<\delta\ \ (\delta>0). \qquad (II.12.1)$$

tenglik o'rinli bo'lsa , u holda $y=f(x)$ funksiya x_0 nuqtada analitik funksiya deyiladi.

Analizdan ma'lumki, masalan, $\sin x, \cos x, e^x$ funksiyalari ixtiyoriy $x_0\in\mathbb{R}$ nuqtada analitik. Agar funksiya (a,b) intervalning har bir nuqtasida analitik bo'lsa, bu funksiya (a,b) intervalda analitik deyiladi.

Ushbu $\sum_{n=0}^{+\infty}a_n(x-x_0)^n$ darajali qatorning R yaqinlashish radiusi uchun $\frac{1}{R}=\overline{\lim_{n\to\infty}}\sqrt[n]{|a_n|}$ Koshi formulasi o'rinli. Yaqinlashish radiusi qator yaqinlashadigan eng katta $|x-x_0|<R$ $(R>0)$ intervalni aniqlaydi, $R=+\infty$ bo'lganda darajali qator $(-\infty,+\infty)$ oraliqda yaqinlashuvchi bo'ladi. (II.12.1) tenglik aslida $|x-x_0|<R$ yaqinlashish intervalida o'rinli bo'ladi. (II.12.1) dagi darajali qatorni uning yaqinlashish intervalida xohlagancha marta hadma-had differensiallash mumkin; bunda qatorning R yaqinlashish radiusi o'zgarmaydi. x_0 nuqtada analitik funksiya (II.12.1) darajali qatorning yaqinlashish intervalida (shu intervalning har bir nuqtasida) ham analitik bo'ladi. (II.12.1) formuladagi darajali qator koeffitsientlari uchun

$$a_n=\frac{f^{(n)}(x_0)}{n!},\ n=0,1,2,...\ (0!=1;\ 1!=1;\ n!=1\cdot 2\cdot ...\cdot n,\ n\in\mathbb{N}),$$

formulalar o'rinli, ya'ni analitik funksiya o'zining Teylor qatori yig'indisidan iborat.

Agar

$$f(x)=\sum_{n=0}^{+\infty}a_n(x-x_0)^n=\sum_{n=0}^{+\infty}\tilde{a}_n(x-x_0)^n,\ |x-x_0|<\delta\ \ (\delta>0),$$

bo'lsa, bu darajali qatorlarning mos koeffitsientlari bir xil, ya'ni $a_n=\tilde{a}_n$ $(n=0,1,2,...)$ (yagonalik xossasi).

Analitik $f(x)=\sum_{n=0}^{+\infty}a_n(x-x_0)^n$, $|x-x_0|<R_1$, va

$g(x)=\sum_{n=0}^{+\infty}b_n(x-x_0)^n$, $|x-x_0|<R_2$, funksiyalar berilgan bo'lsin.

Ularning yig'indisi va ayirmasi ham analitik funksiya va

$$f(x)\pm g(x)=\sum_{n=0}^{+\infty}(a_n\pm b_n)(x-x_0)^n,\ |x-x_0|<\min(R_1,R_2),$$

yoyilma o'rinli; ko'paytma funksiya ham analitik va uning darajali qatorga yoyilmasi berilgan qatorlarni formal ravishda ko'paytirishdan hosil bo'ladi:

$$f(x)g(x)=\left(\sum_{n=0}^{+\infty}a_n(x-x_0)^n\right)\cdot\left(\sum_{n=0}^{+\infty}b_n(x-x_0)^n\right)=\sum_{n=0}^{+\infty}c_n(x-x_0)^n,$$

$$c_n=\sum_{k=0}^{n}a_kb_{n-k}=a_0b_n+a_1b_{n-1}+\ldots+a_nb_0,\ |x-x_0|<\min(R_1,R_2);$$

bundan tashqari, qo'shimcha ravishda $g(x_0)\neq 0$ ham bo'lsa, u holda $f(x)/g(x)$ nisbat x_0 nuqtada analitik funksiya bo'ladi va

$$\frac{f(x)}{g(x)}=\sum_{n=0}^{+\infty}d_n(x-x_0)^n,\ |x-x_0|<\delta,$$

($\delta<\min(R_1,R_2)$ bo'lishi mumkin)

yoyilmaning d_n koeffitsientlari

$$\sum_{n=0}^{+\infty}a_n(x-x_0)^n=\left(\sum_{n=0}^{+\infty}b_n(x-x_0)^n\right)\cdot\left(\sum_{n=0}^{+\infty}d_n(x-x_0)^n\right)=$$

$$=\sum_{n=0}^{+\infty}\Big(\sum_{k=0}^{n}b_kd_{n-k}\Big)(x-x_0)^n,$$

tenglikdagi $(x-x_0)$ ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirish yordamida topilishi mumkin.

Differensial tenglama yechimni darajali qator yig'indisi sifatida topish, ya'ni analitik yechimni qurish jarayoni bilan ikkinchi tartibli chiziqli differensial tenglama misolida tanishamiz.

Birinchi yoki yuqori tartibli chiziqli tenglama yoki chiziqli tenglamalar sistemasi uchun ham analitik yechimni topish jarayoni shunga o'xshash amalga oshiriladi. Analitik yechimni topish uchun tenglama(lar)dagi barcha koeffitsientlar analitik bo'lishi kerak.

Aniqrog'i, ushbu

$$y'' + p_1(x)y' + p_0(x)y = q(x) \qquad (II.12.2)$$

tenglamani qaraylik. Bu yerdagi $p_1(x)$, $p_0(x)$, $q(x)$ funksiyalar x_0 nuqtada, ya'ni x_0 nuqtaning biror atrofida analitik deb hisoblanadi. Yechimni x_0 markazli hozircha noma'lum koeffitsientli darajali qator yig'indisi sifatida yozamiz:

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad |x - x_0| < R_1 \quad (R_1 > 0), \qquad (II.12.3)$$

bu yerda $a_n \ (n = 0, 1, 2, \ldots)$ – noma'lum koeffitsientlar. Qatorni hadma-had ikki marta differensiallaymiz:

$$y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n, \quad |x - x_0| < R_1, \qquad (II.12.4)$$

$$y'' = \sum_{n=1}^{\infty} n(n+1) a_{n+1} (x - x_0)^{n-1} =$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x - x_0)^n, \quad |x - x_0| < R_1. \qquad (II.12.5)$$

Tenglamada berilgan koeffitsientlarni ham x_0 markazli darajali qatorlarga yoyamiz:

$$p_1(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n, \quad |x - x_0| < R_2, \qquad (II.12.6)$$

$$p_0(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n, \quad x - x_0| < R_2, \qquad (II.12.7)$$

$$q(x) = \sum_{n=0}^{\infty} d_n (x - x_0)^n, \quad |x - x_0| < R_2, \qquad (II.12.8)$$

(bu yerda $p_1(x)$, $p_0(x)$, $r(x)$ uchun darajali qatorlar yaqinlashish radiuslarining eng kichigini R_2 bilan belgiladik).

Bu qatorlarni qaralayotgan (II.12.2) differensial tenglamaga qo'yib, zarur soddalashtirishlarni bajaramiz:

$$\sum_{n=0}^{+\infty}(n+1)(n+2)a_{n+2}(x-x_0)^n+\sum_{n=0}^{+\infty}b_n(x-x_0)^n\cdot\sum_{n=0}^{+\infty}(n+1)a_{n+1}(x-x_0)^n+$$

$$+\sum_{n=0}^{+\infty}c_n(x-x_0)^n\cdot\sum_{n=0}^{+\infty}a_n(x-x_0)^n=\sum_{n=0}^{+\infty}d_n(x-x_0)^n;$$

$$\sum_{n=0}^{+\infty}\Big((n+1)(n+2)a_{n+2}+\sum_{k=0}^{n}\big((k+1)a_{k+1}b_{n-k}+a_kc_{n-k}\big)\Big)(x-x_0)^n=$$

$$=\sum_{n=0}^{+\infty}d_n(x-x_0)^n.$$

Bu tenglikning chap va o'ng tomonidagi $(x-x_0)$ ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib, $a_n\ (n=0,1,2,...)$ noma'lumlarni topish uchun cheksiz sistemani hosil qilamiz:

$(x-x_0)^0:\ \ 1\cdot 2\cdot a_2+a_1b_0+a_0c_0=d_0,$

$(x-x_0)^1:\ \ (n+1)(n+2)2\cdot 3\cdot a_3+a_1b_1+a_0c_1+2a_2b_0+a_1c_0=d_1,$

. . . .

$(x-x_0)^n:\ \ (n+1)(n+2)a_{n+2}+\sum_{k=0}^{n}\big((k+1)a_{k-1}b_{n-k}+a_kc_{n-k}\big)=d_n,$

. . . .

Agar $a_0=y(x_0)$ va $a_1=y'(x_0)$ qiymatlarni ixtiyoriy tanlasak (Koshi masalasida ular berilgan bo'ladi), u holda bu yerdagi birinchi tenglamadan a_2, ikkinchisidan a_3 va h.k. barcha qolgan a_n lar bir qiymatli aniqlanadi. Topilgan a_n larga ko'ra (II.12.3) darajali qatorning yaqinlashish radiusini hisoblaymiz va qurilgan analitik funksiya yaqinlashish intervalida (II.12.2) differensial tenglamaning yechimi ekanligini asoslaymiz. Ko'rsatish mumkinki,

$R_1 \geq R_2$ bo'ladi, ya'ni qurilgan (II.12.3) funksiya (II.12.2) tenglamada qatnashgan analitik funksiyalarining umumiy analitiklik intervalida shu tenglamaning (analitik) yechimi bo'ladi. Shunday qilib (II.12.2) tenglamaning a_0, a_1 ikki parametrli analitik yechimlar oilasini hosil qilamiz.

Yuqorida bajarilgan ishlarning umumiy holda qonuniy ekanligini quyidagi teorema asoslaydi [ ].

Teorema 1. *Faraz qilaylik,* $p_1(x), p_0(x)$ *va* $q(x)$ *funksiyalar* $(x_0 - R, x_0 + R)$ *intervalda analitik bo'lsin. U holda har qanday* a_0 *va* a_1 *sonlar uchun ushbu*

$$y'' + p_1(x)y' + p_0(x)y = q(x),\ y(x_0) = a_0,\ y'(x_0) = a_1,$$

Koshi masalasi $(x_0 - R, x_0 + R)$ *intervalda aniqlangan yagona analitik yechimga ega.*

Konkret misollar keltiraylik.

Misol 1. Ushbu

$$y'' - y = 0 \tag{II.12.9}$$

differensial tenglamaning

$$y = \sum_{n=0}^{\infty} a_n x^n \quad (x_0 = 0) \tag{II.12.10}$$

ko'rinishdagi yechimini quring.

Bu yerdagi $a_n\ (n = 0, 1, 2, \ldots)$ – noma'lum sonlarni aniqlashimiz kerak. (II.12.10) dan y'' ni hisoblaymiz:

$$y'' = \sum_{n=2}^{\infty}(n-1)na_n x^{n-2} = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n. \tag{II.12.11}$$

(II.12.11) va (II.12.10) formulalarni (II.12.9) ga qo'yamiz:

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n = \sum_{n=0}^{\infty}a_n x^n .$$

x ning bir xil darajalari koeffitsientlarini tenglashtiramiz:

$$x^0:\ 1 \cdot 2 \cdot a_2 = a_0,$$

$$x^1:\ 2 \cdot 3 \cdot a_3 = a_1,$$

$$x^2: \ 3\cdot 4\cdot a_4 = a_2,$$

....

$$x^n: \ (n+1)(n+2)a_{n+2} = a_n,$$

....

a_0, a_1 larni ixtiyoriy tayinlaymiz. U holda yuqoridagi tenglamalarning toq nomerlilaridan

$$a_2 = \frac{a_0}{1\cdot 2},\ a_4 = \frac{a_2}{3\cdot 4} = \frac{a_0}{1\cdot 2\cdot 3\cdot 4} = \frac{a_0}{4!},\ \ldots,\ a_{2n} = \frac{a_0}{(2n)!}\ (n = 1,2,\ldots),$$

juft nomerlilaridan esa

$$a_3 = \frac{a_1}{2\cdot 3},\ a_5 = \frac{a_3}{4\cdot 5} = \frac{a_1}{2\cdot 3\cdot 4\cdot 5} = \frac{a_1}{5!},\ \ldots,\ a_{2n+1} = \frac{a_1}{(2n+1)!}\ (n = 1,2,\ldots)$$

tengliklarni topamiz. Bularni (II.12.10) ga qo'yib, formal ravishda yechimni topamiz:

$$y = a_0 \sum_{n=1}^{+\infty} \frac{1}{(2n)!} x^{2n} + a_1 \sum_{n=1}^{+\infty} \frac{1}{(2n+1)!} x^{2n+1}. \qquad \text{(II.12.12)}$$

Bu formuladagi darajali qatorlar $(-\infty, +\infty)$ oralig'ida absolyut yaqinlashuvchi (masalan, Dalamber alomatiga asosan). Demak, yuqorida qilingan ishlar (hadma-had differensiallash va h.k.) qonuniy va (II.12.12) formula (II.12.9) tenglamaning $(-\infty, +\infty)$ oraliqda aniqlangan analitik yechimini beradi. Analizdan ma'lumki, giperbolik kosinus va giperbolik sinus funksiyalari uchun

$$\mathrm{ch}x = \sum_{n=1}^{+\infty} \frac{1}{(2n)!} x^{2n},\ \mathrm{sh}x = \sum_{n=1}^{+\infty} \frac{1}{(2n+1)!} x^{2n+1}.$$

Shunday qilib, (II.12.9) tenglamaning ushbu

$$y = a_0 \mathrm{ch}x + a_1 \mathrm{sh}x \quad (a_0, a_1 - \text{ ixtiyoriy o'zgarmaslar})$$

ikki parametrli analitik yechimlar oilasini (umumiy yechimini) topdik. ♘

Yuqoridagi (II.12.2) tenglamaning (II.12.3) yechimidagi $a_n\ (n = 0,1,2,\ldots)$ koeffitsientlarni

$$a_n = \frac{y^{(n)}(x_0)}{n!} \quad (n = 0,1,2,\ldots)$$

formulalarga ko'ra topish ham mumkin. Bunda $a_0 = y(x_0)$, $a_1 = y'(x_0)$ va qolgan koeffitsientlar (II.12.2) tenglamadan aniqlanadi. $y = y(x)$ yechim bo'lgani uchun, u x_0 nuqtaning biror atrofida (II.12.3) tenglamani ayniyatga aylantiradi:

$$y''(x) + p_1(x)y'(x) + p_0(x)y(x) = q(x)\ . \qquad \text{(II.12.13)}$$

Bu ayniyatda $x = x_0$ deb $y''(x_0)$ ni va, demak, $a_2 = \dfrac{y''(x_0)}{2!}$ ni topamiz. (II.12.13) ayniyatni ketma-ket differensiallab va $x = x_0$ deb, $y'''(x_0), y^{IV}(x_0), \ldots$ hamda $a_3, a_4, \ldots$ larni hisoblaymiz:

$$y'''(x) = q'(x) - p_1'(x)y'(x) - p_1(x)y''(x) - p_0'(x)y(x) - p_0(x)y'(x)\ ,$$

bundan $y'''(x_0)$ va $a_3 = \dfrac{y'''(x_0)}{3!}$ lar;

$$y^{IV}(x) = \big(q'(x) - p_1'(x)y'(x) - p_1(x)y''(x) -$$

$$- p_0'(x)y(x) - p_0(x)y'(x)\big)' = q''(x) - \ldots\ ,$$

bundan $y^{IV}(x_0)$ va $a_4 = \dfrac{y^{IV}(x_0)}{4!}$ lar;

....

topiladi.

Biz yuqorida chiziqli tenglamalarning analitik yechimlarini topish bilan shug'ullandik. Nochiziqli tenglamalarning analitik yechimlari ham yuqoridagiga o'xshash topiladi. Bunda quyidagi teorema asos bo'lib xizmat qiladi.

Teorema 2. *Aytaylik,*

$$y'' = f(x, y, y'),\ y(x_0) = y_0,\ y'(x_0) = y_0'$$

(x_0, y_0, y_0' – berilgan sonlar), Koshi masalasi berilgan bo'lsin. Agar $f(x, y, z)$ funksiya $(x_0, y_0, y_0') \in \mathbb{R}^3$ boshlang'ich nuqtaning biror atrofida analitik, ya'ni biror absolyut yaqinlashuvchi $\sum\limits_{k,l,m=0}^{\infty} a_{k,l,m}(x - x_0)^k (y - y_0)^l (z - z_0)^m$ qatorning yig'indisi sifatida ifodalansa, u holda berilgan masala

$x_0 \in \mathbb{R}$ *nuqtaning biror atrofida analitik bo'lgan yagona yechimga ega.*

Qisqacha, lekin noaniqroq: o'ng tomoni analitik bo'lgan differensial tenglamaning yechimlari analitik.

Shunga o'xshash teorema yuqori tartibli differesial tenglama, yoki tenglamalar sistemasi uchun ham o'rinli.

Bunda $y = \sum_{n=0}^{+\infty} a_n (x - x_0)^n$ analitik yechimning $a_n \ (n = 0, 1, 2, \ldots)$ koeffitsientlarini topishni yuqorida aytilgan ikki usuldan biri yordamida amalga oshirish mumkin.

Misol 2. Ushbu

$$y'' = x + y^2, \ y(1) = 1, \ y'(1) = 0,$$

Koshi masalasining analitik yechimi topilsin.

Yechimni

$$y = \sum_{n=0}^{+\infty} a_n (x-1)^n \quad (x_0 = 1),$$

ko'rinishda izlaymiz. $a_n \ (n = 0, 1, 2, \ldots)$ koeffitsientlarni, $a_n = \dfrac{y^{(n)}(1)}{n!}$ formulaga ko'ra, berilgan tenglamani differensiallash yordamida topamiz. Quyidagi hisoblashlarni bajaramiz:

$$a_0 = y(1) = 1, \ a_1 = y'(1) = 0;$$

$$y'' = x + y^2, \ y''(1) = 1, \ a_2 = \frac{y''(1)}{2!} = \frac{1}{2};$$

$$y''' = 1 + 2yy', \ y'''(1) = 1, \ a_3 = \frac{y'''(1)}{3!} = \frac{1}{6};$$

$$y^{IV} = 2y'^2 + 2yy', \ y^{IV}(1) = 1, \ a_4 = \frac{y^{IV}(1)}{4!} = \frac{1}{24};$$

$$y^{V} = 6y'y'' + 2yy''', \ y^{V}(1) = 2, \ a_5 = \frac{y^{V}(1)}{5!} = \frac{1}{60};$$

$$y^{VI}=6y''^2+8y'y'''+2yy^{IV},\ y^{VI}(1)=14,\ a_6=\frac{y^{VI}(1)}{6!}=\frac{7}{420};$$

....

Demak, izlangan yechim

$$y=1+\frac{1}{2}(x-1)^2+\frac{1}{6}(x-1)^3+\frac{1}{24}(x-1)^4+$$
$$+\frac{1}{60}(x-1)^5+\frac{7}{420}(x-1)^6+...$$

ko'rinishga ega. Bu darajali qatorning yaqinlashish radiusi qat'iy musbat bo'ladi. ⊗

Regulyar maxsus nuqta. Frobenius metodi.

Endi (II.12.2) tenglamaga qaraganda umumiyroq

$$p_2(x)y''+p_1(x)y'+p_0(x)y=q(x) \qquad \text{(II.12.14)}$$

differensial tenglamani qaraylik. Bu yerdagi $p_2(x)$, $p_1(x)$, $p_0(x)$, $q(x)$ funksiyalar x_0 nuqtaning biror atrofida analitik deb hisoblanadi. Agar $p_2(x_0)\neq 0$, ya'ni x_0 – tenglamaning regulyar nuqtasi bo'lsa, u holda (II.12.14) tenglama x_0 nuqtaning yetarlicha kichik atrofida ushbu

$$y''+\frac{p_1(x)}{p_2(x)}y'+\frac{p_0(x)}{p_2(x)}y=\frac{q(x)}{p_2(x)}$$

analitik koeffitsientli tenglamaga ekvivalent. Oxirgi tenglamaning x_0 nuqtada analitik yechimini topish bilan yuqorida tanishdik.

Faraz qilaylik, $p_2(x_0)=0$, ya'ni x_0 – (II.12.14) tenglamaning maxsus nuqtası bo'lsın. Bu holda (II.12.14) tenglama, umuman olganda, x_0 nuqtada analitik yechımga ega bo'lmasligi mumkin. Lekin ba'zı hollarda yechimni umumlashgan darajali qator yig'ndisi ko'rinishida ifodalash mumkin.

Ushbu

$$(x-x_0)^2y''+(x-x_0)p_1(x)y'+p_0(x)y=0 \quad \text{yoki}$$

$$y'' + \frac{p_1(x)}{x - x_0} y' + \frac{p_0(x)}{(x - x_0)^2} y = 0 \qquad \text{(II.12.15)}$$

tenglamani qaraylik; bu yerda $p_1(x), p_0(x)$ – x_0 nuqtada analitik funksiyalar. Bu holda x_0 nuqta (II.12.15) tenglama uchun **regulyar maxsus nuqta** deyiladi. Bundan keyin qisqalik uchun $x_0 = 0$ deb hisoblaymiz. Har doim $s = x - x_0$ almashtirish yordamida x_0 nuqtani 0 nuqtaga o'tkazish mumkin.

Teorema 3. *Agar $x_0 = 0$ nuqta (II.12.15) tenglama uchun regulyar maxsus nuqta bo'lsa, u holda (II.12.15) tenglama*

$$y = x^{\mu} \sum_{n=0}^{\infty} a_n x^n \quad (\mu, a_n\ (n = 0, 1, 2, \ldots) - \textit{o'zgarmas sonlar}) \qquad \text{(II.12.16)}$$

ko'rinishdagi kamida bitta yechimga ega; bu umumlashgan darajali qator biror $x \in (0, \rho)$ $(\rho > 0)$ intervalda yaqinlashuvchi bo'ladi.

Teoremada aytilgan $\mu, a_n\ (n = 0, 1, 2, \ldots)$ sonlarni topish uchun, Frobenius metodiga ko'ra, ushbu ($x > 0$)

$$p_0(x) = \sum_{n=0}^{+\infty} c_n x^n, \quad p_1(x) = \sum_{n=0}^{+\infty} b_n x^n;$$

$$y = x^{\mu} \sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} a_n x^{n+\mu}, \quad y' = \sum_{n=0}^{+\infty} (n + \mu) a_n x^{n+\mu-1},$$

$$y'' = \sum_{n=0}^{+\infty} (n + \mu)(n + \mu - 1) a_n x^{n+\mu-2};$$

$$p_0(x) y = \sum_{n=0}^{+\infty} c_n x^n \cdot \sum_{n=0}^{+\infty} a_n x^{n+\mu} = \sum_{n=0}^{+\infty} \sum_{k=1}^{n} a_k c_{n-k} x^{n+\mu};$$

$$x p_1(x) y' = \sum_{n=0}^{+\infty} b_n x^n \cdot \sum_{n=0}^{+\infty} (n + \mu) a_n x^{n+\mu} = \sum_{n=0}^{+\infty} \sum_{k=1}^{n} (k + \mu) a_k b_{n-k} x^{n+\mu};$$

$$x^2 y'' = \sum_{n=0}^{+\infty} (n + \mu)(n + \mu - 1) a_n x^{n+\mu};$$

yoyilmalarni (II.12.15) tenglamaga qo'yib, uni quyidagi ko'rinishga keltiramiz:

$$\sum_{n=0}^{+\infty}\Big((n+\mu)(n+\mu-1)a_n+\sum_{k=0}^{n}a_k\big((k+\mu)b_{n-k}+c_{n-k}\big)\Big)x^{n+\mu}=0$$

yoki

$$a_0\big(\mu(\mu-1)+\mu b_0+c_0\big)x^{\mu}+\sum_{n=1}^{\infty}\Big(\big(a_n(n+\mu)(n+\mu-1)+(n+\mu)b_0+c_0\big)+$$

$$+\sum_{k=0}^{n-1}a_k\big((k+\mu)b_{n-k}+c_{n-k}\big)\Big)x^{n+\mu}=0$$

yoki yana qisqaroq

$$a_0A(\mu)x^{\mu}+\sum_{n=1}^{+\infty}\Big(a_nA(n+\mu)+$$

$$+\sum_{k=0}^{n-1}a_k\big((k+\mu)b_{n-k}+c_{n-k}\big)\Big)x^{n+\mu}=0 \qquad \text{(II.12.17)}$$

bu yerda

$$A(\mu)=\mu(\mu-1)+\mu b_0+c_0\,. \qquad \text{(II.12.18)}$$

(II.12.17) tenglik ayniyat bo'lishi uchun x ning darajalari oldidagi koeffitsientlar nolga teng bo'lishi kerak:

$a_0A(\mu)=0$.

$$a_nA(n+\mu)+\sum_{k=0}^{n-1}a_k\big((k+\mu)b_{n-k}+c_{n-k}\big)=0 \quad (n\geq 1)\,. \qquad \text{(II.12.19)}$$

Bu yerdagi birinchi shartdan a_0 ni ixtiyoriy deb, μ ga nisbatan kvadrat tenglama hosil qilamiz:

$A(\mu)=0$ yoki (II.12.18) ga ko'ra

$$\mu(\mu-1)+\mu b_0+c_0=0 \qquad \text{(II.12.20)}$$

Bu kvadrat tenglama (II.12.15) differensial tenglamaning ***aniqlovchi tenglamasi*** deyiladi. Agar berilgan μ uchun $A(1+\mu)\neq 0,\ A(2+\mu)\neq 0,\dots,A(n+\mu)\neq 0,\dots$ bo'lsa, u holda (II.12.19) tenglamadan tayinlangan a_0 ga ko'ra rekurrent usulda barcha $a_1=a_1(\mu), a_2=a_2(\mu),\dots,a_n=a_n(\mu),\dots$ koeffitsientlarni bir qiymatli topamiz.

Aniqlovchi tenglamaning μ_1, μ_2 ildizlari haqiqiy bo'lsin. Aniqlik uchun $\mu_1 \geq \mu_2$ deylik. Demak, $A(\mu_1)=0$ va $\mu > \mu_1$ bo'lganda $A(\mu) \neq 0$. Shuning uchun (II.12.19) rekurrent munosabatdan $a_0 = 1$ deb, barcha $a_n = a_n(\mu_1)$ $(n \geq 1)$ koeffitsientlarni bir qiymatli aniqlaymiz. Shunday qilib, bu holda (II.12.15) tenglamaning bir dona

$$y_1(x) = x^{\mu_1}\sum_{n=0}^{+\infty} a_n(\mu_1)x^n = x^{\mu_1}\left(1+\sum_{n=1}^{+\infty} a_n(\mu_1)x^n\right) \quad \text{(II.12.21)}$$

yechimini hosil qilamiz. (II.12.15) tenglamaning umumiy yechimini topish uchun uning (II.12.21) yechimga chiziqli bog'liq bo'lmagan yana bir $y_2(x)$ yechimini topishimiz kerak. Bu $y_2(x)$ yechimni qurish usuli aniqlovchi tenglamaning μ_1, μ_2 ildizlariga bog'liq. Quyidagi hollar bo'lishi mumkin.

1^0. $\mu_1 - \mu_2$ ayirma butun son bo'lmasin. U holda, ravshanki, $A(1+\mu_2) \neq 0,\ A(2+\mu_2) \neq 0, \ldots, A(n+\mu_2) \neq 0, \ldots$ va $a_0 = 1$ deb, (II.12.19) rekurrent munosabatdan barcha $a_n = a_n(\mu_2)$ $(n \geq 1)$ koeffitsientlarni bir qiymatli aniqlaymiz. Demak, $y_2(x)$ yechim sifatida quyidagi funksiyani olish mumkin:

$$y_2(x) = x^{\mu_2}\sum_{n=0}^{+\infty} a_n(\mu_2)x^n = x^{\mu_2}\left(1+\sum_{n=1}^{+\infty} a_n(\mu_2)x^n\right). \quad \text{(II.12.22)}$$

2^0. $\mu_1 = \mu_2$ bo'lsin. U holda $A(\mu) = (\mu - \mu_1)^2$. μ ni o'zgaruvchi deb hisoblaymiz va ixtiyoriy a_0 ni tayinlab, (II.12.19) rekurrent munosabatdan barcha $a_n = a_n(\mu)$ $(n \geq 1)$ koeffitsientlarni bir qiymatli aniqlaymiz. Bu koeffitsientkarga ko'ra (II.12.16) formuladagi

$$y = y(x,\mu) = x^{\mu}\sum_{n=0}^{+\infty} a_n x^n = x^{\mu}\left(a_0+\sum_{n=1}^{+\infty} a_n(\mu)x^n\right) \quad \text{(II.12.23)}$$

funksiyani tuzaylik. Bu funksiyaning qurilishiga ko'ra

$$(x-x_0)^2 y'' + (x-x_0)p_1(x)y' + p_2(x)y =$$

$$= a_0 A(\mu)x^{\mu} + \sum_{n=1}^{+\infty}\Big(a_n A(n+\mu) + \sum_{k=0}^{n-1} a_k \big((k+\mu)b_{n-k} + c_{n-k}\big)\Big)x^{n+\mu} =$$

$$= a_0 A(\mu)x^{\mu}.$$

Demak,

$$(x-x_0)^2 y'' + (x-x_0)p_1(x)y' + p_0(x)y = a_0(\mu-\mu_1)^2 x^{\mu}.$$

Oxirgi tenglikni μ bo'yicha differensiallab,

$$(x-x_0)^2\left(\frac{\partial y}{\partial \mu}\right)'' + (x-x_0)p_1(x)\left(\frac{\partial y}{\partial \mu}\right)' + p_0(x)\frac{\partial y}{\partial \mu} =$$

$$= 2a_0(\mu-\mu_1)x^{\mu} + a_0(\mu-\mu_1)^2 x^{\mu}\ln x$$

ekanligini topamiz (bizda $x>0$). Bu tenglikda $\mu=\mu_1$ deb, (II.12.23) formula bilan aniqlangan funksiyaning ushbu

$$y_2(x) = \left.\frac{\partial y(x,\mu)}{\partial \mu}\right|_{\mu=\mu_1}$$

hosilasi (II.12.15) tenglamaning yechimi bo'lishini topamiz. (II.12.23) formulaga ko'ra bu ikkinchi chiziqli erkli yechim

$$y_2(x) = y_1(x)\ln x + x^{\mu_1}\sum_{n=0}^{+\infty} a_n'(\mu_1)x^n \qquad \text{(II.12.24)}$$

ko'rinishga ega bo'ladi. Bu yerdagi $a_n'(\mu_1)$ larni noma'lum koeffitsientlar metodi yordamida, ya'ni (II.12.24) ni (II.12.15) tenglamaga qo'yib, tenglamaning qanoatlanishi shartidan aniqlash mumkin.

3^0. $\mu_1-\mu_2$ ayirma natural sondan iborat bo'lsin. Bu holda ikkinchi $y_2(x)$ yechimni qurish usulini keltirib chiqarish murakkab. Shuning uchun birdaniga natijani keltiramiz. Bu holda $y_2(x)$ yechim

$$y_2(x) = \bar{a}_{-1}y_1(x)\ln x + x^{\mu_2}\sum_{n=0}^{+\infty}\bar{a}_n(\mu_2)x^n \qquad \text{(II.12.25)}$$

ko'rinishda bo'ladi. Bu yechimni qurish uchun dastlab Frobenius metodidan foydalanib, μ_2 ga ko'ra (II.12.15) tenglamaning yechimini qurish kerak. Agar qurilgan $y_2(x)$ yechim $y_1(x)$ (II.12.18) yechimga chiziqli bog'liq bo'lmasa, u (II.12.21) ko'rinishda bo'ladi (bunda $\bar{a}_{-1} = 0$). Aks holda, ya'ni izlangan yechim hosil bo'lmasa, 2^0. banddagidek ish tutib,

$$y_2(x) = \frac{\partial}{\partial\mu}\left((\mu - \mu_2)y(x,\mu)\right)\Big|_{\mu=\mu_2}$$

izlangan yechimni topish kerak. Tushunarliki, (II.12.25) yechimni noma'lum koeffitsientlar metodidan foydalanib ham topish mumkin.

Agar aniqlovchi tenglamaning μ_1, μ_2 ildizlari kompleks sonlardan iborat bo'lsa, u holda $\mu_2 = \bar{\mu}_1$ bo'ladi (tenglamadagi koeffitsienlarning haqiqiyligi uchun) va $e^{\alpha \pm i\beta} = e^{\alpha}(\cos\beta \pm i\sin\beta)$ Eyler formulalaridan foydalanib chiziqli erkli haqiqiy $y_1(x), y_2(x)$ yechimlarni qurish mumkin.

Misol 3. Ushbu

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad (\nu = \text{const} \geq 0) \tag{II.12.26}$$

Bessel tenglamasining chiziqli erkli yechimlarini quring.

Ravshanki, Bessel tenglamasi uchun $x = 0$ nuqta regulyar maxsus nuqtadir. Frobenius metodidan kelib chiqib, yechimni

$$y = x^{\mu}\sum_{n=0}^{\infty} a_n x^n = a_0x^{\mu} + a_1x^{1+\mu} + a_2x^{2+\mu} + \ldots + a_nx^{n+\mu} + \ldots \tag{II.12.27}$$

ko'rinishda izlaymiz; bu yerda μ, a_n $(n = 0,1,2,\ldots)$ – hozircha noma'lum sonlar. Bu funksiyani va uning birinchi va ikkinchi hosilalarni (II.12.22) tenglamaga qo'yamiz:

$$x^2(\mu(\mu-1)a_0x^{\mu-2} + (1+\mu)\mu a_1x^{\mu-1} + \ldots + (n+\mu)(n+\mu-1)a_nx^{n+\mu-2} + \ldots) +$$

$$+ x(\mu a_0x^{\mu-1} + (1+\mu)a_1x^{\mu} + (2+\mu)a_2x^{\mu+1} \ldots + (n+\mu)a_nx^{n+\mu-1} + \ldots) +$$

$$+(x^2-\nu^2)(a_0x^\mu+a_1x^{1+\mu}+a_2x^{2+\mu}+\dots+a_nx^{n+\mu}+\dots)=0$$

Qavslarni ochib chiqib, o'xshash hadlarni ixchamlaymiz:

$$a_0(\mu^2-\nu^2)x^\mu+a_1\left((\mu+1)^2-\nu^2\right)x^{\mu+1}+$$

$$+\dots+\left(a_n\left((\mu+n)^2-\nu^2\right)+a_{n-2}\right)x^{\mu+n}+\dots=0.$$

x ning darajalari oldidagi koeffitsientlarni nolga tenglashtirib, noma'lumlarni topish uchun quyidagi sistemani hosil qilamiz:

$$\begin{cases} a_0(\mu^2-\nu^2)=0, \\ a_1\left((\mu+1)^2-\nu^2\right)=0, \\ \dots\dots\dots\dots\dots\dots \\ a_n\left((\mu+n)^2-\nu^2\right)+a_{n-2}=0 \ \ (n\geq 2), \\ \dots\dots\dots\dots\dots\dots \end{cases} \qquad \text{(II.12.28)}$$

a_0 ni ixtiyoriy deb, ushbu $\mu^2-\nu^2=0$ aniqlovchi tenglamani topamiz. Uning yechimlari $\mu_1=\nu, \mu_2=-\nu$ (ν musbat).

Dastlab $\mu=\nu$ holini qaraymiz. (II.12.28) ning ikkinchi tenglamasida $\mu=\nu$ deb, $a_1=0$ ekanligin topamiz. (II.12.28) sistemadan ushbu

$a_n\left((\mu+n)^2-\nu^2\right)+a_{n-2}=0$, ya'ni

$$a_n=-\frac{1}{(\mu+n+\nu)(\mu+n-\nu)}a_{n-2} \quad (n\geq 2) \qquad \text{(II.12.29)}$$

munosabatni ham topamiz. Bu rekurrent formulani ketma-ket qo'llab, barcha toq indeksli koeffitsientlarning nolga teng ekanligini hosil qilamiz: $0=a_1=a_3=a_5=\dots$. Juft indeksli koeffitsientlar uchun esa

$$a_{2n}=(-1)^n\frac{a_0}{2^{2n}n!(\nu+1)(\nu+2)\dots(\nu+n)} \quad (n\geq 1)$$

formulalarni topamiz. Endi (II.12.27) formulaga topilgan qiymatlarni qo'yib, Bessel tenglamasining

$$y = a_0 x^{\nu}\left(1 + \sum_{n=1}^{+\infty}(-1)^n \frac{x^{2n}}{2^{2n} n!(\nu+1)(\nu+2)\ldots(\nu+n)}\right)$$

yechimini hosil qilamiz. Bu yerdagi ixtiyoriy o'zgarmasni $a_0 = \dfrac{1}{2^{\nu}\Gamma(\nu+1)}$ deb tanlanadi va bunda hosil bo'lgan $y = J_{\nu}(x)$ yechim ν- tartibli birinchi tur Bessel funksiyasi deb ataladi. Kerakli shakl almashtirishlarni bajarib, $J_{\nu}(x)$ Bessel funksiyasini quyidagi ko'rinishda ifodalaymiz:

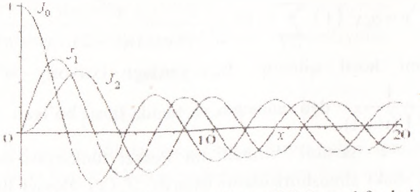
$$J_{\nu}(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu} \qquad \text{(II.12.30)}$$

Osongina ko'rsatish mumkinki, $x^{-\nu} J_{\nu}(x)$ uchun qator ixtiyoriy $x \in [a,b]$ segmentda absolyut va tekis yaqinlashuvchi.

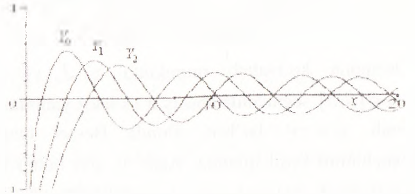
Endi $\mu = -\nu$ bo'lsin. Bunda Bessel tenglamasining $J_{-\nu}(x)$ yechimini hosil qilamiz. Agar ν son butun bo'lmasa, u holda $J_{\nu}(x)$ va $J_{-\nu}(x)$ yechimlar chiziqli erkli bo'ladi. Lekin, agar ν butun son bo'lsa, ular chiziqli bog'langan bo'ladi ($J_{-\nu}(x) = (-1)^{\nu} J_{\nu}(x)$) va bu holda ushbu

$$Y_{\nu}(x) = \frac{\cos\nu\pi \cdot J_{\nu}(x) - J_{-\nu}(x)}{\sin\nu\pi} \qquad \text{(II.12.31)}$$

funksiya kiritiladi ($\nu = k$ butun son bo'lganda (II.12.31) formulani $Y_k(x) = \lim\limits_{\nu \to k} Y_{\nu}(x)$ deb tushunish kerak). Bunda har doim $J_{\nu}(x)$ va $Y_{\nu}(x)$ – chiziqli erkli yechimlar bo'ladi (II.3- va II.4- rasmlar). Bu yerdagi $Y_{\nu}(x)$ funksiya ν- tartibli ikkinchi tur Bessel funksiyasi (yoki Neyman funksiyasi) deb ataladi. ♣



II.3- rasm. Birinchi tur 0-, 1- va 2- tartibli Bessel funksiyalari.



II.4- rasm. Ikkinchi tur 0-, 1- va 2- tartibli Bessel funksiyalari.

Bessel funksiyasi uchun (II.12.30) formuladan quyidagi ikki formula osongina kelib chiqadi:

$$\begin{cases} xJ_\nu'(x)+\nu J_\nu(x)=xJ_{\nu-1}(x), \\ xJ_\nu'(x)-\nu J_\nu(x)=-xJ_{\nu+1}(x). \end{cases} \quad \text{(II.12.32)}$$

Ba'zi tenglamalarning yechimini maxsus almashtirishlar yordamida Bessel funksiyalari orqali ifodalash mumkin.

Masalan, ushbu

$$y'=x+y^2$$

maxsus Rikkati tenglamasini qaraylik. $y=y(x)$ noma'lum funksiya o'rniga yangi $u=u(x)$ noma'lum funksyani

$$y=-\frac{1}{u}\cdot\frac{du}{dx} \quad \text{(II.12.33)}$$

formula bilan kiritib, u funksiya uchun

$$u''+xu=0 \quad \text{(II.12.34)}$$

chiziqli tenglamani hosil qilamiz. Bu tenglamani

$$w = \frac{u}{\sqrt{x}}, \quad z = \frac{2}{3}x^{3/2}, \; w = w(z) \quad (x > 0)$$

almashtirishlar yordamida ushbu

$$z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} + \left[z^2 - \left(\frac{1}{3}\right)^2\right]w = 0$$

Bessel tenglamasiga keltiramiz. $-1/3$ va $1/3$ tartibli birinchi tur $J_{-1/3}(z)$ va $J_{1/3}(z)$ Bessel funksiyalari oxirgi tenglamaning chiziqli erkli yechimlari. Demak, (II.12.34) tenglamaning umumiy yechimi

$$u = c_1\sqrt{x}J_{-1/3}\left(\frac{2}{3}x^{3/2}\right) + c_2\sqrt{x}J_{1/3}\left(\frac{2}{3}x^{3/2}\right) \quad (x > 0)$$

ko'rinishda bo'ladi. Endi (II.12.33) almashtirish va (II.12.32) formulalarga ko'ra berilgan maxsus Rikkati tenglamasining Bessel funksiyalari orqali ifodalangan ushbu

$$y = -\sqrt{x}\,\frac{J_{-2/3}\left(\frac{2}{3}x^{3/2}\right) - cJ_{2/3}\left(\frac{2}{3}x^{3/2}\right)}{cJ_{-1/3}\left(\frac{2}{3}x^{3/2}\right) + J_{1/3}\left(\frac{2}{3}x^{3/2}\right)} \quad (x > 0)$$

yechimini topamiz.

Masalalar

1. Ushbu

$$y'' - xy = 0$$

Eyri[1] tenglamasining (analitik) yechimlari

$$y = a_0\left(1 + \sum_{n=1}^{\infty}\frac{x^{3n}}{2\cdot 3\cdot\ldots\cdot(3n-1)\cdot(3n)}\right) + a_1\left(x + \sum_{n=1}^{\infty}\frac{x^{3n+1}}{3\cdot 4\cdot\ldots\cdot(3n)\cdot(3n+1)}\right)$$

(a_0, a_1 – ixtiyoriy o'zgarmaslar) ko'rinishda va bu yerdagi qatorlarning yaqinlashish radiuslari $R = +\infty$ ekanligini isbotlang.

2. Ermit tenglamasi deb ataluvchi

$$x'' - 2tx' + 2\lambda x = 0 \quad (\lambda \text{ – parametr })$$

tenglamani qaraylik (bu tenglama matematikaning ba'zi sohalarida va kvant mexanikasida uchraydi).

1). Quyidagi belgilashlarni kiritaylik:

$$x_1(t) = 1 - \lambda t^2 + \frac{2^2\lambda(\lambda-2)}{4!}t^4 - \frac{2^3\lambda(\lambda-2)(\lambda-4)}{6!}t^6 + \ldots,$$

$$x_2(t) = t - \frac{2(\lambda-1)}{3!}t^3 + \frac{2^2(\lambda-1)(\lambda-3)}{5!}t^5 - \frac{2^3(\lambda-1)(\lambda-3)(\lambda-5)}{7!}t^7 + \ldots$$

Bu funksiyalar $-\infty < t < +\infty$ oraliqda aniqlangan, chiziqli erkli va ixtiyoriy a_0, a_1 o'zgarmas sonlar uchun

$$x = a_0 x_1(t) + a_1 x_2(t)$$

funksiya Ermit tenglamasining yechimi bo'lishini isbotlang.

2). Ermit tenglamasining $n-$ darajali ko'phaddan iborat bo'lgan yechimi, agar t^n oldidagi koeffitsient 2^n ga teng bo'lsa, ($n-$ darajali) Ermit ko'phadi deb ataladi va $H_n(t)$ bilan belgilanadi. $H_0(t) = 1$, $H_1(t) = 2t$, $H_2(t) = 4t^2 - 2$, $H_3(t) = 8t^3 - 12$ formulalarni isbotlang.

3. Bessel tenglamasiga oid misolda bir nechta tasdiq isbotsiz keltirildi. Shu tasdiqlarni to'liq isbotlang. Agar k butun son bo'lsa, $J_{\pm\frac{k}{2}}(x)$ Bessel funksiyalarining elementar funksiyalar orqali ifodalanishini ko'rsating.

4. Ushbu

$$(1-x^2)y'' - 2xy' + \lambda(\lambda+1)y = 0$$

tenglama Lejandr tenglamasi deb ataladi (λ – berilgan son, parametr).

1). Lejandr tenglamasining umumiy yechimi quyidagi ko'rinishda bo'lishini isbotlang:

$$y = a_0 y_1(x) + a_1 y_2(x) \quad (a_0, a_1 - \text{ixtiyoriy}$$

o'zgarmaslar), bu yerda

$$y_1(x) = 1 - \frac{\lambda(\lambda+1)}{2!}x^2 + \frac{(\lambda-2)\lambda(\lambda+1)(\lambda+3)}{4!}x^4 - \frac{(\lambda-4)(\lambda-2)\lambda(\lambda+1)(\lambda+3)(\lambda+5)}{6!}x^6 + \ldots,$$

$$y_2(x) = x - \frac{(\lambda-1)(\lambda+2)}{3!}x^3 + \frac{(\lambda-3)(\lambda-1)(\lambda+2)(\lambda+4)}{5!}x^5$$

$$\frac{(\lambda-5)(\lambda-3)(\lambda-1)(\lambda+2)(\lambda+4)(\lambda+6)}{7!}x^7+\ldots.$$

2). Bu qatorlarning yaqinlashish radiuslari $R\geq 1$ ekanligini ko'rsating. Ularni $x=\pm 1$ nuqtalarda yaqinlashishga tekshiring. λ butun son bo'lgan taqdirdagina $y_1(x)$ va $y_2(x)$ yechimlar $[-1,1]$ segmentda aniqlangan bo'lishini ko'rsating.

3). Agar $\lambda-$ nomanfiy juft son bo'lsa, u holda $y_1(x)$ ko'phadga aylanadi; bu holda

$$P_\lambda(x)=\frac{y_1(x)}{y_1(1)}$$

ko'phadni kiritamiz. Agar $\lambda-$ musbat toq son bo'lsa, u holda $y_2(x)$ ko'phaddan iborat bo'ladi; bu holda

$$P_\lambda(x)=\frac{y_2(x)}{y_2(1)}$$

deymiz. Kiritilgan $P_\lambda(x)$ $(\lambda=0,1,2,\ldots)$ ko'phadlar Lejandr ko'phadlari deb ataladi. $P_0(x),P_1(x),P_2(x),P_3(x)$ va $P_4(x)$ ko'phadlarning standart ko'rinishini toping. Ushbu

$$P_n(x)=\frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n(x^2-1)^n \quad (n=0,1,2,\ldots)$$

formulalarni isbotlang.

5. Ushbu

$$(1-x^2)y''-xy'+\lambda^2y=0$$

differensial tenglama Chebishyov tenglamasi deb ataladi ($\lambda-$ berilgan son). Bu tenglamaning $x=0$ da analitik yechimlarini toping. λ ning qanday qiymatlarida yechimlar ko'phadlardan iborat bo'ladi? Bu ko'phadlar Chebishyov ko'phadlari deb ataladi.

6. Ushbu

$$x(1-x)y''+[\gamma-(1+\alpha+\beta)xy]y'-\alpha\beta y=0$$

differensial tenglama gipergeometrik tenglama yoki Gauss tenglamasi deb ataladi; bunda $\alpha,\beta,\gamma-$ berilgan sonlar. Quyidagi tasdiqlarni isbotlang:

1). Bu tenglama uchun $x=0-$ regulyar maxsus nuqta.

2). Agar γ soni $0,-1,-2,-3,\ldots$ sonlardan farqli bo'lsa, ushbu

$$F(\alpha,\beta,\gamma;x)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k k!}x^k,\ |x|<1, \qquad (G)$$

funksiya gipergeometrik tenglamaning yechimi bo'ladi; bunda

$$(a)_0=1;\ (a)_n=a(a+1)(a+2)...(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)},\ n\in\mathbb{N}.$$

3). Agar $\gamma=-n,\ n=0,1,2,....,$ va $\alpha\neq-p,\beta\neq-p,p\in\mathbb{N},p<n,$ bo'lsa. (G) qator aniqlanmagan, lekin

$$\lim_{\gamma\to-n}\frac{F(\alpha,\beta,\gamma;x)}{\Gamma(\gamma)}=\frac{(\alpha)_{n+1}(\beta)_{n+1}}{(n+1)!}x^{n+1}F(\alpha+n+1,\beta+n+1,n+2;x)$$

limit mavjud va u Gauss tenglamasining yechimi.

4). Agar $\alpha=-n$ yoki $\beta=-n$ ($n=0,1,2,....$) va $\gamma=-p$, bunda $p=n,\ n+1,\ n+2,...,$ bo'lsa,

$$F(-n,\beta,-p;x)=\sum_{k=0}^{n}\frac{(-n)_k(\beta)_k}{(-p)_k k!}x^k \quad \text{yoki}$$

$$F(\alpha,-n,-p;x)=\sum_{k=0}^{n}\frac{(\alpha)_k(-n)_k}{(-p)_k k!}x^k$$

ko'phad gipergeometrik tenglama yechimi bo'ladi.

Aniqlangan $F(\alpha,\beta,\gamma;x)$ funksiya gipergeometrik funksiya yoki Gauss funksiyasi deb ataladi.

5). Gauss tenglamasining ikkinchi yechimni topish uchun unda $y=x^{1-\gamma}u$ almashtirishni bajaring va hosil bo'lgan tenglamaning kerakli yechimidan foydalaning.

6). Gauss tenglamasi uchun $x=1$ nuqta ham regulyar maxsus nuqta. Bu nuqta atrofida yechimni topish uchun tenglamada $x=1-t$ almashtirishni bajaring. Hosil bo'lgan gipergeometrik tenglamaning kerakli yechimlarini quring.

7. Ushbu

$$\begin{cases} x^2y'-y+x=0 \\ y(0)=0 \end{cases}$$

boshlang'ich masalani yeching. U analitik yechimga egami?

8. Ushbu

$$x^3 y' - 2y = 0$$

differensial tenglamani yeching. Har qanday yechimning barcha hosilalari nol nuqtada nolga teng ekanligini ko`rsating. Tenglamaning nol nuqtada analitik bo`lgan yechimlari nechta?

II.13. Ikkinchi tartibli chiziqli differensial tenglama yechimlarining nollari

Ushbu

$$y'' + a_1(x)y' + a_0(x)y = 0 \qquad \text{(II.13.1)}$$

ikkinchi tartibli chiziqli bir jinsli tenglama berilgan bo'lsin. Biz bu paragrafda uning $y = y(x)$ yechimi ishoralarining o`zgarishini o'rganamiz.

Dastlab (II.13.1) tenglama ko'rinishini soddalashtirish bilan shug'ullanamiz. Aniqrog'i, tenglamani noma'lum funksiya hosilasi qatnashmagan ko'rinishga keltiramiz. Faraz qilaylik, $a_0(x)$, $a_1(x)$ koeffitsiyentlar va $a_1'(x)$ hosila I (a,b) intervalda uzluksiz bo'lsin.

Noma'lum funksiya $y = y(x)$ o'rniga yangi noma'lum funksiya $z = z(x)$ ni $y = uz$ formula bilan kiritaylik. $u = u(x)$ funksiyani tanlash evaziga hosil bo'ladigan differensial tenglamada noma'lum funksiyaning birinchi tartibli hosilasini yo`qotamiz. Buning uchun

$$y = uz, \quad y' = u'z + uz', \quad y'' = u''z + 2u'z' + uz''$$

ifodalarni (II.13.1) tenglamaga qo`yamiz:

$$uz'' + (2u' + a_1(x)u)z' + (u'' + a_1(x)u' + a_0(x)u)z = 0 \qquad \text{(II.13.2)}$$

va bu tenglamadagi z' oldidagi koeffitsiyentni 0 ga tenglashtiramiz:

$$2u' + a_1(x)u = 0.$$

Oxirgi (o'zgaruvchilari ajraladigan) tenglamani yechib, uning ushbu

$$u = \exp\left[-\frac{1}{2}\int_{x_0}^{x} a_1(s)ds\right] \quad (x_0 \in (a;b) - \text{tayinlangan}) \qquad \text{(II.13.3)}$$

yechimini tanlaymiz.

Bu funksiya uchun

$$q(x) \stackrel{def}{=} \frac{1}{u(x)}\left[u''(x) + a_1(x)u'(x) + a_0(x)u(x)\right] =$$

$$= a_0(x) - \frac{a_1^2(x)}{4} - \frac{a_1'(x)}{2} \qquad (II.13.4)$$

munosabatni topib, uni (II.13.2) ga qo'yamiz. Natijada

$$z'' + q(x)z = 0 \qquad (II.13.5)$$

tenglamani hosil qilamiz. Bu yerda $q(x)$ funksiya (II.13.4) formula bilan aniqlanadi. (II.13.5) ning $z = z(x)$ yechimiga ko'ra (II.13.1) ning

$$y = z(x)\exp\left[-\frac{1}{2}\int_{x_0}^{x} a_1(s)ds\right]$$

yechimni topamiz. Demak, quyidagi jumla isbotlandi.

Jumla. *Agar (II.13.1) tenglamada* $\{a_0(x), a_1(x), a_1'(x)\} \subset C((a,b))$ *bo'lsa, u holda* $y = y(x)$ *noma'lum funksiyani* $y = z(x)\exp\left[-\frac{1}{2}\int_{x_0}^{x} a_1(x)ds\right]$ *almashtirish yordamida (II.13.1) tenglamani (II.13.5) «sodda» ko'rinishga keltirish mumkin.*

Agar $a_1'(x)$ ning uzluksizligini talab qilmasak, u holda boshqa. «murakkabroq» almashtirish yordamida (II.13.1) ni (II.13.5) ko'rinishga keltirish mumkin.

Dastlab (II.13.1) tenglamani ushbu

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = 0$$

(II.13.6)

o'z-o'ziga qo'shma differensial tenglama deb ataluvchi tenglama ko'rinishiga keltiraylik. Buning uchun (II.13.1) ning har ikkala tomonini biror silliq $\mu(x)$ funksiyaga ko'paytiramiz va hosil bo'lgan tenglamaning o'z-o'ziga qo'shma bo'lishi shartidan kerakli

$\mu(x)$ ni topamiz:

$$\mu(x)y'' + \mu(x)a_1(x)y' + \mu(x)a_0(x)y = 0.$$

Bu tenglama o'z-o'ziga qo'shma bo'lishi uchun

$$\mu'(x) = \mu(x)a_1(x),$$

ya'ni

$$\mu(x) = \exp\left[\int_{x_0}^{x} a_1(s)ds\right]$$

bo'lishi kifoya. $\mu(x)$ uchun bu ifodani tegishli tenglamaga qo'yib, ushbu

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = 0 \qquad \text{(II.13.6)}$$

o'z-o'ziga qo'shma tenglamani hosil qilamiz; bu yerda

$$p(x) = \exp\left[\int_{x_0}^{x} a_1(s)ds\right] > 0, \quad q(x) = a_0(x)\exp\left[\int_{x_0}^{x} a_1(s)ds\right] \qquad \text{(II.13.7)}$$

Shunday qilib, agar $a_1(x)$, $a_0(x)$ koeffitsiyentlar (a,b) intervalda uzluksiz bo'lsa, u holda (II.13.1) tenglamani (II.13.6) ko'rinishga keltirish mumkin; bu yerda $p \in C^1((a,b))$, $q \in C((a,b))$

Endi (II.13.6) tenglamada x argument o'rniga $\xi = \xi(x)$ o'zgaruvchini ushbu

$$\frac{d\xi}{dx} = \frac{1}{p(x)}$$

tenglamaning yechimi kabi kiritamiz. U holda (II.13.6) tenglama

$$\frac{1}{p(x)}\frac{d}{d\xi}\left[\frac{dy}{d\xi}\right] + q(x)y = 0$$

yoki

$$\frac{d^2y}{d\xi^2} + Q(\xi)y = 0 \qquad \text{(II.13.8)}$$

ko'rinishni oladi; bu yerda $Q(\xi) = p(x)q(x)\big|_{x=x(\xi)}$ ya'ni

$$Q(\xi) = a_0(x)\exp\left[2\int_{\xi}^{x} a_1(s)ds\right]\Bigg|_{x=x(\xi)}$$

(bu yerda $x = x(\xi)$ funksiya $\xi = \xi(x)$ ga teskari funksiyani belgilaydi: u mavjud chunki $\xi'(x) = \dfrac{1}{p(x)} > 0$.)

Endi ikkinchi tartibli chiziqli tenglamaning nollarini o'rganamiz.

Odatdagidek, agar $y(x_0) = 0$ bo'lsa, x_0 ni $y = y(x)$ funksiyaning noli deb ataymiz.

Bundan buyon «soddalashtirilgan»

$$y'' + q(x)y = 0 \tag{II.13.9}$$

tenglamani qaraymiz: bu yerda $q(x) \in C((a,b))$.

Teorema 1. *(II.13.9) tenglamaning har qanday notrivial yechimi ixtiyoriy $[a_1,b_1] \subset (a,b)$ segmentda chekli sondagi nollarga ega xolos.*

⊶ Teskarisini faraz qilaylik. U holda biror $y = y(x)$ notrivial yechim biror $[a_1,b_1] \subset (a,b)$ segmentda cheksiz ko'p nollarga ega bo'ladi. Bu nollardan $x_1, x_2, \ldots, x_n, \ldots$ larini ajrataylik: $y(x_n) = 0$, $x_n \in [a_1,b_1]$, $n \in N$.

Chegaralangan $\{x_n\}$ ketma-ketlikdan Veyyershtrass teoremasiga ko'ra yaqinlashuvchi qismiy ketma-ketlik $\{x_{n_k}\}$ ni ajratamiz; $x_{n_k} \to x_0 \in [a_1,b_1]$ bo'ladi. $y(x)$ yechimning uzluksizligiga ko'ra $y(x_0) = 0$. Hosila ta'rifidan

$$y'(x_0) = \lim_{x\to x_0}\frac{y(x) - y(x_0)}{x - x_0} = \lim_{k\to\infty}\frac{y(x_{n_k}) - y(x_0)}{x_{n_k} - x_0} = 0$$

Demak, $x_0 \in [a_1,b_1]$ nuqtada $y(x_0) = 0$, $y'(x_0) = 0$

Yagonalik xossasiga ko'ra $y(x)\equiv 0$. Bu berilganga zid. Farazimiz noto'g'ri va teorema isbot bo'ldi.

Natija. *Faraz qilaylik, $y(x)$ funksiya (II.13.9) tenglamaning notrivial yechimi bo'lsin. Agar $y(x)$ yechim $(a;b)$ intervalda kamida ikkita nolga ega va ularning biri x_0 bo'lsa ($y(x_0)=0$), u holda $y(x)$ yechimning shunday $x_1 \neq x_0$ noli mavjudki, (x_0,x_1) $(x_0<x_1)$ yoki (x_1,x_0) $(x_1<x_0)$ intervalda $y(x)$ ning noli bo'lmaydi.*

Bu x_0 va x_1 nollar $y(x)$ yechimning **qo'shni (ketma-ket kelgan) nollari** deb ataladi.

Quyidagi tenglamalarni qaraylik

$$y''+q_1(x)y=0, \qquad (II.13.10)$$

$$u''+q_2(x)u=0; \qquad (II.13.11)$$

bu yerda $\{q_1(x),q_2(x)\}\subset C((a;b))$

Teorema 2 (Shturmning taqqoslash teoremasi). *Faraz qilaylik, (a,b) intervalda $q_1(x)\le q_2(x)$, x_0 va x_1 lar esa (II.13.10) tenglamaning notrivial yechimi bo'lmish $y=y(x)$ funksiyaning ketma-ket kelgan ikkita noli bo'lsin ($x_0<x_1$). U holda yo (II.13.11) tenglamaning har qanday $u=u(x)$ notrivial yechimi (x_0,x_1) intervalda kamida bitta nolga ega, yo $u(x_0)=u(x_1)=0$ va (x_0,x_1) intervalda $q_1(x)=q_2(x)$.*

Faraz qilaylik, (II.13.11) tenglamaning notrivial $u(x)$ yechimi (x_0,x_1) intervalda nolga ega bo'lmasin. Demak, $u(x)$ uzluksiz funksiya bu intervalda o'z ishorasini saqlaydi. Umumiylikni buzmasdan (x_0,x_1) intervalda $u(x)>0$ deb hisoblaymiz (aks holda $-u(x)$ yechimni qaraymiz).

Teorema shartiga ko'ra $y(x_0)=y(x_1)=0$ va $\forall x\in(x_0,x_1)$ uchun $y(x)\neq 0$. Umumiylikni buzmasdan (x_0,x_1)

Natija. *Agar* (a,b) *intervalda* $q(x)\le 0$ *bo'lsa, u holda (II.13.9) tenglamaning har qanday notrivial* $y=y(x)$ *yechimi* (a,b) *intervalda ko'pi bilan bitta nolga ega bo'ladi.*

Agar $y(x)$ ning (a,b) intervalda kamida ikkita noli bo'lganda edi, u holda

$$y''+q(x)y=0,\ (q(x)\le 0)$$
$$u''+0\cdot u=0$$

tenglamalarga Shturm teoremasini qo'llab $u=1$ yechimning ham noli bo'lishi kerakligini hosil qilgan bo'lar edik.

Teorema 4 (Knezer). *Agar (II.13.9) tenglamada* $q(x)$ *funksiya uchun* $(x_0;+\infty)$ $(x_0>0)$ *intervalda* $0<q(x)\le\frac{1}{4x^2}$ *tengsizlik o'rinli bo'lsa, u holda (II.13.9) tenglamaning ixtiyoriy notrivial yechimi* $(x_0;+\infty)$ *intervalda ko'pi bilan bitta nolga ega bo'ladi. Agar* $(x_0;+\infty)$ *da* $q(x)\ge\frac{1+\varepsilon}{4x^2}$ $(0<\varepsilon-const)$ *tengsizlik o'rinli bo'lsa, u holda (II.13.9) tenglamaning ixtiyoriy notrivial yechimi* $(x_0;+\infty)$ *intervalda cheksiz ko'p nollarga ega bo'ladi.*

Ushbu

$$y''+\frac{1}{4x^2}y=0 \qquad \text{(II.13.15)}$$

tenglamani qaraylik. Bu tenglama $x>0$ da $y=\sqrt{x}$ yechimga ega. Agar $(x_0;+\infty)$ $(x_0>0)$ intervalda $q(x)\le\frac{1}{4x^2}$ bo'lsa, u holda Shturm teoremasiga ko'ra (II.13.14) tenglama ixtiyoriy yechimining qo'shni ikkita noli orasida (II.13.15) tenglama har qanday notrivial yechimining kamida bitta noli yotadi. Agar (II.13.14) tenglamaning biror yechimi $(x_0;+\infty)$ intervalda ikkita nolga ega bo'lganda edi, u holda (II.13.15) tenglamaning $y=\sqrt{x}$ yechimi bu nollar orasida kamida bir marta nolga aylangan bo'lar edi, lekin $y=\sqrt{x}$ funksiya $(x_0;+\infty)$ $(x_0>0)$ oraliqda nolga teng bo'la olmaydi.

Endi $(x_0;+\infty)$ intervalda $\frac{1+\varepsilon}{4x^2}\le q(x)$, $x>x_0$, $\varepsilon>0$, bo'lgan holni qaraylik. Bu holda taqqoslash uchun ushbu

$$y''+\frac{1+\varepsilon}{4x^2}y=0 \qquad \text{(II.13.16)}$$

tenglamani qaraymiz. Osongina tekshirib ko'rish mumkinki, bu tenglama $x>x_0$ oraliqda $y=\sqrt{x}\cdot\cos\left(\frac{\sqrt{\varepsilon}}{2}\ln x\right)$ yechimga ega (bu tenglamaning umumiy yechimini $t=\ln x$ almashtirish yordamida topish ham mumkin). Ravshanki, bu yechim cheksiz ko'p nollarga ega. Shturm teoremasiga ko'ra (II.13.16) tenglamaning $y=\sqrt{x}\cdot\cos\left(\frac{\sqrt{\varepsilon}}{2}\ln x\right)$ yechimi nollari orasida (II.13.14) tenglamaning har qanday yechimining kamida bitta noli mavjud. Demak, (II.13.16) tenglamaning barcha yechimlari $x>x_0$ oraliqda cheksiz ko'p nollarga ega.

Masalalar

1. Aytaylik, $q(x)\in C\left((a;b)\right)$ va biror $\omega>0$ son uchun

$$q(x)\le\omega^2 \ (x\in(a,b)) \text{ yoki } q(x)\ge\omega^2 \ (x\in(a,b))$$

tengsizlik o'rinli bo'lsin. U holda $y''+q(x)y=0$ tenglamaning har qanday notrivial yechimi qo'shni $x_0<x_1$ nollari uchun ushbu

$$x_1-x_0\ge\frac{\pi}{\omega} \text{ yoki mos ravishda } x_1-x_0\le\frac{\pi}{\omega}$$

baholash o'rinli bo'lishini isbotlang..

2. Ushbu

$$y''+\frac{1}{x}y'+\left(1-\frac{\nu^2}{x^2}\right)y=0,\ x>0\ (\nu=\text{const}\ge 0).$$

Bessel tenglamasi har qanday notrivial yechimining qo'shni nollari orasidagi masofa $0 \le \nu < 1/2$ bo`lganda $\pi$ dan kichik, $\nu = 1/2$ holida $\pi$ ga teng, $\nu > 1/2$ bo`lganda esa π dan katta. Shuni isbotlang.

3. Ushbu

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0, \ x > 0 \ (\nu = \text{const} \ge 0),$$

Bessel tenglamasi ixtiyoriy notrivial yechimining x_n ketma-ket nollari orasidagi masofa uchun $\lim\limits_{n\to\infty}(x_n - x_{n-1}) = \pi$ bo`lishini isbotlang.

4. Ushbu $y'' + xy = 0$ tenglama ixtiyoriy yechimining x_n ketma-ket nollari orasidagi masofa uchun $\lim\limits_{n\to\infty}(x_n - x_{n-1}) = 0$ bo`lishini ko`rsating.

II.14. Ekstremum prinsiplari

Ushbu

$$M[y] \overset{\text{def}}{=} y'' + p(x)y' = f(x), \ a < x < b, \qquad \text{(II.14.1)}$$

$$L[y] \overset{\text{def}}{=} M[y] + q(x)y \overset{\text{def}}{=} y'' + p(x)y' + q(x)y = f(x), \ a < x < b, \qquad \text{(II.14.2)}$$

2-tartibli chiziqli differensial tenglamalarni qaraylik. Bu yerda $p(x), q(x), f(x) - (a,b)$ intervalda aniqlangan berilgan funksiyalar; bundan tashqari $p(x)$ va $q(x)$ koeffitsientlar (a,b) da chegaralangan deb hisoblanadi.

Biz bu paragrafda (II.14.1) va (II.14.2) tenglamalarning $C^2((a,b))$ sinfga tegishli yechimlari tabiatini tekshirishda ishlatiladigan ekstremum prinsiplari bilan tanishamiz.

Dastlab quyidagi ta'riflarni eslataylik. x haqiqiy o`zgaruvchining haqiqiy funksiyasi $y(x)$ berilgan bo`lsin. Agar $x_0 \in \mathbb{R}$ nuqtaning biror atrofida $y(x) \le y(x_0)$ $\left(y(x) \ge y(x_0)\right)$ bo`lsa, $x_0$ nuqta $y(x)$ funksiyaning lokal maksimum (lokal minimum) nuqtasi, $y(x_0)$ esa lokal maksimum (mos ravishda lokal minimum) qiymat deb ataladi. Agar $y(x)$ funksiya $E$ to`plamga tegishli bo`lgan lokal maxsimum (lokal minimum) nuqtaga ega bo`lsa, bu $y(x)$ funksiya E da lokal maksimum (lokal minimum) nuqtaga ega deb gapiriladi.

Kuchsiz maksimum prinsipi.

Lemma 1. *Agar* (a,b) *da* $f(x)>0$ $\left(f(x)<0\right)$ *bo`lsa,* $M[y]=f(x)$ *(II.14.1) tenglamaning har qanday yechimi* $(a,b)$ *da lokal maksimum (lokal minimum) nuqtaga ega bo`lolmaydi.*

Lemmaning lokal maksimumga taa'lluqli qismini isbotlaymiz. Teskarisini faraz qilaylik, ya'ni (II.14.1) ning $y=y(x)$ yechimi $x_0 \in (a,b)$ nuqtada lokal maksimumga ega

bo`lsin. U holda bu nuqtada, analizdan ma'lumki, $y'(x_0)=0$ va $y''(x_0)\le 0$ bo`ladi. Demak, $M[y]\big|_{x=x_0} = y''(x_0)+p(x_0)y'(x_0)=y''(x_0)\le 0$. Lekin, $M[y]\big|_{x=x_0} = f(x_0)>0$. Hosil bo`lgan ziddiyatdan farazimizning noto`g`riligi, lemmaning esa to`g`riligi kelib chiqadi.

Teorema 1 ($M[y]=f(x)$ tenglama uchun kuchsiz maksimum (kuchsiz minimum) prinsipi). Agar (a,b) da $f(x)\ge 0$ $\left(f(x)\le 0\right)$ bo`lsa, $M[y]=f(x)$ (II.14.1) tenglamaning $[a,b]$ segmentda uzluksiz bo`lgan har qanday $y=y(x)$ yechiming $[a,b]$ segmentdagi eng katta (eng kichik) qiymati $\max\{y(a),y(b)\}$ $\left(\min\{y(a),y(b)\}\right)$ ga teng bo`ladi:

$$\sup_{x\in[a,b]} y(x)=\max\{y(a),y(b)\}$$

$$\left(\inf_{x\in[a,b]} y(x)=\min\{y(a),y(b)\}\right). \qquad \text{(II.14.3)}$$

$\gamma>0$ sonni shunday katta tanlaylikki, uning uchun

$$M[e^{\gamma x}]=\gamma\left(\gamma+p(x)\right)e^{\gamma x}>0,\ a<x<b,$$

bo`lsin. Buni bajarish mumkin, chunki berilganga ko`ra $p(x)$ koeffitsient (a,b) da chegaralanganligi uchun

$\gamma + p(x) \geq \gamma + \inf\limits_{a<x<b} p(x) > 0$ tengsizlikni qanoatlantiruvchi $\gamma > 0$ son mavjud. Demak, ixtiyoriy $\varepsilon > 0$ son uchun (a,b) da $M[y(x) + \varepsilon e^{\gamma x}] = M[y(x)] + \varepsilon M[e^{\gamma x}] = f(x) + \varepsilon M[e^{\gamma x}] > 0$.

Ushbu $y(x) + \varepsilon e^{\gamma x}$ funksiya $[a,b]$ segmentda uzluksiz. Demak, u shu segmentda eng katta qiymatga erishadi. Lemmaga ko`ra esa bu $y(x) + \varepsilon e^{\gamma x}$ funksiya $(a,b)$ intervalda maksimumga ega bo`lolmaydi. Demak, u eng katta qiymatini $[a,b]$ ning chetki nuqtasida qabul qiladi:

$$\sup_{x\in[a,b]}\left(y(x) + \varepsilon e^{\gamma x} \right) = \max\{ y(a) + \varepsilon e^{\gamma a}, y(b) + \varepsilon e^{\gamma b} \}.$$

Oxirgi tenglik ixtiyoriy $\varepsilon > 0$ son uchun o`rinli bo`lgani uchun unda $\varepsilon \to 0+$ deb limitga o`tamiz va (II.14.3) tenglikni hosil qilamiz.

Teorema 2 ($L[y] = f(x)$ **tenglama uchun kuchsiz maksimum (kuchsiz minimum) prinsipi).** Agar (a,b) da $q(x) \leq 0$ va $f(x) \geq 0$ $\left(f(x) \leq 0 \right)$ bo`lsa, $L[y] = f(x)$ (II.14.2) tenglamaning $[a,b]$ segmentda uzluksiz bo`lgan har qanday $y = y(x)$ yechimi uchun

$$\sup_{x\in[a,b]} y(x) \leq \max\{0, y(a), y(b)\}$$

$$\left(\inf_{x\in[a,b]} y(x) \geq \min\{0, y(a), y(b)\} \right) \qquad \text{(II.14.4)}$$

bo`ladi.

Agar (a,b) intervalda $y(x) \leq 0$ bo`lsa, $y(x) \in C([a,b])$ bo`lgani uchun $[a,b]$ segmentda ham $y(x) \leq 0$; demak, $\sup\limits_{x\in[a,b]} y(x) \leq 0$ va $\max\{0, y(a), y(b)\} = 0$, ya'ni (II.14.4) tengsizlik o`rinli.

Endi faraz qilaylik, (a,b) intervalning ba'zi nuqtalarida $y(x)$ funksiya musbat qiymatlar qabul qilsin. U holda, ravshanki, biror $x_0 \in [a,b]$ nuqta uchun $\sup\limits_{x\in[a,b]} y(x) = y(x_0) > 0$ bo`ladi. Agar $x_0 = a$ yoki $x_0 = b$ bo`lsa, (II.14.4) tengsizlik o`rinli. Masalan, $x_0 = a$ bo`lganda

$$\sup_{x\in[a,b]} y(x) = y(a) \le \max\{0, y(a), y(b)\}.$$

Agar $a < x_0 < b$ bo`lsa, $I_0 = (a_0, b_0)$ bilan $(a,b)$ da joylashgan shunday eng katta intervalni belgilaylikki, $x_0 \in I_0$ va $I_0$ ning barcha nuqtalarida $y(x) > 0$ bo`lsin. $y(x) \in C([a,b])$ ekanligidan ravshanki, $y(a_0) = y(b_0) = 0$. (a,b) da $L[y] = f(x) \ge 0$ bo`lgani uchun $I_0$ intervalda ham $L[y] = f(x) \ge 0$. Shuning uchun $I_0 = (a_0, b_0)$ intervalda $M[y] = y''(x) + p(x)y'(x) \ge -q(x)y(x) \ge 0$. Demak, teorema 1ga ko`ra $\sup\limits_{x\in I_0} y(x) = \max\{y(a_0), y(b_0)\} = 0$. Bundan $y(x_0) = \sup\limits_{x\in[a,b]} y(x) = \sup\limits_{x\in I_0} y(x) = 0$ ekanligi kelib chiqadi. Lekin $\sup\limits_{x\in[a,b]} y(x) = y(x_0) > 0$ edi. Hosil bo`lgan ziddiyat $a < x_0 < b$ holning bo`lishi mumkin emasligini ko`rsatadi. Boshqa hollarda (II.14.4) tengsizlikning to`g`ri ekanligi isbotlangan edi. 👍

Biz teoremning shartlarida $y(x)$ yechim, agar u o`zining eng katta (maksimal) musbat qiymatini sigmentning chitida qabul qilmasa, segmentning ichki nuqtasida qabul qilolmasligini ko`rsatdik.

Eslatma. Agar $q(x)$ koeffitsient (a,b) intervalda musbat qiymatlar qabul qilsa, $L[y] = f(x)$ tenglama uchun kuchsiz maksimum (kuchsiz minimum) prinsipi o`rinli emas.

Natija 1 ($L[y] = 0$ tenglama uchun kuchsiz ekstremum prinsipi). *Agar (a,b) da $q(x) \le 0$ bo`lsa, $L[y] = 0$ (II.14.2) tenglamaning $y(x) \in C([a,b])$ yechimi uchun ixtiyoriy $x \in [a,b]$*

nuqtada

$$\min\{y(a), y(b)\} \le y(x) \le \max\{y(a), y(b)\}$$

va

$$|y(x)| \le \max\{|y(a)|, |y(b)|\}$$

tengsizliklar o'rinli bo'ladi.

Natija 2. *Aytaylik,* (a,b) *da* $q(x) \le 0$ *va* $f(x) \ge 0$ $(f(x) \le 0)$ *bo'lsin. U holda* $L[y] = f(x)$ *(II.14.2) tenglamaning* $y(x) \in C([a,b])$ *yechimi uchun quyidagilar o'rinli:*

1^0. *agar* $y(a) \ge 0, y(b) \ge 0$ $(y(a) \le 0, y(b) \le 0)$ *bo'lsa,*

har qanday $x \in (a,b)$ *nuqtada* $y(x) \ge 0$ $(y(x) \le 0)$ *bo'ladi;*

2^0. *agar* $y(a) > 0, y(b) > 0$ $(y(a) < 0, y(b) < 0)$ *bo'lsa,*

har qanday $x \in (a,b)$ *nuqtada* $y(x) < 0$ $(y(x) < 0)$ *bo'ladi.*

Natija 3 ($L[y] = f(x)$ **tenglama uchun taqqoslash prinsipi).** *Aytaylik,* (a,b) *da* $q(x) \le 0$, *hamda* $C^2((a,b)) \cap C([a,b])$ *sinfga tegishli bo'lgan* y_1 *va* y_2 *funksiyalar uchun* (a,b)*da* $L[y_1] \le L[y_2]$ *bo'lsin. U holda* $y_1(a) \ge y_2(a)$ *va* $y_1(b) \ge y_2(b)$ *ekanligidan barcha* $x \in [a,b]$ *nuqtalarda* $y_1(x) \ge y_2(x)$ *bo'lishi kelib chiqadi.*

Kuchli maksimum prinsipi.

Kuchsiz maksimum prinsipida yechimni uning interval chetlaridagi qiymatlari orqali baholanadi. Kuchli maksimum prinsipida esa yechimning qiymatlari haqida tenglik bilan ifodalanuvchi xossa tasdiqlanadi.

Lemma 2. *Aytaylik,* (a,b) *da* $q(x) \le 0$ *va* $f(x) \ge 0$ *bo'lsin. Bundan tashqari,* $L[y] = f(x)$ *(II.14.2) tenglamaning* $y = y(x)$ *yechimi uchun quyidagi shartlar ham bajarilsin:*

(i) $x_0 = b$ *(yoki* $x_0 = a$ *) nuqtada* $y(x)$ *funksiya uzluksiz;*

(ii) $y(x_0) > 0$;

(iii) har qanday $x \in (a,b)$ *nuqtada* $y(x) < y(x_0)$.

U holda, agar $y'(b)$ $\left(y'(a)\right)$ *hosila mavjud bo`lsa, ushbu*

$$y'(b) > 0 \ \left(y'(a) < 0\right) \qquad \text{(II.14.5)}$$

qat'iy tengsizlik o`rinli bo`ladi.

Agar $q(x) \equiv 0$ *bo`lsa, (II.14.5) tengsizlik (ii)* $y(x_0) > 0$ *shartsiz ham bajariladi.*

$x_0 = b$ deb hisoblaymiz. Biror $x^* \in (a,b)$ nuqtani olib, ρ_0 sonni $0 < \rho_0 < b - x^*$ shartdan tayinlab (bunda $a < b - \rho_0 < b$), $\bar{I} = [b - \rho_0, b]$ segmentda ushbu

$$u(x) = \exp\left(-\gamma(x - x^*)^2\right) - \exp\left(-\gamma(b - x^*)^2\right)$$

funksiyani aniqlaylik; bu yerda $\gamma -$ hozircha noma`lum o`zgarmas son. Ko`rish qiyin emaski, $u(b) = 0$, $[b - \rho_0, b)$ oraliqda esa $u(x) > 0$. O`zgarmas $\gamma > 0$ sonni shunday tanlaymizki, mos $u = u(x)$ funksiya uchun

$$I = (b - \rho_0, b) \text{ intervalda } L[u] > 0$$

bo`lsin. Bunday tanlashni bajarish mumkinligi $p(x)$ va $q(x)$ funksiyalarning $I = (b - \rho_0, b) \subset (a,b)$ intervalda chegaralanganligidan kelib chiqadi (teorema 1 ning isbotiga qarang). (iii) shartga ko`ra $[b - \rho_0, b)$ oraliqda $y(x) - y(b) < 0$, xususan $y(b) - y(b - \rho_0) > 0$. Ixtiyoriy $\varepsilon > 0$ songa ko`ra ushbu

$$v(x) = y(x) - y(b) + \varepsilon u(x)$$

funksiyani tuzaylik. U $\bar{I} = [b - \rho_0, b]$ segmentda uzluksiz. $x = b$ nuqtadagi uzluksizlik (i) shartdan kelib chiqadi, boshqa nuqtalardagi uzluksizlik ravshan. Har qanday $\varepsilon > 0$ uchun $I = (b - \rho_0, b)$ da

$$\begin{aligned} L[v] &= L[y(x)] - L[y(b)] + \varepsilon L[u(x)] = \\ &= f(x) - q(x)y(b) + \varepsilon L[u(x)] \\ &= \bar{f}(x), \end{aligned}$$

$$\bar{f}(x) \stackrel{def}{=} f(x) - q(x)y(b) + \varepsilon L[u(x)] > 0 .$$

Endi $\varepsilon > 0$ sonni $\bar{I} = [b - \rho_0, b]$ ning chetlarida $v(x) \le 0$ bo`lishidan aniqlaymiz:

$x = b - \rho_0$: $v(b - \rho_0) = y(b - \rho_0) - y(b) + \varepsilon u(b - \rho_0) \le 0$,

$x = b$: $v(b) = y(b) - y(b) + \varepsilon u(b) = 0 \le 0$.

Bu shartlarning bajarilishi uchun $0 < \varepsilon \le \big(y(b) - y(b - \rho_0)\big) / u(b - \rho_0)$ bo`lishi kifoya.

$v(x)$ funksiyaga $\bar{I} = [b - \rho_0, b]$ segmentda maksimum prinsipi (teorema 2) ni qo`llab, ushbu

$$v(x) \le 0 \, , \; x \in [b - \rho_0, b],$$

tengsizlikni hosil qilamiz. $u(b) = 0$ bo`lgani uchun bundan $v(x)$ funksiya $x = b$ nuqtada maksimal qiymatga erishganligi kelib chiqadi. Demak, $v'(b) \ge 0$, ya'ni $y'(b) + \varepsilon u'(b) \ge 0$. Bundan $y'(b) \ge -\varepsilon u'(b) = 2\varepsilon\gamma \exp\big(-\gamma(b - x^*)^2\big)(b - x^*) > 0$, ya'ni (II.14.5) tengsizlik o`rinli. $q(x) \equiv 0$ holi yechimdan o`zgarmasni ayirsak, yana yechim hosil bo'lishidan kelib chiqadi. Lemma isbot bo`ldi. ☝

Teorema 3 (kuchli maksimum (kuchli minimum) prinsipi). *Faraz qilaylik, (a,b) da $q(x) \le 0$ va $f(x) \ge 0$ $\big(f(x) \le 0\big)$ bo`lsin.*

Agar $M[y] = f(x)$ tenglamaning $y = y(x)$ yechimi (a,b) intervalda maksimum (minimum) nuqtaga ega bo`lsa, u $(a,b)$ da o`zgarmas, ya`ni $y(x) = \text{const}$ bo`ladi.

$L[y] = f(x)$ tenglamaning o`zgarmasdan farqli yechimi (a,b) intervalda nomanfiy maksimum (nomusbat minimum) qiymat qabul qilolmaydi.

⊶ Teskarisini faraz qilaylik, ya'ni $y = y(x)$ yechim o`zgarmasdan farqli va (a,b) intervalning biror x_0 nuqtasida

$M = y(x_0) > 0$ maksimum qiymat qabul qilsin. U holda, tushunarliki, biror $(\tilde{a}, x_0) \subset (a,b)$ intervalda $y(x) < y(x_0)$ bo`ladi. Demak, lemma 2 ga ko`ra $y'(x_0) > 0$. Lekin $x_0 \in (a,b)$ maksimum nuqta bo`lgani uchun $y'(x_0) = 0$. Hosil bo`lgan ziddiyat teoremani isbotlaydi.

Teorema 4. *Aytaylik,* (a,b) *da* $q(x) \le 0$ *va* $y = y(x) \in C([a,b])$ *funksiya* $L[y] = 0$ *tenglamaning* (a,b) *intervalda yechimi bo`lsin. U holda, agar* $y'(a)$ *va* $y'(b)$ *hosilalar mavjud hamda* $y'(a) = y'(b) = 0$ *bo`lsa,* $y(x)$ *yechim* (a,b) *da o`zgarmasdan iborat bo`ladi. Bundan tashqari, agar* (a,b) *ning biror nuqtasida* $q(x) < 0$ *ham bo`lsa,* $(a,b)$ *da* $y(x) \equiv 0$ *bo`ladi.*

Teskarisini faraz qilaylik, ya'ni $y(x) \ne \text{const}$ bo lin. U holda $y(x)$ yoki $-y(x)$ funksiya nomanfiy maksimumi M ni (a,b) ning chetida qabul qiladi; bu nuqtani x_0 deylik ($x_0 = b$ yoki $x_0 = a$). Bunda mos funksiyaning (a,b) dagi qiymatlari M dan qat'iy kichik bo`ladi (kuchli maksimum prinsipiga ko`ra). x_0 nuqtaga lemma 2 ni qo'llab, $y'(x_0) \ne 0$ ligini topamiz. Bu esa teoremaning shartiga zid.

Isbotlangan teoremalardan quyidagi chegaraviy masalalar uchun yechimning yagonalik xossasi kelib chiqadi.

Faraz qilaylik, (a,b) da $q(x) \le 0$ va $q(x) \not\equiv 0$ bo`lsin. $(a,b)$ da $L[y] = 0$ tenglamani qaraymiz. Bu tenglama uchun quyidagi chegaravish masalalarni qo`yaylik.

I. Ushbu $y(a) = 0$ va $y(b) = 0$ chegaraviy shartlarni qanoatlantiruvchi $y = y(x) \in C([a,b])$ yechimni toping.

II. $y'(a) = 0$ va $y'(b) = 0$ chegaraviy shartlarni qanoatlantiruvchi $y = y(x) \in C^1([a,b])$ yechimni toping.

III. Shunday $y = y(x) \in C^1([a,b])$ yechimni topingki, u ushbu

$$\alpha_1 y'(a) + \alpha_0 y(a) = 0,\ \alpha_1\alpha_0 < 0;$$

$$\beta_1 y'(b) + \beta_0 y(b) = 0,\ \ \beta_1\beta_0 > 0;$$

chegaraviy shartlarni qanoatlantirsin.

Jumla. *Qo`yilgan I, II, va III chegaraviy masalalarning har biri yagona* $y(x) \equiv 0,\ x \in [a,b]$, *yechimga ega.*

Masala I uchun jumlaning rostligi maksimum prinsipidan ravshan. Masala II uchun jumla teoreama 4 dan bevosita kelib chiqadi. Endi masala III ni qaraylik. Bu masalaning $y = y(x),\ x \in [a,b]$, yechimini olaylik. Faraz qilaylik, (a,b) intervalda yechim $y(x) \not\equiv 0$ bo`lsin. U holda biror $\bar{x} \in (a,b)$ nuqtada $y(\bar{x}) \neq 0$ bo`ladi. Aniqlik uchun $y(\bar{x}) > 0$ deb hisoblaymiz. Tushunarliki, $\sup\limits_{x\in[a,b]} y(x) = y(x_0) > 0,\ \ x_0 \in [a,b]$. Maksimum prinsipiga ko`ra $x_0 = a$ yoki $x_0 = b$. Agar $x_0 = a$ bo`lsa, $y'(a) = -y(a)\alpha_0/\alpha_1 > 0$. Bunday bo`lishi mumkin emas, chunki bu chegaraviy maksimum nuqtada $y'(a) \le 0$. Agar $x_0 = b$ bo`lsa, $y'(b) = -y(b)\beta_0/\beta_1 < 0$. Lekin bu chegaraviy maksimum nuqtada $y'(b) \ge 0$ bo`lishi kerak. Hosil bo`lgan ziddiyatlar jumlaning isbotini tugatadi.

Masalalar

1. Natija 1-3 larni qat'iy isbotlang.
2. Aytaylik, L operatorda $q(x) < 0$ bo`lsin. Agar $y \in C^2((a,b)) \cap C([a,b])$ funksiya (a,b) da $L[y] = f(x)$ tenglamani qanoatlantirsa,

$$\sup\limits_{(a,b)} |y(x)| \le \max\{|y(a)|, |y(b)|\} + \sup\limits_{(a,b)} |f(x)/q(x)|$$

bo`lihini isbotlang.

3. Faraz qilaylik, L operatorda $q(x) \le 0$

$y \in C^2((a,b)) \cap C([a,b])$ va $L[y] \ge f(x)$ $(L[y] = f(x))$

bo`lsin. U holda

$$\sup_{(a,b)} y(x) \le \max\{0, y(a), y(b)\} + C \sup_{(a,b)} |f^-(x)|$$

$$\left(\sup_{(a,b)} |y(x)| \le \max\{|y(a)|, |y(b)|\} + C \sup_{(a,b)} |f(x)| \right),$$

bunda

$$f^-(x) = \min\{0, f(x)\},\ C = e^{(p_0+1)(b-a)} - 1,\ p_0 = \sup|p(x)|,$$

baholashni isbotlang.

II. 15. Chegaraviy masalalar

Chegaraviy masala tushunchasi. Bir jinsli chegaraviy masala.

Ushbu

$$L[y] \stackrel{def}{=} p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x) \tag{II.15.1}$$

2-tartibli chiziqli differensial tenglamani qaraylik; bu yerda $\{p_2, p_1, p_0, f\} \subset C([a,b])$ va $[a,b]$ da $p_2(x) \ge \omega = \text{const} > 0$ deb hisoblanadi.

Biz (II.15.1) tenglama uchun Koshi masalasi bilan tanishdik; bunda tayinlangan $x_0 \in [a,b]$ nuqtada $y(x_0)$ va $y'(x_0)$ qiymatlar beriladi va yechim biror $I \ni x_0$ oraliqda izlanadi.

Bu tenglama uchun berilgan segmentning chegaralari a va b nuqtalarda ham shartlar qo'yish mumkin. (II.15.1) tenglamaning ushbu

$$l_1(y,a) \stackrel{def}{=} \alpha_1 y'(a) + \alpha_0 y(a) = \alpha,\ l_2(y,b) \stackrel{def}{=} \beta_1 y'(b) + \beta_0 y(b) = \beta \tag{II.15.2}$$

chiziqli shartlarni qanoatlantiruvchi $y = y(x) \in C^2([a,b])$ yechimini topish masalasini qaraylik; bu yerda $\alpha_1, \alpha_0, \alpha, \beta_1, \beta_0, \beta$ – o'zgarmaslar va $|\alpha_1| + |\alpha_0| \ne 0$,

$|\beta_1|+|\beta_0| \neq 0$. (II.15.2) shartlar chegaraviy shartlar, (II.15.1),(II.15.2) masala esa - chegaraviy masala deb ataladi va quyidagicha yoziladi:

$$\begin{cases} L[y]=f(x), \\ l_1(y,a)=\alpha, \\ l_2(y,b)=\beta. \end{cases} \qquad \text{(II.15.1),(II.15.2)}$$

Shuni ta'kidlaylikki, (II.15.2) chegaraviy shartlardan $y(x)$ va $y'(x)$ qiymatlar bir vaqtda na $x=a$, na $x=b$ nuqtada topiladi. Shuning uchun (II.15.1),(II.15.2) chegaraviy masala Koshi masalasiga bevosita keltirilmaydi.

(II.15.2) chegaraviy shartlar ajralgan chegaraviy shartlar deb ataladi; birinchi shart $x=a$, ikkinchisi esa $x=b$ nuqtada qo`yilgan. (II.15.1) tenglama uchun ajralmagan chegaraviy shartlar ham qo'yilishi mumkin. Masalan, davriylik shartlari:

$$y(a)=y(b), \quad y'(a)=y'(b).$$

(II.15.1),(II.15.2) chegaraviy masala yechimining mavjudligi va yagonaligi nazariya uchun ham, amaliyot uchun ham katta ahamiyatga ega. Variatsion hisob, matematik fizika va fizikaning bir qancha masalalari chegaraviy masalalarni yechishga keltiriladi.

Agar (II.15.2) chegaraviy shartlarda $\alpha_1=\beta_1=0$ bo`lsa, mos shartlar I tur (tip) shartlar; agar $\alpha_0=\beta_0=0$ bo`lsa, – II tur; agar α_i va β_i lar noldan farqli bo`lsa, – III tur shartlar deb ataladi. Bu shartlarga mos chegaraviy masalalar esa I, II va III chegaraviy masalalar deb yuritiladi. Shunday qilib,

$$\begin{cases} p_2(x)y''+p_1(x)y'+p_0(x)y=f(x), \\ \qquad y(a)=\alpha/\alpha_0, \\ \qquad y(b)=\beta/\beta_0 \end{cases} \qquad (1)$$

$$\begin{cases} p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x), \\ y'(a) = \alpha/\alpha_1, \\ y'(b) = \beta/\beta_1 \end{cases} \tag{II}$$

$$\begin{cases} p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x), \\ \alpha_1 y'(a) + \alpha_0 y(a) = \alpha \ (\alpha_1\alpha_0 \neq 0), \\ \beta_1 y'(b) + \beta_0 y(b) = \beta \ (\beta_1\beta_0 \neq 0) \end{cases} \tag{III}$$

Misol 1. Ushbu

$$y'' + y = 0 \tag{II.15.3}$$

tenglama berilgan bo'lsin. Quyidagi I tur chegaraviy shartlarni qaraylik:

$$y(0) = 0, \ y(\pi) = 0; \tag{II.15.4}$$

$$y(0) = 0, \ y(\pi) = 1; \tag{II.15.5}$$

$$y(0) = 0, \ y(1) = 1; \tag{II.15.6}$$

Ma'lumki, (II.15.3) tenglamaning umumiy yechimi

$$y = c_1 \cos x + c_2 \sin x \tag{II.15.7}$$

formula bilan beriladi.

(II.15.4) chegaraviy shartlarni qanoatlantiramiz:

$$c_1 \cos 0 + c_2 \sin 0 = 0, \quad c_1 \cos \pi + c_2 \sin \pi = 0$$

Bundan $c_1 = 0$, c_2 esa ixtiyoriy ekanligini topamiz. Demak, (II.15.3),(II.15.4) chegaraviy masala cheksiz ko'p $y = c \sin x$, $c = \text{const}$, yechimlarga ega.

Endi (II.15.3), (II.15.5) masalani qaraylik. (II.15.5) shartlardan

$$c_1 \cos 0 + c_2 \sin 0 = 0, \ c_1 \cos \pi + c_2 \sin \pi = 1$$

tengliklarni hosil qilamiz. Bu tengliklarni c_1, c_2 larning hech qanday qiymatlarida qanoatlantirib bo'lmaydi. Demak, (II.15.3), (II.15.5) chegaraviy masala yechimga ega emas.

(II.15.6) chegaraviy shartning bajarilishi uchun

$$c_1 \cos 0 + c_2 \sin 0 = 0, \quad c_1 \cos 1 + c_2 \sin 1 = 1$$

bo'lishi kerak. Bu munosabatlar

$$c_1 = 0, \; c_2 = \frac{1}{\sin 1}$$

bo'lgandagina o'rinli ekanligini topamiz. Demak, (II.15.3), (II.15.6) chegaraviy masala $y = \dfrac{\sin x}{\sin 1}$ yagona yechimga ega.

Bir jinsli bo'lmagan (II.15.2) chegaraviy shartlarni noma'lum funksiyani almashtirish yordamida bir jinsli ko'rinishga keltirish mumkin. (II.15.2) chegaraviy shartlarni qanoatlantiruvchi biror $y = u(x)$ funksiyani topaylik. Osongina tekshirib ko'rib ishonch hosil qilish mumkinki, agar $\alpha_0\beta_0 \neq \dfrac{\beta_0\alpha_1 - \beta_1\alpha_0}{b-a}$ bo'lsa, $u(x)$ funksiya sifatida biror $u(x) = lx + m$ chiziqli funksiyani olish mumkin; aks holda esa $u(x)$ funksiyani $u(x) = kx^2 + m$ ko'rinishdagi kvadratik funksiylar orasidan tanlasa bo'ladi ($k, l, m -$ o'zgarmaslar). Endi (II.15.1),(II.15.2) masaladagi $y = y(x)$ noma'lum funksiya o'rniga yangi $u(x) + y(x)$ noma'lum funksiyani kiritib, (II.15.2) chegaraviy shartlarni bir jinsli, ya'ni

$$l_1(y,a) = 0, \; l_2(y,b) = 0 \qquad \text{(II.15.8)}$$

ko'rinishga keltirish mumkin. Bunda (II.15.1) tenglamaning o'ng tomoni o'zgaradi xolos:

$$L[y] = g(x) \; \left(p_2(x)y'' + p_1(x)y' + p_0(x)y = g(x)\right). \qquad \text{(II.15.9)}$$

(II.15.1) va/yoki (II.15.9) chiziqli differensial tenglamaga mos bir jinsli differensial tenglama

$$L[y] = 0 \; \left(p_2(x)y'' + p_1(x)y' + p_0(x)y = 0\right) \qquad \text{(II.15.10)}$$

ko'rinishda bo'ladi.

(II.15.10) bir jinsli tenglamaning (II.15.8) bir jinsli chegaraviy shartlarni qanoatlantiruvchi yechimini topish masalasi, ya'ni

$$\begin{cases} L[y]=0, \\ l_1(y,a)=0, \\ l_2(y,b)=0 \end{cases} \qquad \text{(II.15.10),(II.15.8)}$$

chegaraviy masala bir jinsli masaladir, chunki (II.15.10) differensial tenglama ham, (II.15.8) chegaraviy shartlar ham bir jinsli. Bu masala (II.15.1),(II.15.2) (bir jinslimas) chegaraviy masalaga mos bir jinsli masala deb ataladi. Ravshanki, bir jinsli chegaraviy masala har doim trivial $y(x)\equiv 0$ yechimga ega.

Teorema 1. *Quyidagi alternativa o`rinli:*

yo (II.15.10),(II.15.8) bir jinsli masala yagona trivial yechimga ega, bunda mos (II.15.1),(II.15.2) masala tenglama va chegaraviy shartlardagi o`ng tomonlarning ixtiyoriy qiymatlarida yagona yechimga ega, yoki (II.15.10),(II.15.8) bir jinsli masala cheksiz ko`p yechimga ega, bunda mos (II.15.1),(II.15.2) bir jinslimas masala o`ng tomonlarning ba'zi qiymatlarida birorta ham yechimga ega emas, qolgan barcha qiymatlarida esa cheksiz ko`p yechimga ega.

$L[y]=0$ (II.15.10) tenglamaning chiziqli erkli yechimlari $y_1=y_1(x)$ va $y_2=y_2(x)$ bo`lsin. Uning umumiy yechimi

$$y=c_1y_1+c_2y_2 \qquad \text{(II.15.11)}$$

ko`rinishda bo`ladi. $L[y]=f(x)$ (II.15.1) tenglamaning biror xususiy yechimini $y_{xus}=y_{xus}(x)$ bilan belgilab, uning umumiy yechimini

$$y=c_1y_1+c_2y_2+y_{xus} \qquad \text{(II.15.12)}$$

ko'rinishda ifodalaylik. Bir jinsli (II.15.10),(II.15.8) chegaraviy masalani yechish uchun (II.15.11) ni (II.15.8) shartlarga qo`yib, c_1, c_2 noma'lumlarga nisbatan ushbu

$$\begin{cases} c_1l_1(y_1,a)+c_2l_1(y_2,a)=0, \\ c_1l_2(y_1,b)+c_2l_2(y_2,b)=0 \end{cases} \qquad \text{(II.15.13)}$$

bir jinsli chiziqli algebraik tenglamalar sistemasini hosil qilamiz.

(II.15.1),(II.15.2) chegaraviy masalaning yechimini topish uchun esa (11) ni (II.15.2) chegaraviy shartlarga qo`yamiz va $c_1$, $c_2$ noma`lumlarga nisbatan

$$\begin{cases} c_1 l_1(y_1,a) + c_2 l_1(y_2,a) = \alpha - l_1(y_{xus},a), \\ c_1 l_2(y_1,b) + c_2 l_2(y_2,b) = \beta - l_2(y_{xus},b) \end{cases} \qquad \text{(II.15.14)}$$

chiziqli bir jinslimas algebraik tenglamalar sistemasiga kelamiz.

Algebradan ma'lumki, (II.15.14) va mos bir jinsli (II.15.13) chiziqli algebraik sistemalar yechimlarining soni ushbu

$$\Delta = \begin{vmatrix} l_1(y_1,a) & l_1(y_2,a) \\ l_2(y_1,b) & l_2(y_2,b) \end{vmatrix} \qquad \text{(II.15.15)}$$

determinant qiymatining nolga teng yoki tengmasligi bilan aniqlanadi.

$\Delta \neq 0$ bo`lganda (II.15.11) bir jinsli sistema ((II.15.10),(II.15.8) bir jinsli masala ham) faqat trivial yechimga ega, (II.15.14) sistema ((II.15.1),(II.15.2) masala) esa o`ng tomoni ixtiyoriy bo`lganda ham yagona yechimga ega. $\Delta = 0$ bo`lganda (II.15.11) bir jinsli sistema ((II.15.10),(II.15.8) bir jinsli masala) notrivial yechimlarga ega, (II.15.14) sistema ((II.15.1),(II.15.2) masala) esa o`ng tomonning ba'zi qiymatlarida birorta ham yechimga ega emas qolgan barcha qiymatlarida esa cheksiz ko`p yechimga ega. 👍

Chegaraviy masala yechimining yagonaligi.

Bir jinsli chegaraviy masalaning faqat trivial yechimga ega bo`lishi uchun yetarli shartlar ekstremum prinsiplaridan keltirib chiqarilgan edi (II.14- bandga qarang). Bu yerda yechimning yagonaligini o`z-o`ziga qo`shma tenglama uchun integral tengliklar yordamida o`rganamiz.

Ushbu

$$\mathcal{L}[y] \stackrel{def}{=} \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y, \qquad \text{(II.15.16)}$$

bunda

$$\begin{gathered} \{p(x), p'(x), q(x)\} \in C([a,b]); \\ p(x) \geq \omega = \text{const} > 0,\ q(x) \leq 0,\ x \in [a,b], \end{gathered} \qquad \text{(II.15.17)}$$

o`z-o`ziga qo`shma operator orqali tuzilgan

$$\mathcal{L}[y]=g(x)\quad \left(\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y=g(x)\right) \qquad \text{(II.15.18)}$$

chiziqli tenglama uchun (II.15.2) chegaraviy shartlarni qo`yaylik. Mos bir jinsli tenglama

$$\mathcal{L}[y]=0\quad \left(\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y=0\right) \qquad \text{(II.15.19)}$$

ko`rinishga ega. Bu (II.15.19) bir jinsli tenglama uchun (II.15.8) bir jinsli chegaraviy shartlarni qo`yib, mos bir jinsli chegaraviy masala (II.15.19),(II.15.8) ni hosil qilamiz.

Tushunarliki, teorema 1 L operatorni o`z-o`ziga qoshma operator $\mathcal{L}$ bilan almashtirganda ham o`z kuchini saqlaydi.

(II.15.19),(II.15.8) bir jinsli masalaning trivial yechimdan boshqa yechimga ega bo`lmasligi uchun yetarli shartlar quyidagi teoremada keltirilgan.

Teorema 2. *Faraz qilaylik, (II.15.17) shartlar bajarilsin. U holda (II.15.19),(II.15.8) bir jinsli chegaraviy masala uchun quyidagi tasdiqlar o`rinli:*

1) *agar* $\alpha_1=\beta_1=0$ *bo`lsa, bir jinsli I chegaraviy masala faqat trivial yechimga ega;*
2) *agar* $\alpha_0=\beta_0=0$ *va* (a,b) *da* $q(x)\not\equiv 0$ *bo`lsa, bir jinsli II chegaraviy masala faqat trivial yechimga ega;*
3) *agar* $\alpha_0=\beta_0=0$ *va* (a,b) *da* $q(x)\equiv 0$ *bo`lsa, bir jinsli II chegaraviy masalaning yechimlari o`zgarmaslardan iborat bo`ladi;*
4) *agar* $\alpha_1\alpha_0<0$ *va* $\beta_1\beta_0>0$ *bo`lsa, bir jinsli* III chegaraviy masala faqat trivial yechimga ega.

🗝 Aytaylik, $y=y(x)\in C^2([a,b])$ funksiya (II.15.19), (II.15.8) bir jinsli chegaraviy masalaning yechimi bo`lsin. Ravshanki,

$$y\mathcal{L}[y]=y\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y^2=0\,,\quad x\in[a,b]\,.$$

Bu ayniyatni x bo`yicha a dan b gacha integrallab topamiz:

$$\int_a^b y(x)\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)dx + \int_a^b q(x)y^2(x)dx = 0.$$

Oxirgi tenglikning chap tomonidagi birinchi integralni bo`laklab integrallaymiz va quyidagi tenglikni hosil qilamiz:

$$\int_a^b \left(p(x)y'^2(x) - q(x)y^2(x)\right)dx +$$

$$+p(a)y(a)y'(a) - p(b)y(b)y'(b) = 0 \qquad \text{(II.15.20)}$$

Agar $\alpha_1 = \beta_1 = 0$ bo`lsa, $y(a) = y(b) = 0$ va $q(x) \le 0$ shartga ko`ra (II.15.20) dan

$$\int_a^b p(x)y'^2(x)dx = 0 \qquad \text{(II.15.20)}$$

tenglikni topamiz. Bu tenglikdan $p(x) \ge p_0 > 0$ shartga ko`ra $y'(x) = 0$, $x \in [a,b]$, ekanligi kelib ciqadi. Demak, $y(x) \equiv c = \text{const}$, $x \in [a,b]$. $y(x) \in C([a,b])$ va $y(a) = y(b) = 0$ bo`lgani uchun $c = 0$, ya'ni $y(x) \equiv 0$, $x \in [a,b]$. Bu bir jinsli I chegaraviy masala faqat trivial yechimga ega ekanligini isbotlaydi. 1) qism isbot bo`ldi.

Agar $\alpha_0 = \beta_0 = 0$ bo`lsa, $y'(a) = y'(b) = 0$ chegaraviy shartlar qanoatlangan va (II.15.20) tenglikka ko'ra $q(x) \le 0$ bo`lganda yana (II.15.20) munosabat hosil bo`ladi. Bundan $y(x) \equiv c = \text{const}$, $x \in [a,b]$, ekanligi kelib chiqadi. $y(x) \equiv c$ ni (II.15.19) tenglamaga qo`yib, $q(x)c \equiv 0$, $x \in [a,b]$, shartga kelamiz. Bundan $q(x) \not\equiv 0$ bo`lganda $c = 0$ ekanligi kelib chiqadi, ya'ni yechim $y(x) \equiv c = 0$. 2) qism isbotlandi. $q(x) \equiv 0$ bo`lganda yechim $y(x) \equiv c = \text{const}$, $x \in [a,b]$ 3) qism isbotlandi.

Endi $\alpha_0\beta_0 < 0$ va $\alpha_1\beta_1 > 0$ bo`lgan holni qaraylik. Bu holda (II.15.20) tenglikdan (II.15.8) chegaraviy shartlarga ko`ra quyidagini topamiz:

$$\int_a^b \left(p(x)y'^2(x) - q(x)y^2(x)\right)dx - p(a)\frac{\alpha_0}{\alpha_1}y^2(a) + p(b)\frac{\beta_0}{\beta_1}y^2(b) = 0.$$

Bundan $y(x) \equiv c = \text{const}$, $x \in [a,b]$, va $y(a) = y(b) = 0$ ekanligi kelib chiqadi. $y(x) \in C([a,b])$ bo`lgani uchun $y(x) \equiv c = 0$. 4) qism isbotlandi.

Endi chegaraviy masala uchun ***Grin funksiyasi*** tushunchasini kiritamiz.

Chegaraviy masala (II.15.9),(II.15.8) uchun Grin funksiyasi deb, quyidagi uchta xossaga ega bo`lgan $G(x,\xi)$ funksiyaga aytiladi:

1. $G(x,\xi)$ funksiya $x \in [a,b]$, $\xi \in [a,b]$ bo`lganda aniqlangan va uzluksiz: $G(x,\xi) \in C([a,b] \times [a,b])$.

2. Tayinlangan $\xi \in [a,b]$ uchun $y(x) = G(x,\xi)$ funksiya x bo`yicha $x \neq \xi$ nuqtalarda mos bir jinsli $L[y] = 0$ tenglamani, $x = a$ va $x = b$ nuqtalarda esa (II.15.8) chegaraviy shartlarni qanoatlantiradi.

3. Tayinlangan $\xi \in (a,b)$ uchun $G(x,\xi)$ funksiyaning birinchi tartibli hosilasi $x = \xi$ nuqtada sakrashga ega va

$$\left.\frac{\partial G(x,\xi)}{\partial x}\right|_{x=\xi+0} - \left.\frac{\partial G(x,\xi)}{\partial x}\right|_{x=\xi-0} = \frac{1}{p_2(\xi)}. \qquad \text{(II.15.21)}$$

Terema 3 (Grin funksiyasining mavjudligi haqidagi). *Agar (II.15.10), (II.15.8) bir jinsli chegaraviy masala faqat trivial yechimga ega bo'lsa, u holda (II.15.9),(II.15.8) masala uchun Grin funksiyasi mavjud va u quyidagi ko'rinishga ega:*

$$G(x,\xi) = \begin{cases} c_1(\xi)y_1(x), & \text{agar } a \le x \le \xi \text{ bo'lsa}, \\ c_2(\xi)y_2(x), & \text{agar } \xi \le x \le b \text{ bo'lsa}, \end{cases} \qquad \text{(II.15.22)}$$

bu yerda y_1 *va* y_2 *– bir jinsli tenglama (II.15.10) ning mos ravishda* $l_1(y_1,a) = 0$ *va* $l_2(y_2,b) = 0$ *shartlarni qanoatlantiruvchi notrivial yechimlari,* $c_1(\xi)$ *va* $c_2(\xi)$ *lar esa ushbu*

$$\begin{cases} c_1(\xi)y_1(\xi) - c_2(\xi)y_2(\xi) = 0, \\ c_2(\xi)y_2'(\xi) - c_1(\xi)y_1'(\xi) = 1/p_0(\xi) \end{cases} \qquad \text{(II.15.23)}$$

sistemadan aniqlanadi.

Izoh. (II.15.23) dagi birinchi shart (II.15.22) Grin funksiyasining uzluksizligini, ikkinchi shart esa uning (II.15.21) xossasini ta'minlaydi.

8→ y_1 va y_2 funksiyalarni quyidagi Koshi masalalarining yechimlari sifatida tanlaylik:

$$L[y_1] = 0,\ y_1(a) = \alpha_1,\ y_1'(a) = -\alpha_0,$$

$$L[y_2] = 0,\ y_2(b) = \beta_1,\ y_1'(b) = -\beta_0.$$

Ravshanki, $l_1(y_1, a) = 0$ va $l_2(y_2, b) = 0$. Bundan tashqari, y_1 va y_2 yechimlar notrivial ($|\alpha_1| + |\alpha_0| \neq 0$, $|\beta_1| + |\beta_0| \neq 0$) va chiziqli erkli, chunki aks holda $y_2(x) \equiv cy_1(x)$ ($c \neq 0$, $l_1(y_2, a) = cl_1(y_1, a) = 0$) va, demak, teoremaning shartiga zid ravishda (II.15.10).(II.15.8) masala y_1 bilan birgalikda y_2 notrivial yechimga ham ega bo'lardi. Shunday qilib, $L[y] = 0$ tenglamaning ixtiyoriy yechimi $y = c_1y_1 + c_2y_2$ ko'rinishda bo'ladi. Grin funksiyasining ta'rifidan uning (II.15.22) ko'rinishda bo'lishi kerakligi kelib chiqadi. y_1 va y_2 yechimlar chiziqli erkli bo'lgani uchun ularning Vronskiani noldan farqli, va demak, (II.15.23) sistema $c_1(\xi)$ va $c_2(\xi)$ larni bir qiymatli aniqlaydi, chunki mos determinant

$$\begin{vmatrix} y_1(\xi) & -y_2(\xi) \\ -y_1'(\xi) & y_2'(\xi) \end{vmatrix} = W[y_1(\xi), y_2(\xi)] \neq 0.$$

Ravshanki (II.15.23) sistemadan topilgan $c_1(\xi)$ va $c_2(\xi)$ funksiyalar $\xi \in [a,b]$ da uzluksiz. Demak, (II.15.23) dagi birinchi shartga ko'ra (II.15.22) formula bilan aniqlangan funksiya $G(x,\xi) \in C([a,b] \times [a,b])$. Tayinlangan $\xi \in (a,b)$ uchun $G(x,\xi)$ funksiya x bo'yicha $x \neq \xi$ nuqtalarda ikki marta differensiallanuvchi va uning birinchi tartibli hosilasi $x=\xi$ nuqtada (II.15.23) dagi 2- formulaga ko'ra qiymati $1/p_2(\xi)$ ga teng bo'lgan sakrashga ega.

Eslatma. Teoremaning shartlari bajarilganda Grin funksiyasi bir qiymatli aniqlanadi. Agar y_1 va y_2 yechimlar c_3y_1 va c_4y_2 bilan almashtirilsa, (II.15.23) dan ravshanki, (II.15.22) dagi $G(x,\xi)$ funksiya o'zgarmaydi.

Bu yerda yana shuni e'tirof etaylikki, biz teoremani konstruktiv isbotladik, ya'ni Grin funksiyasini qurish usulini keltirdik.

Yuqorida qurilgan Grin funksiyasi (II.15.9), (II.15.8) chegaraviy masalaning yechimini topishga imkon beradi.

Teorema 4. *Faraz qilaylik, (II.15.10), (II.15.8) masala faqat trivial yechimga ega, $G(x,\xi)$ uning Grin funksiyasi va $g(x)\in C([a;b])$ bo'lsin. U holda ushbu*

$$\begin{cases} p_2(x)y''+p_1(x)y'+p_0(x)y=g(x), \\ \alpha_1y'(a)+\alpha_0y(a)=0 \ \ (|\alpha_1|+|\alpha_0|\neq 0), \\ \beta_1y'(b)+\beta_0y(b)=0 \ \ (|\beta_1|+|\beta_0|\neq 0) \end{cases} \qquad \text{(II.15.24)}$$

chegaraviy masalaning yagona yechimi

$$y(x)=\int_a^b G(x,\xi)g(\xi)d\xi \qquad \text{(II.15.25)}$$

formula bilan ifodalanadi.

Biz (II.15.25) formula bilan aniqlangan funksiya (II.15.24) masalaning yechimi ekanligini ko'rsatishimiz kerak.

Ixtiyoriy $x\in(a,b)$ nuqta uchun

$$y(x)=\int_a^x G(x,\xi)g(\xi)d\xi+\int_x^b G(x,\xi)g(\xi)d\xi$$

bundan (integralni differensiallash haqidagi Leybnits formulasiga ko'ra)

$$y'(x) = G(x,\xi)g(\xi)\Big|_{\xi=x-0} + \int_a^x \frac{\partial G(x,\xi)}{\partial x} g(\xi)d\xi - G(x,\xi)g(\xi)\Big|_{\xi=x+0}$$

$$+\int_x^b \frac{\partial G(x,\xi)}{\partial x} g(\xi)d\xi = \int_a^x \frac{\partial G(x,\xi)}{\partial x} g(\xi)d\xi + \int_x^b \frac{\partial G(x,\xi)}{\partial x} g(\xi)d\xi \,.$$

(II.15.26)

chunki $G(x,\xi)$ uzluksiz funksiya: $G(x,\xi)\Big|_{\xi=x-0} = G(x,\xi)\Big|_{\xi=x+0}$.

Endi ikkinchi tartibli hosilani hisoblaymiz ((II.15.26) dan foydalanamiz):

$$y'' = \int_a^b \frac{\partial^2 G(x,\xi)}{\partial x^2} g(\xi)d\xi + \left[\frac{\partial G}{\partial x}\Big|_{\xi=x-0} - \frac{\partial G}{\partial x}\Big|_{\xi=x+0}\right] g(x) =$$

$$= \int_a^b \frac{\partial^2 G(x,\xi)}{\partial x^2} g(\xi)d\xi + \frac{1}{p_2(x)} g(x)\,, \qquad \text{(II.15.27)}$$

chunki (II.15.21) ga ko'ra $\frac{\partial G}{\partial x}\Big|_{\xi=x-0} - \frac{\partial G}{\partial x}\Big|_{\xi=x+0} = \frac{1}{p_2(x)}$.

(II.15.26) va (II.15.27) formulalarga ko'ra (II.15.25) formula bilan aniqlangan funksiya (II.15.9) tenglamani qanoatlantirishi kelib chiqadi:

$$p_2(x)y'' + p_1(x)y' + p_0(x)y =$$

$$= p_2(x)\int_a^b \frac{\partial^2 G(x,\xi)}{\partial x^2} g(\xi)d\xi + g(x) + p_1(x)\int_a^b \frac{\partial G(x,\xi)}{\partial x} g(\xi)d\xi +$$

$$+ p_0(x)\int_a^b G(x,\xi)g(\xi)d\xi = g(x).$$

Endi chegaraviy shartlarning qanoatlanganligini tekshiramiz. (II.15.25) formulalarga ko'ra

$$\alpha_1 y'(x)+\alpha_0 y(x)=\alpha_1\int_a^x\frac{\partial G(x,\xi)}{\partial x}g(\xi)d\xi+\alpha_1\int_x^b\frac{\partial G(x,\xi)}{\partial x}g(\xi)d\xi+$$

$$+\alpha_0\int_a^x G(x,\xi)g(\xi)d\xi+\alpha_0\int_x^b G(x,\xi)g(\xi)d\xi$$

Bu tenglikda $x\to a+0$ deb limitga o'tamiz va Grin funksiyasining 2- xossasiga ko'ra (II.15.24)dagi chegaraviy shartlarning birinchisining bajarilishini isbotlaymiz:

$$\alpha_1 y'(a)+\alpha_0 y(a)=\alpha_1\int_a^b\frac{\partial G(a,\xi)}{\partial x}g(\xi)d\xi+\alpha_0\int_a^b G(a,\xi)g(\xi)d\xi=$$

$$=\int_a^b\left(\alpha_1\frac{\partial G(a,\xi)}{\partial x}+\alpha_0 G(a,\xi)\right)g(\xi)d\xi=\int_a^b 0\cdot g(\xi)d\xi=$$

$$=0.$$

Shunga o'xshash $\beta_1 y'(b)+\beta_0 y(b)=0$ ekanlgi ham tekshiriladi.

Misol 2. Ushbu

$$y''+y=g(x),\ y(0)=0,\ y(\pi/2)=0,$$

chegaraviy masala uchun Grin funksiyasini quring va yechimini yozing.

Mos bir jinsli chegaraviy masala

$$y''+y=0,\ y(0)=0,\ y(\pi/2)=0,$$

faqat trivial yechimga ega (tekshirib ko'ring), ya'ni Grin funksiyasining mavjudlik sharti bajariladi. Bir jinsli $y''+y=0$ tenglamaning $y_1=\sin x$ va $y_2=\cos x$ yechimlari $y_1(0)=0$ va $y_2(\pi/2)=0$ shartlarni qanoatlantiradi. Demak, (II.15.22) ga ko'ra izlanayotgan Grin funksiyasi

$$G(x,\xi)=\begin{cases}c_1(\xi)\sin x, & \text{agar } 0\le x\le\xi \text{ bo'lsa},\\ c_2(\xi)\cos x, & \text{agar } \xi\le x\le\pi/2 \text{ bo'lsa},\end{cases}$$

ko'rinishga ega. (II.15.23) sistemadan $c_1(\xi)$ va $c_2(\xi)$ larni topishimiz kerak. Qaralayotgan holda bu sistema quyidagi ko'rinishda:

$$\begin{cases} c_1(\xi)\sin\xi - c_2(\xi)\cos\xi = 0, \\ -c_2(\xi)\sin\xi - c_1(\xi)\cos\xi = 1. \end{cases}$$

Bu sistemani yechib, $c_1(\xi) = -\cos\xi$, $c_2(\xi) = \sin\xi$ ekanligini topamiz. Demak, berilgan chegaraviy masala uchun Grin funksiyasi

$$G(x,\xi) = \begin{cases} -\cos\xi\sin x, \text{ agar } 0 \le x \le \xi \text{ bo'lsa}, \\ -\sin\xi\cos x, \text{ agar } \xi \le x \le \pi/2 \text{ bo'lsa}, \end{cases}$$

formula bilan beriladi. Berilgan masala yechimi endi

$$y(x) = \int_a^b G(x,\xi)g(\xi)d\xi$$

(II.15.25) formula bilan aniqlanadi. ✍

Quyidagi λ parametrli bir jinsli chegaraviy masalani qaraylik:

$$\begin{cases} L[y] - \lambda y = 0, \\ \alpha_1 y'(a) + \alpha_0 y(a) = 0, \\ \beta_1 y'(b) + \beta_0 y(b) = 0 \end{cases}$$

Agar bu masala berilgan λ uchun notrivial yechimga ega bo'lsa, bu λ qaralayotgan masalaning xos soni, notrivial yechim esa (shu xos songa mos) xos funksiyasi deyiladi. Barcha xos sonlar va mos xos funksiyalarni topish **Shturm-Liuvill masalasi** deb ataladi. Bu masalani o'rganish matematik fizikada katta ahamiyatga ega.

Misol 3. Ushbu

$$\begin{cases} y'' - \lambda y = 0, \\ y'(0) = 0, \\ y'(\pi) = 0 \end{cases}$$

masalaning xos sonlari va xos funksiyalarini toping.

⊶ $y'' - \lambda y = 0$ o'zgarmas koeffitsientli tenglamaning umumiy yechimi osongina quriladi:

$\lambda > 0$ bo'lganda $y = c_1\exp(\sqrt{\lambda}x) + c_2\exp(-\sqrt{\lambda}x)$;

$\lambda = 0$ bo`lganda $y = c_1 + c_2 x$.

Qo`yilgan $y'(0) = 0$, $y'(\pi) = 0$ shartlardan ikkala holda ham $c_1 = c_2 = 0$ ekanligini, ya'ni yechimning trivial bo`lishini topamiz. Demak, notrivial yechimlar $\lambda < 0$ bo`lganda mavjud bo`lishi mumkin. $\lambda = -\mu^2, \mu > 0$, deylik. U holda berilgan tenglamaning umumiy yechimi $y = c_1 \cos \mu x + c_2 \sin \mu x$ ko`rinishda ifodalanadi. Chegaraviy shartlarga ko`ra yechim noldan farqli bo`lishi uchun $c_2 = 0$ va $\sin \mu\pi = 0 \Rightarrow \mu = \mu_k = k, k \in \mathbb{N}$, bo`lishi kerakligini topamiz. Demak, xos sonlar $\lambda = \lambda_k = -\mu_k^2 = -k^2$, mos xos funksiyalar esa $y = y_k = c \cos kx$, $k = 1, 2, \dots$. ✍

Izoh. Grin funksiyasi umumlashgan ma'noda ushbu

$$L_x[G(x,\xi)] = \delta(x-\xi),\ a < x < b,$$

tenglamani qanoatlantiradi. Bu yerda $\delta(x-\xi)-$ Dirakning delta-funksiyasi. Uni $x = \xi$ nuqtada qo`yilgan intensivligi 1 ga teng bo`lgan impuls deb tasavvur qilish mumkin. $\delta(x-\xi)$ ni noldan farqli qiymatlari ξ nuqtaning kichik atroflarida joylashgan uzluksiz va

$$\lim_{n\to\infty} \varphi_n(x-\xi) = 0, x \neq \xi,$$

$$\int_a^b \varphi_n(x-\xi)dx = 1$$

shartlarni qanoatlantiruvchi $\varphi_n(x-\xi)$ funksiyalar limiti deb tushunish kerak. U holda ixtiyoriy $f(x) \in C([a,b])$ funksiya uchun

$$\lim_{n\to\infty} \int_a^b \varphi_n(x-\xi) f(x)dx = \int_a^b \varphi_n(x-\xi) f(\xi)dx = f(\xi)$$

bo`ladi. Delta-funksiya umumlashgan funksiyadir. Umumlashgan funksiyalar matematik fizikaning zamonaviy qurolilaridan biri hisoblanadi.

Eslatma. Biz

$$p_n(x)y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = f(x), a < x < b. \tag{II.15.26}$$

n- tartibli chiziqli differensial tenglama va

$$l_j(y) \stackrel{def}{=} \sum_{k=0}^{n-1} \alpha_k^{(j)} y^k(a) + \sum_{k=0}^{n-1} \beta_k^{(j)} y^k(b) = \alpha_j, \ j = \overline{1,n}. \tag{II.15.27}$$

n dona chiziqli chegaraviy shartlarni qo'yib, (II.15.26) tenglamaning (II.15.27) chegaraviy shartlarni qanoatlantiruvchi yechimini topish haqidagi chegaraviy masalani hosil qilamiz. Bu masala ham yuqoridagiga o'xshash o'rganiladi.

Masalalar

1. Agar $y(a) = y(b) = 0$, barcha $x \in (a,b)$ lar uchun $y(x) > 0$ va $y''(x) + y(x) > 0$ bo'lsa,

$b - a > \pi$ bo'lishini isbotlang.

2. Masala uchun Grin funksiyasini quring:

$$xy'' + y' = g(x), \ y'(1) = 0, \ y(2) = 0.$$

3. Masalaning xos son va xos funksiyalarini toping (Shturm-Liuvill masalasini yeching):

$$(1 + x)^2 y'' + 2(1 + x)y' = \lambda y \quad (0 < x < 1), \ y(0) = y(1) = 0.$$

4. Grin funksiyasi yordamida berilgan masalani integral tenglamani yechishga keltiring:

$$e^x y'' + e^x y' = \lambda y, \ y(0) - y'(0) = 0, \ y(1) + y'(1) = 0$$

5. Agar $x^2 y'' + 2xy' - 2y = f(x)$ tenglamaning yechimi $x \to 0+$ va $x \to +\infty$ $f \to +\infty$ da chegaralangan bo'lsa, shu yechimni va uning hosilasini, $0 \le f(x) \le m$, $f(x) \in C((0, +\infty))$, deb faraz qilib, baholang.

Javoblar, ko`rstmalar va yechimlar

0.1.

1. $a \neq b$ bo'lganda $y(x) = -\dfrac{\cos(a+b)x}{2(a+b)} - \dfrac{\cos(a-b)x}{2(a-b)} + c$,

$a = b \neq 0$ bo'lganda esa $y(x) = -\dfrac{1}{4a}\cos(2ax) + c$,

$a = b = 0$ bo'lganda esa $y(x) = c$.

2. Ko`rsatma.

$$f(x) + g(x) = \text{const} \Rightarrow f(x) + g(x) = f(0) + g(0) = \frac{\pi}{4}, \ x \in \mathbb{R};$$

$$f(+\infty) + g(+\infty) = \frac{\pi}{4} \Rightarrow \left(\int_0^{+\infty} e^{-s^2} ds \right)^2 = \frac{\pi}{4}$$

I.1.

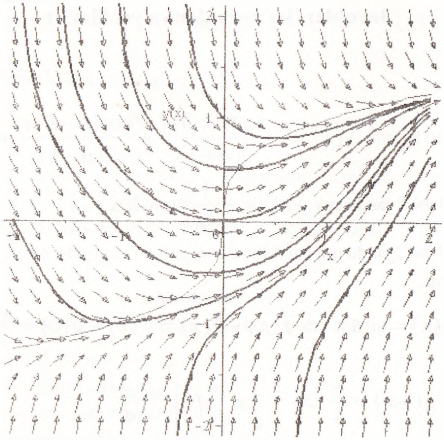
1. $x > 0$ oraliqda yechimlar $y = \int_1^x \frac{e^s}{s} ds + c$, $x < 0$ oraliqda yechimlar $y = \int_{-1}^x \frac{e^s}{s} ds + c$.

2. $y = \dfrac{1}{2}x|x| + c$. 3. Chunki $y \notin C^1((0;1))$ $(x = 1/2$ nuqtada y' hosila mavjud emas). 4. Ko`rsatma. Berilgan funksiya berilgan tenglamani hech qanday oraliqda ayniyatga aylantirmasligini ko`rsating.

I.3.

1. Tenglamani $x^2 - y^2 > 0$ va $x^2 - y^2 < 0$ to`plamlarda qarang.

2. J.1- rasmga qarang.



I.1- rasm.

I.4.

1. $x' = -\sin t,\ y' = \cos t$ hosilalar bir vaqtda nolga tengmas va $xdx + ydy = \cos t \cdot (-\sin t) \cdot dt + \sin t \cdot \cos t \cdot dt = 0$.

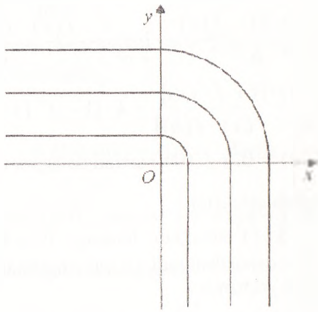
2. Ko`rsatma. Tenglamani $x \neq 0$ da $1/x^2$ ga $y \neq 0$ da $1/y^2$ ga ( yoki $(0;0)$ nuqtadan tashqarida $1/(x^2+y^2)$ ga) ko`paytiring.

I.5.

2. $y = c\exp(\frac{x|x|}{2})$. 3. $y(x) = \begin{cases} x^2/2, & \text{agar } x \in [0;2] \text{ bo'lsa} \\ 2e^{x-2}, & \text{agar } x \in (2;+\infty) \text{ bo'lsa} \end{cases}$

4. $y(x)=\begin{cases} -\dfrac{1}{2}+e^{2x}, \text{ agar } x\le -\dfrac{\ln 2}{2} \text{ bo'lsa};\\ \dfrac{1}{2}-\dfrac{1}{4}e^{-2x}, \text{ agar } x> -\dfrac{\ln 2}{2} \text{ bo'lsa}.\end{cases}$

5. $x\ge 0$, $y\ge 0$ chorak tekislikda berilgan tenglama $2xdx+2ydy=0$ ko`rinishga keladi. Uning yechimlari $x^2+y^2=c$;



J.2-

$x>0, y\le 0$ bo`lganda berilgan tenglama $2xdx=0$, yechimlari $x=c$;

$x\le 0, y>0$ bo`lganda berilgan tenglama $2ydy=0$, yechimlari $y=c$;

$x\le 0$, $y\le 0$ bo`lganda differensial tenglama aniqlanmagan.

Yechimlar grafiklari J.2- rasmda ko`rsatilgan.

6. Yechilishi.

(*) da $y=0$ deylik. U holda

$$f(u)=\frac{f(u)+f(0)}{1-f(u)\cdot f(0)}\Rightarrow f(0)\cdot[1+f^2(u)]=0\Rightarrow f(0)=0.$$

Berilganga ko'ra ushbu

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)}{h}=k$$

hosila mavjud

Endi (*) ga ko'ra $f'(x)$ ni hisoblaymiz.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\dfrac{f(x)+f(h)}{1-f(x)\cdot f(h)}-f(x)}{x}=$$

$$=\lim_{h\to 0}\frac{f(h)\cdot[1+f^2(x)]}{h\cdot[1-f(x)\cdot f(h)]}=k\cdot[1+f^2(x)],$$

chunki $f(h)\underset{h\to 0}{\to} f(0)=0$. ($f'(0)$ mavjud bo'lagini uchun $f(x)$ funksiya $x=0$ nuqtada uzluksiz).

Demak, $y=f(x)$ noma'lum funksiya $y'=k\cdot(1+y^2)$ differensial tenglamani qanoatlantiradi. Oxirgi tenglamani o'zgaruvchilarni ajratib yechamiz:

$$\frac{dy}{1+y^2}=k\cdot dx\;\Rightarrow\;\operatorname{arctg}y=kx+c$$

$y|_{x=0}=f(0)=0$ bo'lgani uchun $c=0$ bo'lishi kerak. Demak, agar noma'lum funksiya $y=f(x)$ mavjud bo'lsa, u $\operatorname{arctg}y=kx$ munosabatni qanoatlantiradi. Oxirgi tenglikdan $y=\operatorname{tg}kx$ ekanligini topamiz. Tangens funksiyaning xossalariga ko'ra bu $y=\operatorname{tg}kx$ funksiyaning (*) funktsional tenlamani qanoatlantirishini va $y'|_{x=0}=k$ hosilaga ega ekanligin ko'rish qiyin emas.

Shunday qilib, qo'yilgan masalaning yechimlari

$$f(x)=\operatorname{tg}kx\quad(k=\mathrm{const})$$

formula bilan beriladi.

Yuqoridagi fikr yuritishlardan ravshanki, $-\frac{p}{2}<kx<\frac{p}{2}$, ya'ni

$k \neq 0$ bo'lganda $f(x) = \mathrm{tg}kx$ yechim $|x| < a = \dfrac{\pi}{2|k|}$ oraliqda aniqlangan. $k = f'(0) = 0$ bo'lganda esa yechim $f(x) = 0$ va u $x \in (-\infty; +\infty)$ oraliqda aniqlangan.

I.6.

2. $y = x^2\left(1 - \dfrac{1}{\ln x - c}\right)$. Ko'rsatma. Tenglamada $y = z^m$ almashtirishni bajaring va $m = 2$ da o'zgaruvchilariga nisbatan bir jinsli tenglama hosil qiling.

I.7.

2. Ko'rsatma. $v(t) = u(t) - x(t)$ deylik. Berilganga ko'ra $v' - p(t)v \geq 0$, $v(0) \geq 0$. Demak, $\left(v \cdot \exp(-\int\limits_0^t p(s)ds)\right)' \geq 0$. Buni integrallab, kerakli tengsizlikni topamiz:

$$v(t) \geq v(0) + \exp(\int\limits_0^t p(s)ds) \geq 0.$$

3. $y = \ln \dfrac{a-1}{x + cx^a}$. $e^{-y} = z$ deng.

4. $\cos y = \dfrac{c - x^3}{3(x^2 - 1)}$. $z = \cos y$ almashtirish bajaring.

5. $z' = c(a-b)z - c$.

6. Ushbu

$$a(x) = \frac{y_2(x) - y(x)}{y_2(x) - y_1(x)},\ b(x) = \frac{y_3(x) - y_1(x)}{y_3(x) - y(x)}$$

funksiyalarni kiritib, $(a(x)b(x))' = a'(x)b(x) + a(x)b'(x)$ hosilaning nolga teng ekanligini bevosita tekshirish yo'li bilan isbotlang.

7. Oldingi masaladan foydalaning.

8. $y = \dfrac{3}{\sqrt{x}\,(3\sqrt[3]{x}\,\mathrm{tg}(c - 3\sqrt[3]{x}) - 1)}$.

I.8.

1. Teskarisini faraz qiling va $du(x,y) = \dfrac{y}{x^2+y^2}dx - \dfrac{x}{x^2+y^2}dy$ tenglikda $x = \cos\varphi,\ y = \sin\varphi$ deb, $du(\cos\varphi, \sin\varphi) = -d\varphi$ tenglikni hosil qiling. Oxirgi tenglikda φ ni 0 dan 2π gacha o`zgartirib, ziddiyatga keling.

2. Aniqlangan $u(x,y)$ funksiyaning to`la differensialini integral ostida differensiallash qoidasiga ko`ra hisoblab, D da $du(x,y) = M(x,y)dx + N(x,y)dy$ bo`lishini tekshiring.

3. $2x^3 + y^2 = cx + y$, $u = y^2 - y$ almashtirish bajaring.

I.9.

1. Teskarisini faraz qiling

2. $y(0) = 0 \Rightarrow y(x) \equiv 0$, $x \in (-a; a)$.

3. $|y_1(x) - y_0(x)| \le M \Rightarrow |y_2(x) - y_1(x)| \le \sqrt[3]{M}\,|x|$,

$$|y_3(x) - y_2(x)| \le M^{1/9}\frac{3}{4}|x|^{4/3},$$

$$|y_4(x) - y_3(x)| \le M^{1/27}\frac{3^2}{4\cdot 7}|x|^{7/3}, \ldots$$

baholashlarni hosil qiling. Ushbu $y_0(x) + \big(y_1(x) - y_0(x)\big) + \big(y_2(x) - y_1(x)\big) + \ldots$ qator va $\{y_n(x)\}$ ketma-ketlik tekis yaqinlashuvchi bo`ladi. Limit funksiya berilgan Koshi masalasining yechimidan iborat.

I.10.

1. $y = \varphi(x)$ yechim chegaralangan bo`lsin: $\exists m > 0\ \forall x \in \mathbb{R}\ |\varphi(x)| < m$. Agar bu yechimni

$[-m',m']\times[-a,a]$ ($m'>m, a-$ ixtiyoriy musbat son) to`rtburchakda davom ettirsak, yechim to`rtburchakning yon tomonlaridan chiqib ketadi (yuqori va quyi tomonlariga yetib borolmaydi).

2. Har qanday $y=y(x)$ yechim uchun x nuqtada

$$y(x)=y_0+\int_0^x \frac{ds}{1+s^2+y^{2002}(s)}.$$

Demak,

$$|y(x)|\le|y_0|+\left|\int_0^x\frac{ds}{1+s^2}\right|\le|y_0|+\int_0^\infty\frac{ds}{1+s^2}=\text{const}.$$

I.12.

1. $y^2=cx^2+c^2$ ($c-$ ixtiyoriy o'zgarmas).
2. $y=\sin(x+c)$, $y=\pm1$.

II.1.

1. Ixtiyoriy $y=y(x)$ yechim ($y''(x)+y(x)=0$) ga ko`ra ushbu

$$Y_1(x)=y(x)\cos x-y'(x)\sin x,$$
$$Y_2(x)=y(x)\sin x+y'(x)\cos x$$

funksiyalarni tuzaylik. Ularning hosilasi nolga teng:

$$\begin{aligned}Y_1'(x)&=y'(x)\cos x-y(x)\sin x-y''(x)\sin x-y'(x)\cos x=\\&=-(y(x)+y''(x))\sin x=\\&=0,\end{aligned}$$

$$Y_2'(x)=\ldots=0.$$

Demak, $Y_1(x)$ va $Y_2(x)$ lar o`zgarmas:

$$\begin{cases}c_1=y(x)\cos x-y'(x)\sin x\\ c_2=y(x)\sin x+y'(x)\cos x\end{cases}\quad (c_1,c_2-\text{const})$$

Oxirgi sistemani yechib, $y(x)=c_1\cos x+c_2\sin x$ ekanligini topamiz.

2. Ox o'qni tik yuqoriga yo'naltirylik O – Yer sathida. Jismga (moddiy nuqtaga) faqat og'irlik kuchi mg ta'sir etadi deb faraz qilamiz. $x = x(t)$ – moddiy nuqtaning t paytagi koordinatasi bo'lsin. U holda $v = \frac{dx}{dt}$ – tezlik $a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = x''$ – tezlznish va Nyutonnig ikkinchi qonuniga ko'ra

$$ma = -mg,\ x'' = -g.$$

Boshlang'ich shartlar: $x(0) = h,\ x'(0) = v_0$. Hosil bo'lgan Koshi masalasining yechimi:

$$x = h + v_0 t - \frac{gt^2}{2}$$

II.3.

1. $y = c_1,\ y = 2c_1\left(c_1 \pm \sqrt{c_1^2 - x - c_2}\right) - x - c_2$.

2. $y = c_1 \exp(-c_2 x^3) x^{-1/3}$

3. $x = te^{-t} + c_1,\ y = (t^2 + t + 1)e^{-t} + c_2$. Yechilishi:

$$y' = t,\ \frac{dt}{dx} = \frac{e^t}{1-t} \Rightarrow x = te^{-t} + c_1;$$

$$\frac{dy}{dx} = t,\ dy = td(te^{-t} + c_1) \Rightarrow y = (t^2 + t + 1)e^{-t} + c_2.$$

4. $y = -x + e^{x^2/2}(1 + c_1)\int e^{-x^2/2}dx + c_2 e^{x^2/2}$.

II.4.

1. Yo'q. Yechimning yagonalik xossasidan foydalaning.

2. Ha. 3. $a)\ 2p' + p^2 = 4q$; $b)\ q' + 2pq = 0$. y_1 va xy_1 ning yechim ekanligidan $2y_1' + p(x)y_1 = 0 \Rightarrow y_1 = \exp(-\frac{1}{2}\int p(x)dx)$ kelib chiqadi. Bu y_1 ni tenglamaga qo'yib, $2p' + p^2 = 4q$ shartni hosil qilamiz. ... .

II.5.

1. Yo`q, bunday funksiyalar (nol-funksiya qatnashgani uchun) har doim chiziqli bog`langan.

II.6.

1 – 2 mos funksiyalarni chiziqli erkli yechimlar ekanligini isbotlang.

II.7.

1. *a*) $y'' - \dfrac{2\varphi'(x)}{\varphi(x)}y' + \left(\dfrac{2\varphi'^2(x)}{\varphi^2(x)} - \dfrac{\varphi''(x)}{\varphi(x)}\right)y = 0$;

b) $y'' - \left(\dfrac{2\varphi'(x)}{\varphi(x)} + \dfrac{\varphi''(x)}{\varphi'(x)}\right)y' + \dfrac{2\varphi'^2(x)}{\varphi^2(x)}y = 0$.

2. Izlanayotgan $y = y(x)$ yechimni Ostrogradskiy-Liuvill formulasiga ko`ra topish mumkin:

$$\begin{vmatrix} \varphi_1(x) & y \\ \varphi_1'(x) & y' \end{vmatrix} = \exp\left(-\int a_1(x)dx\right),$$

$$\varphi_1(x)y' - \varphi_1'(x)y = \exp\left(-\int a_1(x)dx\right),$$

$$\frac{\varphi_1(x)y' - \varphi_1'(x)y}{\varphi_1^2(x)} = \frac{1}{\varphi_1^2(x)}\exp\left(-\int a_1(x)dx\right),$$

$$\left(\frac{y}{\varphi_1(x)}\right)' = \frac{1}{\varphi_1^2(x)}\exp\left(-\int a_1(x)dx\right),$$

$$y = \varphi_1(x)\int\frac{1}{\varphi_1^2(x)}\exp\left(-\int a_1(x)dx\right)dx.$$

3. $\int_{x_0}^{+\infty} a_1(x)dx = +\infty$. Ostrogradskiy-Liuvill formulasidan foydalaning.

II.8.

1. $y=\dfrac{c_1}{x}+\dfrac{c_2\ln x}{x}+\dfrac{x+\mathrm{Si}(x)}{x}$, $\mathrm{Si}(x)=\int_0^x \dfrac{\sin t}{t}dt$.

2. Berilgan differensial tenglamaning ixtiyoriy $x=x(t)$ yechimi

$$x''+x=-\varphi(t)x \tag{1}$$

tenglikni qanoatlantiradi. Ushbu

$$x''+x=f(t) \tag{2}$$

tenglama uchun Koshi formulasiga ko`ra

$$x(t)=A\sin(t-\varphi)+\int_{t_0}^{t}\sin(t-s)f(s)ds \quad (A,\varphi-\mathrm{const}). \tag{3}$$

(3) formulada $f(t)=-\varphi(t)x(t)$ deb, berilgan (1) tenglamaning yechimi uchun

$$x(t)=A\sin(t-\varphi)-\int_{t_0}^{t}\sin(t-s)\varphi(s)x(s)ds \tag{4}$$

integral tenglamaga kelamiz. $M(t)=\sup\limits_{t_0\le s\le t}|x(s)|$ deylik. $|x(s)|$ uzluksis funksiya bo`lgani uchun bu supremum biror $\tau=\tau(t)$, $t_0\le\tau\le t$, nuqtada erishiladi, ya'ni $M(t)=|x(\tau)|$ bo`ladi. Demak, (4) integral munosabatdan yetarlicha katta t lar (aniqrog`i $t>t_0>c$) uchun quyidagi baholashlarni hosil qilamiz:

$$M(t)=|x(\tau)|\le|A|+M(\tau)\int_{t_0}^{\tau}|\varphi(s)|ds\le$$

$$\le|A|+M(t)\int_{t_0}^{t}\frac{c}{s^2}ds\le$$

$$\le|A|+cM(t)\left(\frac{1}{t_0}-\frac{1}{t}\right)$$

tengsizlikni hosil qilamiz. Bundan o`sha t lar uchun

$$M(t) \leq \frac{|A|}{1+\frac{c}{t}-\frac{c}{t_0}},$$

ya'ni

$$M(t) \leq \frac{|A|}{1-\frac{c}{t_0}} \quad (t > t_0 > c).$$

Demak, $t \to +\infty$ da $|x(t)|$ yuqoridan $\frac{|A|}{1-\frac{c}{t_0}}$ son bilan chegaralangan.

II.9.

6. $\operatorname{Re} y(x)$ va $\operatorname{Im} y(x)$ haqiqiy funksiyalarni qarang.

7. $y'(x) = zy(x) \Leftrightarrow y'(x)e^{-zx} - y(x)ze^{-zx} = 0 \Leftrightarrow$

$$\Leftrightarrow \left(y(x)e^{-zx}\right)' = 0 \Leftrightarrow y(x)e^{-zx} = c = \mathrm{const}.$$

II.10.

1. $y = c_1 e^x + c_2 xe^x + c_3 x^2 e^x$.

2. $y = c_1 e^x + c_2 e^{-x} + c_3 e^{x/2}\cos\frac{\sqrt{3}x}{2} + c_4 e^{x/2}\sin\frac{\sqrt{3}x}{2} +$

$$+c_5 e^{-x/2}\cos\frac{\sqrt{3}x}{2} + c_6 e^{-x/2}\sin\frac{\sqrt{3}x}{2}.$$

3. $y = c_1 e^x \cos x + c_2 e^x \sin x + c_3 xe^x \cos x + c_4 xe^x \sin x$.

4. $y = c_1 e^{2x} + c_2 e^x \cos x + c_3 e^x \sin x + c_4 xe^x \cos x + c_5 xe^x \sin x$.

II.11.

1. $a)$ $y = xe^{2x} + c_1 e^{2x} + c_2 e^x \cos 2x + c_3 e^x \sin 2x$;

   $b)$ $y = c_1 x^2 + c_2 x + c_3 x \ln x$.

7. Berilgan tenglamaning yechimlari

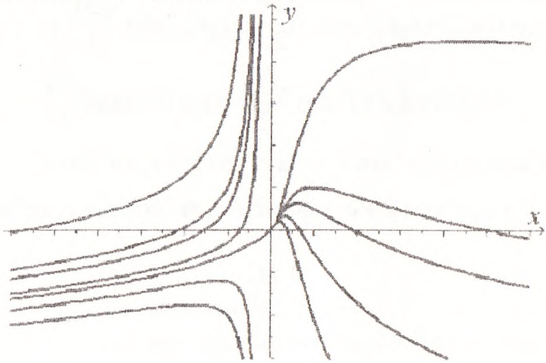
$x<0$ da $y=e^{-1/x}\left(c+\int_{-1}^{1/x}\frac{e^s}{s}ds\right)$, $x>0$ da esa $y=e^{-1/x}\left(c+\int_{1}^{1/x}\frac{e^s}{s}ds\right)$ formulalar bilan beriladi. Berilgan boshlang`ich masalaning yechimi cheksiz ko`p (J.3- rasm)

$$y=y(x)=\begin{cases}\varphi(x),\ x<0;\\ 0,\ x=0;\\ e^{-1/x}\left(c+\int_{1}^{1/x}\frac{e^s}{s}ds\right),\ x>0.\end{cases}$$

$$\varphi(x)\stackrel{def}{=}e^{-1/x}\left(-\int_{-1}^{-\infty}\frac{e^s}{s}ds+\int_{-1}^{1/x}\frac{e^s}{s}ds\right)=e^{-1/x}\int_{-\infty}^{1/x}\frac{e^s}{s}ds,$$

$$\lim_{x\to 0-}\varphi(x)=0,\ \lim_{x\to 0+}e^{-1/x}\left(c+\int_{1}^{1/x}\frac{e^s}{s}ds\right)=0,$$

analitik yechimi esa mavjud emas.



J.3- rasm.

8. $y = y(x) = c\exp(-\frac{1}{x^2})$.

$$\lim_{x\to 0} y^{(n)}(x) = c\lim_{x\to 0} P(\frac{1}{x})\exp(-\frac{1}{x^2}) = 0,$$

$n = 0,1,2,\ldots$, $P(t)$ – ko'phad. Tenglamaning nol nuqtada analitik bo'lgan yechimi bitta, u ham bo'lsa $y(x) \equiv 0$.

II.13.

1. Dastlab ushbu

$$u'' + \omega^2 u = 0$$

tenglamaning har qanday yechimi $u = A\cos(\omega x + \varphi)$ (A, φ – o'zgarmaslar) ko'rinishda bo'lgani uchun uning har qanday notrivial yechimining ixtiyoriy qo'shni nollari orasidagi masofa π/ω ga teng ekanligini e'tirof etaylik. Endi $y'' + q(x)y = 0$ va $u'' + \omega^2 u = 0$ tenglamalarga Shturm teoremasini qo'llaymiz.

Faraz qilaylik, $q(x) \le \omega^2$ $(x \in (a,b))$ shart bajarilsin. U holda berilgan tenglamaning har qanday notrivial yechimining qo'shni $x_0 < x_1$ nollari uchun $x_1 - x_0 \ge \frac{\pi}{\omega}$ bo'lishi kerak, chunki aks holda $x_1 - x_0 < \frac{\pi}{\omega}$, ya'ni $x_0 < x_1 < x_0 + \frac{\pi}{\omega}$ bo'lardi va Shturm teoremasiga ko'ra $u'' + \omega^2 u = 0$ tenglamaning x_0 da nolga aylanuvchi yechimi x_1 dan kichik yoki teng nolga ega bo'lib, bu nollar orasidagi masofa π/ω dan kichik bo'lib qolardi.

Endi faraz qilaylik, $q(x) \ge \omega^2$ $(x \in (a,b))$ tengsizlk o'rinli bo'lsin. U holda berilgan tenglamaning har qanday notrivial $y = y(x)$ yechimining qo'shni $x_0 < x_1$ nollari uchun $x_1 - x_0 \le \frac{\pi}{\omega}$ bo'lishi kerak. Buni isbotlash uchun aksini faraz qiling, $y = y(x)$ va $u'' + \omega^2 u = 0$

tenglamaning x_0 va $x_0 + \dfrac{\pi}{\omega}$ nuqtalarda nolga aylanuvchi yechimiga Shturm teoremasini qo`llab, ziddiyat hosil qiling.

2. Ushbu

$$y = z(x)\exp\left(-\frac{1}{2}\int\frac{1}{s}ds\right) = \frac{z(x)}{\sqrt{x}}$$

almashtirishni bajaring. U holda $z = z(x)$ yangi noma'lum funksiya uchun ushbu

$$z'' + \left(1 - \frac{\nu^2 - 1/4}{x^2}\right)z = 0$$

tenglamani hosil qilamiz. z oldidagi koeffitsient $\nu > 1/2$ bo`lganda 1 dan kichik, $0 \le \nu < 1/2$ bo`lganda esa ≥ 1. Demak, oldingi masalaga ko'ra Bessel tenglamasi har qanday notrivial yechiming qo'shni nollari orasidagi masofa $\nu > 1/2$ holida π dan katta, $0 \le \nu < 1/2$ holida esa π dan kichik. Agar $\nu = 1/2$ bo`lsa, Bessel tenglamasining umumiy yechimi . ($A, \varphi -$ ixtiyoriy o'zgarmaslar) elementar funksiyadan iborat va notrivial yechimning qo'shni nollari orasidagi masofa π ga teng.

3. Oldingi masalaga qarang.

4. Sturmning taqqoslash teoremasidan foydalaning.

II.14.

2. Teorema 2 dan foydalaning.

3. $\alpha = p_0 + 1$ va $x \in (a,b)$ bo`lsin. U holda

$$M[e^{\alpha(x-a)}] = (\alpha^2 + \alpha p(x))e^{\alpha(x-a)} \ge \alpha(\alpha - p_0)e^{\alpha(x-a)} \ge \alpha \ge 1.$$

$z = \max\{0, y(a), y(b)\} + \left(e^{\alpha(b-a)} - e^{\alpha(x-a)}\right)\sup\limits_{(a,b)}|f^-|$ nomanfiy funksiyani kiritaylik.

$$L[z] \le M[z] = -M[e^{\alpha(x-a)}]\sup_{(a,b)}|f^-| \le -\sup_{(a,b)}|f^-|,$$

$L[z-y] \le -\left(\sup\limits_{(a,b)}|f^-(x)| + f(x)\right) \le 0$, a va b nuqtalarda $z - y \ge 0$. Natija 3 ga ko`ra

(a,b) da ham $z-y\geq 0$. Demak,

$$\sup_{(a,b)}|y|\leq \sup_{(a,b)}|z|\leq \max\{0, y(a), y(b)\}+\left(e^{a(b-a)}-1\right)\sup_{(a,b)}|f|.$$

y ni $-y$ bilan almashtirib, $L[y]=f(x)$ holdagi tasdiqni isbotlaymiz.

II.15.

2. $G(x,\xi)=\begin{cases}\ln\dfrac{\xi}{2},\ 1\leq x\leq \xi,\\[2mm] \ln\dfrac{x}{2},\ \xi\leq x\leq 2.\end{cases}$

3. $\lambda_k=-\dfrac{1}{4}-\left(\dfrac{\pi k}{\ln 2}\right)^2,\ y_k=\dfrac{1}{\sqrt{1+x}}\sin\dfrac{\pi k\ln(1+x)}{\ln 2},\ k\in\mathbb{N}.$

Tenglama yechimini $y=(1+x)^a$ ko'rinishda izlang. Umumiy yechimni topib, chegaraviy shartlarni yozing. $(1+x)^{i\varphi}=\cos(\varphi\ln(1+x))+i\sin(\varphi\ln(1+x))$ formuladan foydalanib λ_k va y_k larni aniqlang.

4. $y(x)=\lambda\displaystyle\int_0^1 G(x,\xi)d\xi,\ G(x,\xi)=\begin{cases}\dfrac{1}{2}(e^{x}-2)e^{-\xi},\ 0\leq x\leq \xi,\\[2mm] \dfrac{1}{2}(e^{\xi}-2)e^{-x},\ \xi\leq x\leq 1.\end{cases}$

5. $-\dfrac{m}{2}\leq y(x)\leq 0,\ -\dfrac{m}{3x}\leq y'(x)\leq \dfrac{m}{3x}.$ Ekvivalent integral tenglamaga o'ting.

ADABIYOTLAR

1. Арнольд В.И. Обыкновенные дифференциальные уравнения. М.:Наука, 1984.
2. Бибиков Ю.Н. Курс обыкновенных дифференциальных уравнений. М.:Высшая школа, 1991.
3. Босс. В. Лекции по математике: дифференциальные уравнения. М.: УРСС, 2004.
4. Dilmurodov N. Differensial tenglamalardan mustaqil ishlar. Qarshi. 2010.
5. Краснов М.Л., Киселёв А.И., Макаренко Г.И. Обыкновенные дифференциальные уравнения. Задачи и примеры с подробными решениями. М.: УРСС, 2002.
6. Маматов М. Дифференциал тенгламалардан масалалар тўплами. Тошкент: Университет, 1995.
7. Петровский И.Г. Лекции по теории обыкновенных дифференциальных уравнений. М.: Изд-во МГУ, 1984; М.: УРСС, 2002.
8. Понтрягин Л.С. Обыкновенные дифференциальные уравнения. М.: Наука, 1983.
9. Салохитдинов М.С., Насритдинов Ғ.Н. Оддий дифференциал тенгламалар. Тошкент: Ўзбекистон, 1994.
10. Самойленко А.М., Перестюк Н.А., Кривошея С.А. Дифференциальные уравнения: примеры и задачи. М.: Высшая школа, 1989.
11. Степанов В.В. Курс дифференциальных уравнений. М.: Гиз.физ-мат. литературы, 1958
12. Тихонов А.Н., Васильева А.Б., Свешникова А.Г. Дифференциальные уравнения. М.: Наука, 1985.
13. Федорюк М.В. Обыкновенные дифференциальные уравнения. М.: Наука, 1985.
14. Филиппов А.Ф. Введение в теорию дифференциальных уравнений. М.: УРСС, 2004.
15. Эльсгольц Л.Е. Дифференциальные уравнения и вариационное исчисление. М.: Наука., 1969; М.: УРСС, 2002.
16. Филиппов А.Ф. Сборник задач по дифференциальным уравнениям. М.: Наука, 1998; Ижевск: Изд-во РХД, 2000.

O'quv-uslubiy nashr

N. Dilmurodov

DIFFERENSIAL TENGLAMALAR KURSI

I jild

Texnik muharrir: **M. Raxmatov**

Muharrir: **E. Jabborov**

Terishga 25.10.2012 yilda berildi. Bosishga 28.12.2012 yilda ruxsat etildi. Bichimi 84x108 1/16. Nashr bosma tabog'i 10.25. № 37 – буюртма. 100 nusxada. Erkin narxda

Qarshi Davlat universiteti
kichik bosmaxonasida bosildi

Qarshi shahri, Ko'chabog' ko'chasi 17-uy