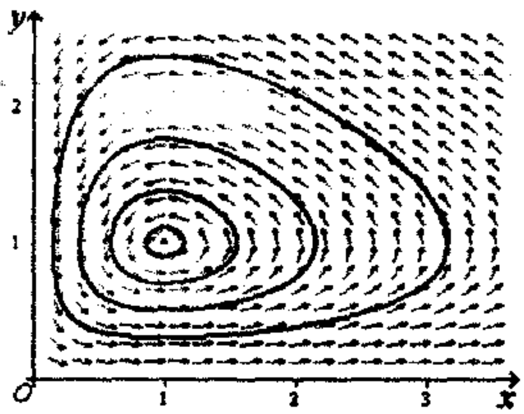
N. DILMURODOV

DIFFERENSIAL TENGLAMALAR KURSI

II
jild



$x' = x(1-y),\ y' = 0,5(x-1)y$

Mazkur kitob differensial tenglamalar kursini o`rganish uchun yozilgan qo`llanmaning ikkinchi jildidir.

Unda differensial tenglamalarning nochiziqli va chiziqli normal sistemalar, avtonom sistemalar, Lyapunov bo`yicha turg'unlik va yechimning parametrga silliq bog`liqligi va uning tatbiqlari kabi sohalari o`rganilgan.

Bundan tashqari, bilimlarni chuqrlashtirish va mustahkamlash uchun masalalar berilgan. Bu masalalarning ko`pini yechish uchun ko`rsatmalar va/yoki javoblar ham berilgan.

Qo`llanma "matematika" va "amaliy matematika va informatika" yo`nalishlari bo`yicha tahsil oluvchi bakalavriat talabalari uchun differensial tenglamalar kursi dasturini to`la qamrab olgan. Kitobdan differensial tenglamalarni mustaqil o`rganmoqchi bo`lgan barcha xohlovchilar unumli foydalanishlari mumkin.

Ushbu o'quv-uslubiy qo'llanma O'zbekiston Respublikasi Matbuot va axborot agentligi, Oliy va o'rta maxsus ta'lim vazirligi hamda Qarshi davlat universiteti tomonidan 2012 yilda tuzilgan uch yoqlama shartnoma rejasiga asosan nashrga tavsiya etilgan.

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MUNDARIJA

So`z boshi

Mazkur ikki jildli kitob "matematika" va "amaliy matematika va informatika" yo`nalishlari bo`yicha tahsil oluvchi bakalavriat talabalari uchun differensial tenglamalar kursi dasturining barcha mavzularini o`z ichiga qamrab olgan bo`lib, u shu kursni o`rganish uchun o`quv qo`llanma sifatida yozilgan.

Kitobning ikkinchi jildida nochiziqli normal sistemalar, chiziqli normal sistemalar, avtonom sistemalar, Lyapunov bo`yicha turg'unlik, yechimning parametrga silliq bog`liqligi va uning tatbiqlari, xususan, birinchi tartibli chiziqli va kvazichiziqli xususiy hosilali differensial tenglamalar o`rganilgan.

Mavzularga oid misollar to`la yechimlari bilan birgalikda keltirilgan. Bundan tashqari, paragraflar oxirida mustaqil yechish uchun masalalar taklif etilgan. Bu masalalarning yechimlari va javoblari ham berilgan..

Qo`shimcha misol va masalalarni muallifning "Differensial tenglamalardan mustaqil ishlar" (Qarshi, 2010) kitobidan topish mumkin. Bundan tashqari, kitob oxirida keltirilgan adabiyotlardan ham foydalanish maqsadga muvofiq bo`ladi.

Muallif qo`llanmadagi o`quv materiallarini aprobatsiyadan o`tkazishda yordam bergan barcha shogirdlari va talabalaridan hamda kasbdoshlaridan, bundan tashqari, taqrizchilardan ham, minnatdor ekanligini mamnunlik bilan e'tirof etadi.

Kitob haqidagi fikr va mulohazalaringizni nosir_d@mail.ru elektron manzilga yozsangiz, muallif sizdan minnatdor bo`ladi.

Asosiy belgilashlar ro`yxati

$\forall$ — har qanday, ixtiyoriy, har bir (umumiylik kvantori).

$\exists$ — mavjud, kamida bitta mavjud (mavjudlik kvantori).

$\Rightarrow$ — kelib chiqadi (implikatsiya belgisi).

$\Leftrightarrow$ — teng kuchli (ekvivalent).

$\overset{def}{\Leftrightarrow}$ — ta'rifga ko`ra ekvivalent (teng kuchli).

$\overset{def}{=}$ — ta'rifga ko`ra teng.

$\{x\in E \mid P(x)\}$ — E to`plamning $P(x)$ xossaga ega bo`lgan barcha x elementlari to`plami.

$\mathbb{N}$ — natural sonlar to`plami.

$\mathbb{R}$ — haqiqiy sonlar to`plami.

$\mathbb{C}$ — kompleks sonlar to`plami.

$\mathbb{R}^n$ — n o`lchamli haqiqiy Evklid fazosi.

$c, c_1, c_2, \ldots$ — ixtiyoriy o`zgarmaslar (doimiylar).

const — o`zgarmas (doimiy) miqdor.

$(a,b)\overset{def}{=}\{x\in\mathbb{R} \mid a<x<b\}$ $(a<b)$ — interval.

$[a,b]\overset{def}{=}\{x\in\mathbb{R} \mid a\le x\le b\}$ $(a<b)$ — segment.

$(a,b]\overset{def}{=}\{x\in\mathbb{R} \mid a<x\le b\}$ $(a<b)$ — yarim segment.

$[a,b)\overset{def}{=}\{x\in\mathbb{R} \mid a\le x<b\}$ $(a<b)$ — yarim segment.

$\mathbb{R}_+\overset{def}{=}[0,+\infty)$.

I — sonli oraliq (ichi bo`sh bo`lmagan bog`lanishli sonli to`plam).

D — soha, ya'ni ochiq va bog`lanishli to`plam.

$\max E$ — E sonli to`plamning maksimumi (eng katta elementi).

$\min E$ — E sonli to`plamning minimumi (eng kichik elementi).

$\sup E$ — E sonli to`plamning supremumi (yuqori chegaralarning eng kichigi, aniq yuqori chegara).

$\inf E$ — E sonli to`plamning infimumi (quyi chegaralarning eng kattasi, aniq quyi chegara).

$\|\ \|$ — norma (yoki matritsa) belgisi.

∂E — E to`plamning chegarasi.

E^c — E to`plamning (qaralayotgan fazogacha) to`ldiruvchisi.

$B_\delta(a)$ — δ radiusli a markazli (ochiq) shar.

$B_\delta = B_\delta(o)$

$X\times Y$ — to`plamlarning to`g`ri (Dekart) ko`paytmasi.

$\cup$, $\cap$, $\setminus$ — mos ravishda to`plamlar birlashmasi, kesishmasi, ayirmasi.

$f: X \to Y$ — X to`plamda aniqlangan, qiymatlari $Y$ to`plamda joylashgan f funksiya (akslantirish).

$D(f)$ — f funksiyaning aniqlanish to`plami (sohasi).

$f|_E$ — f funksiyaning E to`plamga torayishi.

$f|_a = f(a)$

$g \circ f$ — f va g funksiyalar kompozitsiyasi (ketma-ket bajarilishi).

$f(x) = o(g(x))$, $x \to a$, — asimptotik tenglik (kichik o); u $f(x) = \varepsilon(x) \cdot g(x)$, $\lim\limits_{x \to a} \varepsilon(x) = 0$, ekanligini anglatadi.

$f(x) = O(g(x))$, $x \to a$, — (katta o); u $f(x)$ funksiya $g(x)$ ni a nuqtaning biror atrofida chegaralangan $h(x)$ funksiyaga ko`paytirishdan hosil bo`lishini ($f(x) = h(x) \cdot g(x)$) anglatadi.

$C(X;Y)$ — barcha uzluksiz $f: X \to Y$ funksiyalar to`plami.

$C(X) = C(X, \mathbb{R})$.

$C^k(X;Y)$ — barcha k- tartibli hosilalari (demak, undan past tartiblilari ham) uzluksiz bo`lgan $f: X \to Y$ funksiyalar sinfi.

dist(X, Y) — to`plamlar orasidagi masofa (distance – masofa).

dim$X$ — X fazoning o`lchami (dimension – o`lcham).

deg$P$ — P ko`phadning darajasi (degree – daraja).

$\mathrm{M}_{n \times n}(\mathbb{R})$ — haqiqiy sonlardan tuzilgan $n \times n$ o`lchamli matritsalar to`plami.

$\mathrm{M}_{n \times n}(\mathbb{C})$ — kompleks sonlardan tuzilgan $n \times n$ o`lchamli matritsalar to`plami.

$\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{c}, \boldsymbol{h}, \boldsymbol{f}, \boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{q}, \ldots$ (qalin harflar) — vektorlar.

MYaT — mavjudlik va yagonalik teoremasi.

DT — differensial tenglama.

ODT — oddiy differensial tenglama.

🗝 — masala (misol) yechilishining, teorema (jumla) isbotining boshlanishi belgisi.

✍ — masala (misol) yechilishining, teorema (jumla) isbotining tugallanganligi belgisi.

III BOB. NOCHIZIQLI NORMAL SISTEMALAR

III.1. Yordamchi ma'lumotlar. $\mathbb{R}^n$ fazoda analiz elementlari

1⁰. $\mathbb{R}^n$ fazo. Haqiqiy sonlar to'plami $\mathbb{R}$ ni n marta o'zini o'ziga to'g'ri ko'paytirishdan hosil bo'lgan to'plamni odatdagidek $\mathbb{R}^n$ bilan belgilaymiz: $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n \text{ marta}}$. $\mathbb{R}^n$ ning elementlarini ustun ko'rinishida yozilgan vektor deb tushunamiz:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (x_1, x_2, \ldots, x_n)^T, \quad x_j \in \mathbb{R}, \ j = \overline{1, n}.$$

Bu yerda ^{T} – transpozitsiya belgisi.

$\mathbb{R}^n$ fazoda **vektorlarni qo'shish, vektorni songa ko'paytirish** odatdagicha kiritilgan deb hisoblaymiz, ya'ni:

$$\boldsymbol{x} + \boldsymbol{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}, \quad \{\boldsymbol{x}, \boldsymbol{y}\} \subset \mathbb{R}^n;$$

$$\lambda \boldsymbol{x} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}, \quad \lambda \in \mathbb{R}, \ \boldsymbol{x} \in \mathbb{R}^n.$$

$\boldsymbol{x}\lambda = \lambda\boldsymbol{x}$ deb tushuniladi. Ma'lumki, bu amallarga nisbatan $\mathbb{R}^n$ to'plam n o'lchamli vektor (chiziqli) fazoni tashkil etadi. Bunda nol vektor $0 = (0, 0, \ldots, 0)^T \in \mathbb{R}^n$ kabi tasvirlanadi.

$\mathbb{R}^n$ ni nuqtaviy (affin) fazo sifatida ham qarash mumkin.

Bunda nuqtaning koordinatalarini satr bo'ylab yozamiz. $x=(x_1,x_2,...,x_n)\in\mathbb{R}^n$ va $y=(y_1,y_2,...,y_n)\in\mathbb{R}^n$ nuqtalardan boshi x da oxiri y da bo'lgan $y-x=(y_1-x_1,y_2-x_2,...,y_n-x_n)^T$ vektorni tuzish mumkin.

$\mathbb{R}^n$ fazoda

$$\|x\|=\sqrt{x_1^2+x_2^2+...+x_n^2} \qquad \text{(III.1.1)}$$

norma (vektor uzunligi) kiritilgan deb hisoblaymiz. Ma'lumki, bunda $\mathbb{R}^n$ ushbu $\rho(x,y)=\|x-y\|$ masofa bilan birgalikda **metrik fazoga** aylanadi. (III.1.1) norma **Evklid normasi** deb ataladi. Evklid normasi ushbu $x\cdot y\equiv(x,y)=\sum_{j=1}^{n}x_jy_j$ **skalyar ko'paytmadan** induksiyalanadi (hosil bo'ladi), chunki $\|x\|^2=(x,x)$. Skalyar ko'paytma uchun ushbu

$$|(x,y)|\le\|x\|\cdot\|y\| \quad \left(\{x,y\}\subset\mathbb{R}^n\right)$$

Koshi-Bunyakovskiy tengsizligi o'rinlidir.

$a\in\mathbb{R}^n$ nuqtaning δ - **atrofi** deb ushbu

$$B_\delta(a)=\left\{x\in\mathbb{R}^n \mid \|x-a\|<\delta\right\}$$

sharga aytiladi. Qisqalik uchun $B_\delta=B_\delta(0)$ deymiz.

Biror $E\subset\mathbb{R}^n$ to'plam berilgan bo'lsin. Agar a nuqta o'zining biror atrofi bilan birgalikda E to'plamda joylashsa, bu a nuqta E ning **ichki nuqtasi** deyiladi. Barcha nuqtalari ichki nuqtalardan iborat bo'lgan to'plam **ochiq to'plam** deyiladi. Ma'lumki, har qanday $B_\delta(a)$ shar ochiq to'plamdir. Ushbu $E_i=\{x=(x_1,...,x_i,...,x_n)^T\in\mathbb{R}^n \mid x_i>0\}$ $(i=1,2,...,n)$ yarim fazo ham ochiq to'plam. Ochiq to'plamlarning ixtiyoriy birlashmasi va cheklita ochiq to'plamlar kesishmasi ham ochiq to'plamdir.

Agar E to'plam to'laligicha biror sharda joylashsa, ya'ni

$$\exists a\in\mathbb{R}^n \ \exists\rho>0 \ E\subset B_\rho(a)$$

bo'lsa, u holda E to'plam **chegaralangan to'plam** deyiladi.

Norma quyidagi xossalarga ega:

ixtiyoriy $\lambda\in\mathbb{R}$, $x\in\mathbb{R}^n$, $y\in\mathbb{R}^n$ lar uchun

$$\left.\begin{array}{c}\|x\|\ge0;\ \|x\|=0\Leftrightarrow x=0,\\ \|x+y\|\le\|x\|+\|y\|,\\ \|\lambda x\|=|\lambda|\cdot\|x\|.\end{array}\right\} \qquad \text{(III.1.2)}$$

Bundan tashqari, $\big|\|x\|-\|y\|\big|\le\|x-y\|$ tengsizlik ham o'rinli.

$\mathbb{R}^n$ da (III.1.1) Evklid normasidan boshqa norma ham kiritish mumkin. Lekin $\mathbb{R}^n$ dagi har qanday $\|\cdot\|_*$ norma qaralayotgan (III.1.1) Evklid normasiga ekvivalent bo'ladi, ya'ni shunday $c_1>0$ va $c_2>0$ sonlar mavjud bo'ladiki, ixtiyoriy $x\in\mathbb{R}^n$ uchun

$$c_1\|x\|_*\le\|x\|\le c_2\|x\|_* \qquad \text{(III.1.3)}$$

qo'sh tengsizlik bajariladi. Ekvivalent normalar bir xil limit (yaqinlashish) tushunchasiga olib keladi.

Agar $\mathbb{R}^n$ dagi x^k $(x^k\in\mathbb{R}^n,\ k\in\mathbb{N})$ ketma-ketlik va $\xi\in\mathbb{R}^n$ uchun $\|x^k-\xi\|\underset{k\to\infty}{\to}0$ bo'lsa, u holda x^k **ketma-ketlik** ξ **ga yaqinlashadi** (yoki x^k **ning limiti** ξ **ga teng**) deyiladi va $x^k\to\xi$ (yoki $\lim\limits_{k\to\infty}x^k=\xi$) ko'rinishda yoziladi .

Agar $x^k=(x_1^k,x_2^k,....,x_n^k)^T$, $\xi^k=(\xi_1^k,\xi_2^k,....,\xi_n^k)^T$ desak, ushbu

$$|x_j^k-\xi_j|\le\|x^k-\xi\|\le\sum_{i=1}^{n}|x_i^k-\xi_i|,\ j=\overline{1,n},$$

tengsizliklardan $\mathbb{R}^n$ dagi yaqinlashishning koordinatalar bo'yicha yaqinlashish ekanligini ko'ramiz:

$$\lim_{k\to\infty}x^k=\xi \ \Leftrightarrow \lim_{k\to\infty}x_j^k=\xi_j,\ j=\overline{1,n}. \qquad \text{(III.1.4)}$$

Biror $E \subset \mathbb{R}^n$ to'plam va $\xi \in \mathbb{R}^n$ nuqta berilgan bo'lsin. Agar E ning ξ dan farqli nuqtalaridan tuzilgan va ξ nuqtaga intiluvchi ketma-ketlik $\{x^k\}$ ($x^k \in E$, $x^k \neq \xi$, $x^k \to \xi$) mavjud bo'lsa, u holda ξ nuqta E to'plamning **limit nuqtasi** deyiladi. Barcha limit nuqtalari o'ziga tegishli bo'lgan to'plam **yopiq to'plam** deyiladi. $F \subset \mathbb{R}^n$ yopiq to'plamning fazogacha to'ldiruvchisi, ya'ni $F^c \overset{def}{=} \mathbb{R}^n \setminus F$ ochiq to'plamdir va aksincha, ya'ni ochiq to'plamning to'ldiruvchisi yopiq.

$E \subset \mathbb{R}^n$ to'plam bilan uning barcha limit nuqtalarini birlashtirish natijasida hosil bo'lgan to'plam E ning **yopilmasi** deyiladi va $\bar{E}$ bilan belgilanadi. Analizdan bilamizki, $\bar{E}$ ning yopilmasi E ni qoplovchi eng kichik (tor) yopiq to'plamdan iborat:

$$\bar{E} = \bigcap_{\substack{F \supset E \\ F - \text{yopiq}}} F .$$

Ma'lumki, $\mathbb{R}^n$ to'la fazo, ya'ni $\mathbb{R}^n$ dagi x^k ketma-ketlikning yaqinlashuvchi bo'lishi uchun uning fundamental bo'lishi zarur va yetarlidir. **Ketma-ketlikning fundamental ekanligi** quyidagini anglatadi:

ixtiyoriy $\varepsilon > 0$ songa ko'ra shunday ν natural son topiladiki, barcha $k > \nu$ va $j > \nu$ nomerlar uchun $\| x^k - x^j \| < \varepsilon$ tengsizlik o'rinli bo'ladi, ya'ni $\| x^k - x^j \| \xrightarrow[k,j \to \infty]{} 0$.

Agar $K \subset \mathbb{R}^n$ to'plamdan olingan ixtiyoriy ketma-ketlikdan limiti shu K da joylashgan yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin bo'lsa , u holda K to'plam **kompakt to'plam** deyiladi. Ma'lumki, $\mathbb{R}^n$ ning K qismi kompakt bo'lishi uchun uning chegaralangan va yopiq bo'lishi zarur va yetarli. $K_1 \subset \mathbb{R}^n$ va $K_2 \subset \mathbb{R}^m$ kompaktlarning $K_1 \times K_2 \subset \mathbb{R}^{n+m}$ to'g'ri ko'paytmasi ham kompaktdir.

Jumla. *Agar $K \subset \mathbb{R}^n$ kompakt, $F \subset \mathbb{R}^n$ yopiq to'plam va $K \cap F = \varnothing$ bo'lsa, u holda ular orasidagi masofa qat'iy musbat bo'ladi, ya'ni*

$$\operatorname{dist}(K,F) \overset{def}{=} \inf\left\{ \| x - y \| \,\middle|\, x \in K, y \in F \right\} > 0 .$$

Teskarisini faraz qilamiz, ya'ni jumlaning shartlari o'rinli, lekin $\operatorname{dist}(K,F) = 0$ bo'lsin. Aniq quyi chegara (inf) ta'rifiga ko'ra shunday $\{x^j\} \subset K$ va $\{y^j\} \subset F$ ketma-ketliklar mavjudki, ular uchun $\lim_{j\to\infty} \| x^j - y^j \| = 0$ bo'ladi. K kompakt bo'lganligi uchun $\{x^j\} \subset K$ ketma-ketlikdan K ning biror ξ elementiga yaqinlashuvchi qismiy ketma-ketlik ajratish mumkin. Shuning uchun umumiylikni buzmasdan $\{x^j\} \subset K$ ketma-ketlikning o'zi yaqinlashuvchi deb hisoblaymiz: $\lim_{j\to\infty} x^j = \xi \in K$.

U holda $\lim_{j\to\infty} y^j = \xi \in F$ ham bo'ladi, chunki

$0 \le \| y^j - \xi \| \le \| y^j - x^j \| + \| x^j - \xi \|$ va $\lim_{j\to\infty} \| x^j - y^j \| = 0$, $\lim_{j\to\infty} \| x^j - \xi \| = 0$. Shunday qilib, $\xi \in K \cap F$. Bu esa $K \cap F = \varnothing$ ekanligiga zid. Demak farazimiz noto'g'ri.

$\mathbb{R}^n$ fazoda **qator** deb ushbu $\sum_{k=1}^{\infty} x^k$, $x^k \in \mathbb{R}^n$, formal yig'indiga aytiladi. Agar uning xususiy yig'indilaridan tuzilgan $s^k = \sum_{j=1}^{k} x^j$, ketma-ketlik yaqinlashuvchi, ya'ni $\lim_{k\to\infty} s^k = s \in \mathbb{R}^n$ bo'lsa, u holda $\mathbb{R}^n$ dagi $\sum_{k=1}^{\infty} x^k$ qator ham yaqinlashuvchi deyiladi va bu $\sum_{k=1}^{\infty} x^k = s$ kabi yoziladi.

2^0. Skalyar argumentning vektor-funksiyasi. $E \subset \mathbb{R}$ va $f : E \to \mathbb{R}^n$, ya'ni har bir $t \in E$ songa bittadan $f(t) \in \mathbb{R}^n$ vektor mos keltirilgan bo'lsin. Bu holda $f : E \to \mathbb{R}^n$ akslantirish

(n o'lchamli) **vektor-funksiya** (yoki qisqacha: funksiya) deyiladi. $f(t)=(f_1(t),f_2(t),...,f_n(t))^T$, $f_j:E\to\mathbb{R}, j=\overline{1,n}$, deylik. Bu yerdagi f_j lar f vektor-funksiyaning **koordinata funksiyalari** deb ataladi.

$f(t)$ vektor-funksiyaning $t_0\in E$ nuqtadagi uzluksizligi quyidagini anglatadi: ixtiyoriy $\varepsilon>0$ songa ko'ra shunday $\delta>0$ son topiladiki, $|t-t_0|<\delta$ tengsizlikni qanoatlantiruvchi barcha $t\in E$ $(t\in B_\delta(t_0)\cap E)$ lar uchun $\|f(t)-f(t_0)\|<\varepsilon$ (ya'ni $f(t)\in B_\varepsilon(f(t_0))$) bo'ladi.

Tushunarliki, $f(t)$ vektor-funksiyaning t_0 nuqtada uzluksiz bo'lishi uchun uning barcha $f_j(t), j=\overline{1,n}$, koordinata funksiyalari shu t_0 nuqtada uzluksiz bo'lishi yetarli va zarurdir.

Agar $f:E\to\mathbb{R}^n$ vektor-funksiya E ning har bir nuqtasida uzluksiz bo'lsa, u E to'plamda uzluksiz deyiladi va bu $f\in C(E,\mathbb{R}^n)$ ko'rinishda yoziladi.

Agar $G\subset\mathbb{R}^n$ to'plamga uning ixtiyoriy ikki $x\in G$ va $y\in G$ nuqtasi bilan birgalikda shu nuqtalarni tutashtiruvchi kesma $\{x+s(y-x)|\ 0\le s\le 1\}$ ham qarashli bo'lsa, u holda G to'plam **qavariq to'plam** deyiladi.

Agar $G\subset\mathbb{R}^n$ ning ixtiyoriy ikki x va y nuqtalarini G da joylashgan uzluksiz chiziq bilan tutashtirish mumkin bo'lsa, ya'ni $u:[0;1]\to G$, $u(0)=x$, $u(1)=y$, xususiyatlarga ega bo'lgan uzluksiz $u(\cdot)$ funksiya mavjud bo'lsa, u holda G to'plam **bog'lanishli (chiziqli bog'lanishli) to'plam** deyiladi.

Bog'lanishli ochiq to'plam **soha** deb ataladi.

Agar $f(t)$ vektor-funksiya t_0 nuqtaning biror atrofida aniqlangan va biror $\xi\in\mathbb{R}^n$ hamda barcha yetarli kichik $h\in\mathbb{R}$ lar uchun

$$f(t_0+h)-f(t_0)=\xi h+\varepsilon(h)h,\quad \varepsilon(h)\xrightarrow[h\to0]{}0, \qquad (III.1.5)$$

munosabat o'rinli bo'lsa, u holda $f(t)$ vektor-funksiya t_0 nuqtada **differensiallanuvchi**, ξh va ξ esa uning shu nuqtadagi **differensiali** $df(t_0)$ va mos ravishda **hosilasi** $f'(t_0)$ deyiladi: $df(t_0)=f'(t_0)h$, $df(t_0)=\xi h$, $f'(t_0)=\xi$. Ravshanki,

$$\xi=f'(t_0)=\left(\frac{df_1(t_0)}{dt},\frac{df_2(t_0)}{dt},...,\frac{df_n(t_0)}{dt}\right)^T,$$

ya'ni

$$df(t_0)=\begin{pmatrix}\frac{df_1(t_0)}{dt}\\ ...\\ \frac{df_n(t_0)}{dt}\end{pmatrix}h=\begin{pmatrix}\frac{df_1(t_0)}{dt}h\\ ...\\ \frac{df_n(t_0)}{dt}h\end{pmatrix}. \qquad (III.1.6)$$

Vektor-funksiyaning differensiallanuvchiligi uning barcha koordinata funksiyalarining differensiallanuvchiligiga ekvivalent.

$f:[a,b]\to\mathbb{R}^n$ vektor-fuksiyaning $[a,b]$ segment bo'yicha integrali skalyar funksiyaning integraliga o'xshash kiritiladi. $[a,b]$ segmentni $a=t_0<t_1<t_2<...<t_k=b$ nuqtalar bilan k ta $[t_0,t_1]$, $[t_1,t_2],...,[t_{k-1},t_k]$ bo'lakchalarga ajratamiz va $[t_{i-1},t_i]$ bo'lakchadan ixtiyoriy α_i nuqta tanlab, quyidagi integral yig'indini tuzamiz:
$$y=\sum_{i=1}^{k}f(\alpha_i)\Delta t_i,\quad \Delta t_i=t_i-t_{i-1}.$$

Ravshanki,
$$y=(\sigma_1,\sigma_2,...,\sigma_n)^T\in\mathbb{R}^n;\ \sigma_j=\sum_{i=1}^{k}f_j(\alpha_i)\Delta t_i,\ j=\overline{1,n}.$$

Agar $d\overset{def}{=}\max\limits_{1\le i\le n}\Delta t_i\to0$ bo'lganda y integral yig'indining limiti $\alpha_i\in[t_{i-1},t_i]$ nuqtalarning tanlanishiga bog'liqsiz holda mavjud bo'lsa, u holda $f(t)$ vektor-funksiya $[a,b]$ segmentda

integrallanuvchi deyiladi va uning shu segment bo'yicha integralini odatdagidek $\int_a^b f(t)dt$ bilan belgilanadi:

$$\int_a^b f(t)dt = \lim_{d\to 0} y \quad \left(\left\| y - \int_a^b f(t)dt \right\| \xrightarrow[d\to 0]{} 0 \right). \quad \text{(III.1.7)}$$

Ravshanki, $f(t)$ vektor-funksiyaning $[a,b]$ da integrallanuvchiligi uning $f_1(t), f_2(t), ..., f_n(t)$ koordinata funksiyalarining shu segmentda integrallanuvchanligiga ekvivalent va

$$\int_a^b f(t)dt = \left(\int_a^b f_1(t)dt, \int_a^b f_2(t)dt, ..., \int_a^b f_n(t)dt \right)^T. \quad \text{(III.1.8)}$$

Jumla. *Agar* $f(t)$ *vektor-funksiya* $[a,b]$ *da integrallanuvchi bo'lsa,* $\|f(t)\|$ *haqiqiy funksiya ham* $[a,b]$ *da integrallanuvchi va quyidagi tengsizlik o'rinli:*

$$\left\| \int_a^b f(t)dt \right\| \le \int_a^b \|f(t)\| dt. \quad \text{(III.1.9)}$$

⊸ Haqiqatan ham, normaning xossalariga ko'ra

$$\|y\| = \left\| \sum_{i=1}^k f(\alpha_i)\Delta t_i \right\| \le \sum_{i=1}^k \|f(\alpha_i)\| \Delta t_i.$$

Bu tengsizlikda $d = \max_{1\le i\le n} \Delta t_i \to 0$ deb limitga o'tsak,

$$\|y\| \to \left\| \int_a^b f(t)dt \right\| \quad \left(\text{chunki} \left| \|y\| - \left\| \int_a^b f(t)dt \right\| \right| \le \left\| y - \int_a^b f(t)dt \right\| \right)$$

bo'lganligi uchun, (III.1.9) tengsizlikni hosil qilamiz. ♁

Agar $f:[a,b] \to \mathbb{R}^n$ vektor-funksiya $[a,b]$ segmentda uzluksiz bo'lsa, ushbu

$$\frac{d}{dt}\int_a^t f(s)ds = f(t) \quad (t \in [a,b])$$

formula (Nyuton-Leybnits formulasi) o'rinli bo'ladi (chunki u har bir koordinata funksiyasi f_i uchun o'rinli).

$I \subset \mathbb{R}$ oraliqda aniqlangan $f^k : I \to \mathbb{R}^n$, $k \in \mathbb{N}$, vektor-funksiyalar ketma-ketligi berilgan bo'lsin. Agar $f : I \to \mathbb{R}^n$ vektor-funksiya uchun ushbu $\|f^k(t) - f(t)\|$ skalyar funksiyalar ketma-ketligi I da nolga tekis intilsa, ya'ni $\sup_{t\in I} \|f^k(t) - f(t)\| \xrightarrow[k\to\infty]{} 0$ bo'lsa, u holda $f^k(t)$ vektor-funksiyalar ketma-ketligi $f(t)$ vektor-funksiyaga I da **tekis intiladi** deyiladi va $f^k(t) \overset{I}{\rightrightarrows} f(t)$ kabi yoziladi. Shunday qilib,

$$f^k(t) \overset{I}{\rightrightarrows} f(t) \Leftrightarrow \|f^k(t) - f(t)\| \overset{I}{\rightrightarrows} 0$$

$$(f^k(t) \overset{I}{\rightrightarrows} f(t) \Leftrightarrow \sup_{t\in I} \|f^k(t) - f(t)\| \xrightarrow[k\to\infty]{} 0).$$

Vektor-funksiyalar ketma-ketligining tekis yaqinlashishi koordinata funksiyalarining tekis yaqinlashishiga ekvivalent:

$$f^k(t) \overset{I}{\rightrightarrows} f(t) \Leftrightarrow f_j^k \overset{I}{\rightrightarrows} f_j, \; j = \overline{1,n}.$$

Ma'lumki, agar $f^k(t)$ lar I da uzluksiz va $f^k(t) \overset{I}{\rightrightarrows} f(t)$ bo'lsa, $f(t)$ ham I da uzluksiz bo'ladi, ya'ni uzluksiz funksiyalarning tekis limiti uzluksizdir.

Ushbu $\sum_{k=1}^{\infty} f^k(t)$ funksional qatorning tekis yaqinlashishi $s^k(t) = \sum_{j=1}^{k} f^j(t)$ funksional ketma-ketlikning tekis yaqinlashishini anglatadi. $\sum_{k=1}^{\infty} f^k(t)$ funksional qatorning I da tekis yaqinlashuvchi bo'lishi uchun uning Koshi mezonini I da tekis qanoatlantirishi yetarli va zarurdir:

$$\forall \varepsilon > 0 \ \exists \nu \ \forall k > \nu \ \forall p \in \mathbb{N} \ \forall t \in I \ \left\| \sum_{j=k}^{k+p} f^j(t) \right\| < \varepsilon .$$

Tekis yaqinlashish uchun Veyershtrass alomati: agar shunday a_k sonlari mavjud bo'lib, $\forall t \in I$ uchun $\|f^k(t)\| \le a_k$ va $\sum_k a_k < +\infty$ bo'lsa, u holda $\sum_{k=1}^{\infty} f^k(t)$ funksional qator I da tekis yaqinlashuvchi bo'ladi.

Agar $\sum_{k=1}^{\infty} f^k(t)$ funksional qator I oraliqda tekis yaqinlashuvchi va uning barcha hadlari I da uzluksiz bo'lsa, u holda bu funksional qatorning yig'indisi ham I da uzluksizdir.

3^0. Skalyar argumentning matritsaviy funksiyasi. a_{ij}, $i=\overline{1,n}$, $j=\overline{1,m}$, haqiqiy (yoki kompleks) sonlardan tuzilgan ushbu

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \quad \text{(qisqaroq } A=[a_{ij}] \text{)}$$

jadval haqiqiy (yoki kompleks) **matritsa** deb ataladi. a_{ij} lar uning elementlari: i – satr nomeri, j– ustun nomeri. Bu matritsaning o'lchami $n \times m$. **Matritsa songa ko'paytirilganda** uning barcha elementlari shu songa ko'paytiriladi, ya'ni $\lambda[a_{ij}] = [\lambda a_{ij}]$; $\lambda[a_{ij}] = [a_{ij}]\lambda$ deb qabul qilinadi. Bir xil $n \times m$ o'lchamli $[a_{ij}]$ va $[b_{ij}]$ **matritsalar qo'shilganda** ularning mos elementlari qo'shiladi, ya'ni $[a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$. Bu amallarga nisbatan barcha tayin $n \times m$ o'lchamli matritsalar chiziqli fazoni tashkil etadi. Bu fazoni $\mathbb{M}_{n\times m}(\mathbb{R})$ (yoki $\mathbb{M}_{n\times m}(\mathbb{C})$) bilan belgilaymiz. $\mathbb{M}_{n\times m}(\mathbb{R})$ haqiqiy fazoning o'lchami $n \cdot m$ ga teng; shunga o'xshash $\mathbb{M}_{n\times m}(\mathbb{C})$ kompleks fazoning o'lchami ham $n \cdot m$ ga teng. $A \in \mathbb{M}_{n\times m}(\mathbb{C})$ va $B \in \mathbb{M}_{m\times l}(\mathbb{C})$ **matritsalarning ko'paytmasi** $C = AB \in \mathbb{M}_{n\times l}(\mathbb{C})$ aniqlangan. Bu $C = AB$ matritsaning elementlari

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} \ , \ i=\overline{1,n} \ , \ j=\overline{1,l} \ ,$$

formula bilan aniqlanadi (A matritsaning i-satri B matritsaning j-ustuniga skalyar ko'paytirilgan).

Barcha elementlari 0 dan iborat bo'lgan matritsa nol matritsa deyiladi va 0 bilan belgilanadi. Ushbu

$$\delta_{ij} = \begin{cases} 1, \text{ agar } i = j \text{ bo'lsa} \\ 0, \text{ agar } i \ne j \text{ bo'lsa} \end{cases}$$

belgi **Kroneker belgisi** deyiladi. Elementlari Kroneker belgisidan iborat $a_{ij} = \delta_{ij}$ bo'lgan kvadrat matritsa **birlik matritsa** deyiladi va u E bilan belgilanadi. Agar $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ kvadrat matritsaning determinanti noldan farqli bo'lsa ($\det A \ne 0$), u holda bu matritsa **teskarilanuvchi** bo'ladi, ya'ni A^{-1} matritsa mavjud va $A^{-1}A = AA^{-1} = E$. Bir xil o'lchamli A va B kvadrat matritsalar uchun $\det AB = \det A \cdot \det B$ bo'ladi.

$\mathbb{M}_{n\times m}(\mathbb{R})$ fazoni (unga izomorf bo'lgan) $n \cdot m$ o'lchamli Evklid fazosi $\mathbb{R}^{nm}$ bilan tenglashtirib ($\mathbb{M}_{n\times m}(\mathbb{R}) \equiv \mathbb{R}^{nm}$), $\mathbb{M}_{n\times m}(\mathbb{R})$ fazoda yaqinlashish, uzluksizlik, hosila, integral va shunga o'xshash tushunchalarni odatdagicha kiritamiz. $A \in \mathbb{M}_{n\times m}(\mathbb{R})$ matritsaning Evklid normasi $\|A\| = \sqrt{\sum_{i,j} a_{ij}^2}$. Har qanday $x \in \mathbb{R}^m$ vektor uchun Koshi–Bunyakovskiy tengsizligiga ko'ra

$$\|Ax\|^2 = \sum_{j=1}^{n}(Ax)_j^2 = \sum_{j=1}^{n}(\sum_{k=1}^{m} a_{jk}x_k)^2 \le \sum_{j=1}^{n}\sum_{k=1}^{m} a_{jk}^2 \sum_{k=1}^{m} x_k^2 =$$

$$= \sum_{j,k} a_{jk}^2 \|x\|^2 = \|A\|^2 \|x\|^2,$$

ya'ni

$$\|Ax\| \le \|A\| \cdot \|x\| .$$

Yana Koshi–Bunyakovskiy tengsizligiga asosan $\|AB\| \le \|A\| \cdot \|B\|$ ekanligini ham ko'rsatish qiyin emas.

Ixtiyoriy $A \in \mathbb{M}_{n\times n}(\mathbb{R})$ kvadrat matritsa tayinlangan bo'lsin. Har qanday $x \in \mathbb{R}^n$ vektorga $y = Ax \in \mathbb{R}^n$ vektorni mos keltirib, $A : \mathbb{R}^n \to \mathbb{R}^n$ chiziqli operatorni aniqlaymiz. Agar biror $\lambda \in \mathbb{R}$ son va biror $x \ne 0$ vektor uchun $Ax = \lambda x$ bo'lsa, bu $x \ne 0$ vektor A matritsaning λ **xos son** (qiymat)iga mos keluvchi **xos vektori** deb ataladi. Xos sonlar ushbu $\det(A - \lambda E) = 0$ **xarakteristik tenglamaning** ildizlari sifatida topiladi.

A matritsaning (operatorning) $\mathbb{R}^n$ fazodagi normadan induksiyalangan $\|A\|^*$ normasi quyidagi formula bilan aniqlanadi:

$$\|A\|^* = \sup_{x\ne 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\| .$$

Bu normani hisoblash uchun quyidagicha ish tutish mumkin. $\|Ax\|^2 = (Ax)^T Ax = x^T A^T Ax = x^T Sx$, $S \overset{def}{=} A^T A$, kvadratik formani qaraylik. Ma'lumki, $S = A^T A$ simmetrik matritsani biror Q ortogonal ($Q^{-1} = Q^T$) matritsa yordamida diagonallashtirish mumkin, ya'ni $QSQ^{-1} = \Lambda$ – diagonal matritsa, $S = Q^{-1}\Lambda Q$. S va Λ matritsalarning xos sonlari bir xil. S matritsaning xos sonlari nomanfiy, chunki $x^T Sx = \|Ax\|^2 \ge 0$ va $Sy = \lambda y$ uchun $y^T Sy = \lambda \|y\|^2$. Demak, $\Lambda = \mathrm{diag}(s_1^2, s_2^2, \dots, s_n^2)$ $(s_j \ge 0,\ j = 1,\dots,n)$. $x = Q^T z$ almashtirish yordamida $\|Ax\|^2$ ifodani kvadratlar yig'indisi ko'rinishiga keltirish mumkin:

$$\|Ax\|^2 = x^T Sx = (Q^T z)^T S(Q^T z) = z^T QQ^{-1}\Lambda QQ^T z =$$

$$= z^T \Lambda z = \sum_{j=1}^{n} s_j^2 z_j^2 .$$

Tushunarliki, $\|x\|^2 = \|z\|^2 = 1$ bo'lganda $\|Ax\|^2$ ning eng katta qiymati $\max_{1\le j\le n} s_j^2$ dan iborat. Shunday qilib,

$$\|A\|^* = \max_{1\le j\le n} s_j$$

formula o'rinli.

$\Phi : I \to \mathbb{M}_{n\times m}(\mathbb{R})$ akslantirish $t \in I \subset \mathbb{R}$ skalyar argumentning (haqiqiy) matritsaviy funksiyasi deyiladi. U har bir $t \in I$ songa $\Phi(t) \in \mathbb{M}_{n\times m}(\mathbb{R})$ matritsani mos keltiradi. Ushbu

$$\frac{d\Phi(t)}{dt} = \Phi'(t) = \lim_{h\to 0} \frac{1}{h}[\Phi(t+h) - \Phi(t)]$$

formula bilan Φ ning hosilasi kiritiladi. $\mathbb{R}^{nm}$ fazodagi yaqinlashish kordinatalar bo'yicha bo'lgani uchun

$$\frac{d}{dt}\begin{pmatrix} \varphi_{11}(t) & \dots & \varphi_{1m}(t) \\ \dots & \dots & \dots \\ \varphi_{n1}(t) & \dots & \varphi_{nm}(t) \end{pmatrix} = \begin{pmatrix} \frac{d\varphi_{11}(t)}{dt} & \dots & \frac{d\varphi_{1m}(t)}{dt} \\ \dots & \dots & \dots \\ \frac{d\varphi_{n1}(t)}{dt} & \dots & \frac{d\varphi_{nm}(t)}{dt} \end{pmatrix}.$$

Ravshanki, agar $A : I \to \mathbb{M}_{n\times m}(\mathbb{R})$ va $B : I \to \mathbb{M}_{m\times l}(\mathbb{R})$ matritsaviy funksiyalar hosilaga ega bo'lsa, u holda $(AB)' = A'B + AB'$ bo'ladi. $\Phi(t)$ matritsaviy funksiyaning integrali vektor-funksiyaning integrali kabi kiritiladi. Bunda, agar $\Phi(t)$ matritsaviy funksiya $[a;b]$ segmentda integrallanuvchi bo'lsa, baholashlarda ishlatiladigan ushbu

$$\left\|\int_a^b \Phi(t)dt\right\| \le \int_a^b \|\Phi(t)\|dt$$

tengsizlik ham o'rinli bo'ladi ((III.1.9) tengsizlikka qarang).

4^0. Ko'p o'zgaruvchining vektor-funksiyalari. $G \subset \mathbb{R}^m$ – ochiq to'plam bo'lsin. $f: G \to \mathbb{R}^n$ akslantirish G da aniqlangan m ta *haqiqiy o'zgaruvchining n o'lchamli vektor-funksiyasi* deyiladi. f ning koordinata funksiyalari $f_i: G \to \mathbb{R}$, $i = \overline{1,n}$, m ta haqiqiy o'zgaruvchining skalyar funksiyalaridan iborat bo'ladi.

Agar berilgan vektor-funksiyaning barcha koordinata funksiyalari G da chegaralangan bo'lsa, u holda bu vektor-funksiya G da chegaralangan deyiladi. Vektor-funksiyaning chegaralanganligi $\|f(x)\|$ normaning chegaralanganligiga ekvivalent.

$f: G \to \mathbb{R}^n$ funksiyaning uzluksizligi uning barcha koordinata funksiyalarining uzluksizligiga ekvivalent.

Kompaktning uzluksiz aksi kompaktdir, ya'ni agar $K \subset \mathbb{R}^m$ kompakt va $f: K \to \mathbb{R}^n$ uzluksiz funksiya bo'lsa, u holda K ning $f(K)$ aksi ham kompaktdir.

$f: G \to \mathbb{R}^n$ vektor-funksiya va $x^0 \in G$ berilgan bo'lsin. Agar biror $A \in \mathbb{M}_{n\times m}(\mathbb{R})$ matritsa uchun ushbu

$$\|f(x^0 + h) - f(x^0) - Ah\| = o(\|h\|),\ h \to 0,$$

asimptotik tenglik o'rinli bo'lsa, $f: G \to \mathbb{R}^n$ funksiya x^0 nuqtada ***hosilaga ega (differensiallanuvchi)*** *va bu hosila A matritsaga* teng deyiladi va $f'_x(x^0) = A$ ($f'(x^0) = A$) ko'rinishda yoziladi. $f: G \to \mathbb{R}^n$ vektor-funksiyaning x^0 nuqtadagi hosilasi ushbu

$$f'_x(x^0) = \begin{pmatrix} \dfrac{\partial f_1(x^0)}{\partial x_1} & \dfrac{\partial f_1(x^0)}{\partial x_2} & \cdots & \dfrac{\partial f_1(x^0)}{\partial x_m} \\ \dfrac{\partial f_2(x^0)}{\partial x_1} & \dfrac{\partial f_2(x^0)}{\partial x_2} & \cdots & \dfrac{\partial f_2(x^0)}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial f_n(x^0)}{\partial x_1} & \dfrac{\partial f_n(x^0)}{\partial x_2} & \cdots & \dfrac{\partial f_n(x^0)}{\partial x_m} \end{pmatrix}$$

xususiy hosilalardan tuzilgan matritsadan iborat. U **Yakobi matritsasi** deyiladi. G ochiq to'plamning har bir nuqtasida **differensiallanuvchi** (hosilaga ega) funksiya G da **differensiallanuvchi** (hosilaga ega) deyiladi.

Agar $f: G \to \mathbb{R}^n$ funksiyaning barcha koordinata funksiyalari $x^0 \in G$ nuqtaning biror atrofida barcha birinchi tartibli xususiy hosilalarga ega va bu hosilalar shu x^0 nuqtada uzluksiz ham bo'lsa, u holda f funksiya x^0 nuqtada differensiallanuvchi bo'ladi.

Agar $f: G \to \mathbb{R}^n$ funksiyaning barcha koordinata funksiyalari G sohada barcha birinchi tartibli uzluksiz $\dfrac{\partial f_k}{\partial x_i}$, $k = \overline{1,n}$, $i = \overline{1,m}$, xususiy hosilalarga ega bo'lsa, u holda f funksiya G da uzluksiz differensiallanuvchi funksiya deyiladi va bu $f \in C^1(G, \mathbb{R}^n)$ kabi yoziladi. Sohada uzluksiz differensiallanuvchi funksiya shu sohada differensiallanuvchi hamdir.

Differensiallanuvchi f va u funksiyalar kompozitsiyasi $f \circ g$ ham differensiallanuvchi hamda $(f \circ u)' = f'u'$ matritsaviy tenglik o'rinli. Bu tasdiqdan foydalanmagan holda matritsalarni ko'paytirish qoidasiga ko'ra quyidagi tasdiqni osongina tekshirib ko'rish mumkin:

agar $G\subset\mathbb{R}^m$, $D\subset\mathbb{R}^l$, $u\in C^1(G,D)$, $f\in C^1(D,\mathbb{R}^n)$ bo'lsa, u holda $g=f\circ u\in C^1(G,\mathbb{R}^n)$ va $g'_x(x)=f'_u(u(x))\cdot u'_x(x)$ matritsaviy tenglik o'rinli bo'ladi.

$G\subset\mathbb{R}^m$ – qavariq soha va $f\in C^1(G,\mathbb{R}^n)$ bo'lsin. $\{x,y\}\subset G$ uchun $u(s)=x+s(y-x)$, $0\le s\le 1$, funksiyani qaraylik. G qavariq bo'lgani uchun $s\in[0;1]\Rightarrow u(s)\in G$. Ravshanki,

$$f(y)-f(x)=\int_0^1\frac{df(u(s))}{ds}ds.$$

Lekin $\frac{df(u(s))}{ds}=f'_x(u(s))\cdot(y-x)$. Demak,

$$f(y)-f(x)=\int_0^1 f'_x(x+s(y-x))ds\cdot(y-x) \qquad \text{(III.1.10)}$$

Bu formula chekli orttirmalar formulasi deyiladi. U $f(y)-f(x)$ chekli orttirmani hisoblashga hamda baholashga imkon beradi:

$$\|f(y)-f(x)\|\le\sup_{0\le s\le1}\|f'_x(x+s(y-x))\|\cdot\|y-x\|$$

Agar f'_x chegaralangan, ya'ni $\|f'_x(x)\|\le c$, $c>0$, bo'lsa, u holda, ravshanki,

$$\|f(y)-f(x)\|\le\sup_{0\le s\le1}\|f'_x(x+s(y-x))\|\cdot\|y-x\|\le c\|y-x\|$$

baholash o'rinli bo'ladi.

5⁰. Lipshits sharti. Differensial tenglamalar sistemasi uchun Koshi masalasi yechimining mavjudligi va yagonaligi to'g'risidagi teoremani ifodalashda Lipshits sharti kerak bo'ladi. Shu tushunchani kiritaylik.

$E\subset\mathbb{R}^{1+m}$ – ixtiyoriy to'plam, $G\subset\mathbb{R}^{1+m}$ – soha bo'lsin. $\mathbb{R}^{1+m}$ fazoning nuqtalarini (t,x), $t\in\mathbb{R}$, $x\in\mathbb{R}^m$, ko'rinishda belgilaymiz.

$$f:E\to\mathbb{R}^n,\ (t,x)\to f(t,x),$$

vektor-funksiya berilgan bo'lsin. Agar shunday $L>0$ soni mavjud bo'lib, $\forall(t,x^1)\in E$, $\forall(t,x^2)\in E$ nuqtalar uchun,

$$\|f(t,x^1)-f(t,x^2)\|\le L\|x^1-x^2\| \qquad \text{(III.1.11)}$$

tengsizlik o'rinli bo'lsa, u holda f funksiya E to'plamda "x" – o'zgaruvchi bo'yicha (global) **Lipshits shartini qanoatlantiradi** deyiladi. Bu tengsizlikda yozish mumkin bo'lgan eng kichik L soni **Lipshits doimiysi** deyiladi. U ushbu

$$L=\sup\frac{\|f(t,x^1)-f(t,x^2)\|}{\|x^1-x^2\|},\ x^1\ne x^2,\{(t,x^1),(t,x^2)\}\subset E,$$

formula bilan hisoblanishi mumkin.

Endi $G\subset\mathbb{R}^{1+m}$ sohada berilgan $f:G\to\mathbb{R}^n$, $(t,x)\to f(t,x)$, funksiyani qaraylik. Agar har bir $(t_0,x^0)\in G$ nuqtaning biror atrofida bu funksiya Lipshits shartini "x" bo'yicha qanoatlantirsa, u holda $f(t,x)$ funksiya G sohada "x" bo'yicha **lokal Lipshits shartini qanoatlantiradi** deyiladi. Ravshanki, agar $f(t,x)$ funksiya G da (global) Lipshits shartini ("x" bo'yicha) qanoatlantirsa, u G da lokal Lipshits shartini ham ("x" bo'yicha) qanoatlantiradi.

Normaning ta'rifidan ravshanki, $f(t,x)$ funksiyaning Lipshits shartini qanoatlantirishi uning barcha koordinata funksiyalari $f_i(t,x), i=\overline{1,n}$,

$$(f(t,x)=(f_1(t,x),f_2(t,x),\dots,f_n(t,x))^T\in\mathbb{R}^n)$$

ning Lipshits shartini qanoatlantirishiga teng kuchli.

Misol. Chegaralangan matritsaviy funksiya $A:I\to\mathbf{M}_{n\times m}(\mathbb{R})$ orqali tuzilgan $f:I\times\mathbb{R}^m\to\mathbb{R}^n$, $f(t,x)=A(t)x$, $x\in\mathbb{R}^m$, vektor-funksiya $I\times\mathbb{R}^m$ to'plamda x bo'yicha Lipshits shartini qanoatlantiradi. Haqiqatan ham, A chegaralangan bo'lgani uchun $\exists L>0\ \forall t\in I\ \|A(t)\|\le L$. Endi ravshanki,

$$\| f(t,x^1)-f(t,x^2)\| = \| A(t)(x^1-x^2)\| \le \| A(t)\|\cdot\| x^1-x^2\| \le$$
$$\le L\| x^1-x^2\|$$

Jumla. *Agar* $f: G \to \mathbb{R}^n$, $G \subset \mathbb{R}^{1+m}$, $(t,x)\to f(t,x)$, $f(t,x)=(f_1(t,x), f_2(t,x),\dots,f_n(t,x))^T$, *funksiyaning koordinata funksiyalari* $x_1, x_2,\dots,x_m$ *o'zgaruvchilar bo'yicha* G *da uzluksiz birinchi tartibli xususiy hosilalarga ega bo'lsa, u holda bu funksiya* G *da* $x=(x_1,x_2,\dots,x_m)$ *bo'yicha lokal Lipshits shartini qanoatlantiradi.*

Berilgan funksiya ixtiyoriy $(t_0,x^0)\in G$ nuqtaning biror atrofida x bo'yicha Lipshits shartini qanoatlantirishini ko'rsatish kerak. G ochiq bo'lgani uchun $(t_0,x^0)\in G$ nuqta o'zining biror sharsimon yopiq atrofi F bilan birgalikda G da joylashadi. $f(t,x)$ funksiya F da x bo'yicha uzluksiz differensiallanuvchi bo'lgani uchun uning f'_x hosilaviy matritsasi $\frac{\partial f_i}{\partial x_j}$ chegaralangan xususiy hosilalardan tuzilgan. Demak, $\exists L>0\ \forall (t,x)\in F\ \ \| f'_x(t,x)\|\le L$ $(L=L(F))$. Endi chekli orttirmalar formulasidan $\forall \{(t,x^1),(t,x^2)\}\subset F$ uchun

$$\left\| f(t,x^1)-f(t,x^2)\right\| = \left\| \int_0^1 f'_x(t,x^1+s(x^1-x^2))ds(x^1-x^2)\right\| \le$$
$$\le \int_0^1 \left\| f'_x(t,x^1+s(x^1-x^2))\right\| ds\cdot \left\| x^1-x^2\right\| \le L\,\| x^1-x^2\| .$$

ekanligini topamiz.

Teorema. *Agar* $f(t,x)$ *funksiya* ($f: G\to\mathbb{R}^n$) G *da* x *bo'yicha lokal Lipshits shartini qanoatlantirsa, u* G *ning ixtiyoriy kompakt qismida* x *bo'yicha global Lipshits shartini ham qanoatlantiradi.*

Teskarisini faraz qilaylik. U holda biror $K\subset G$ kompaktda $f(t,x)$ funksiya x bo'yicha Lipshits shartini qanoatlantirmaydi, ya'ni

$$\forall j\in\mathbb{N}\ \ \exists\{(t_j,x^j),(t_j,y^j)\}\subset K\ \ \| f(t_j,x^j)-f(t_j,y^j)\| \ge j\,\| x^j-y^j\| \tag{III.1.12}$$

K – kompakt bo'lgani uchun $\{(t_j,x^j)\}\subset K$ va $\{(t_j,y^j)\}\subset K$ ketma-ketliklardan yaqinlashuvchi qismiy ketma-ketliklarni ajratishimiz mumkin. Yozuvda qisqalik uchun ularning o'zi yaqinlashuvchi deb hisoblaymiz. Aytaylik, $j\to\infty$ da $t_j\to t_0$, $x^j\to x^0$, $y^j\to y^0$ bo'lsin. K kompakt (demak, yopiq) bo'lganligi uchun $(t_0,x^0)\in K$ va $(t_0,y^0)\in K$. Shunday qilib, $t_j\to t_0$, $x^j\to x^0$, $y^j\to y^0$ ketma-ketliklar uchun

$$\| f(t_j,x^j)-f(t_j,y^j)\| \ge j\,\| x^j-y^j\| \tag{III.1.13}$$

tengsizlik o'rinli.

Agar $x^0=y^0$ bo'lsa, bu tengsizlik $f(t,x)$ funksiya $(t_0,x^0)=(t_0,y^0)\in G$ nuqta atrofida x bo'yicha Lipshits shartini qanoatlantirmasligini anglatadi. Bu berilganga zid.

Endi $x^0\ne y^0$ bo'lsin. (t_0,x^0) va (t_0,y^0) nuqtalarning yetarli kichik atrofida f ning normasi 1 soni bilan yuqoridan chegaralangan (Bu f ning shu nuqtalar atrofida Lipshits shartini qanoatlantirishidan ravshan). (III.1.13) tengsizlikdan yetarli katta j lar uchun ushbu

$$2\ge j\left\| x^j-y^j\right\|$$

tengsizlikni hosil qilamiz. Bu tengsizlikda $j\to\infty$ deb limitga o'tamiz. Bunda $\left\| x^j-y^j\right\|\to\left\| x^0-y^0\right\|>0$ bo'lgani uchun yana ziddiyatga ($2\ge\infty$?) kelamiz.

Shunday qilib, farazimiz noto'g'ri va teorema isbotlandi.

6^0. ***Oshkormas funksiya to'g'risida.***

Ba'zan ushbu

$$\begin{cases} F_1(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) = 0, \\ F_2(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ F_n(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) = 0 \end{cases} \qquad \text{(III.1.14)}$$

sistemadan $y_1, y_2, \dots, y_n$ o'zgaruvchilarni $x_1, x_2, \dots, x_m$ o'zgaruvchilarning silliq funksiyalari sifatida topilishi uchun yetarli shartlar kerak bo'ladi. Qisqalik uchun, odatdagi belgilashlardan foydalanib, (III.1.14) sistemani vektorli ko'rinishda yozaylik

$$F(x, y) = 0. \qquad \text{(III.1.15)}$$

Teorema (oshkormas funksiya to'g'risidagi). *Aytaylik,*

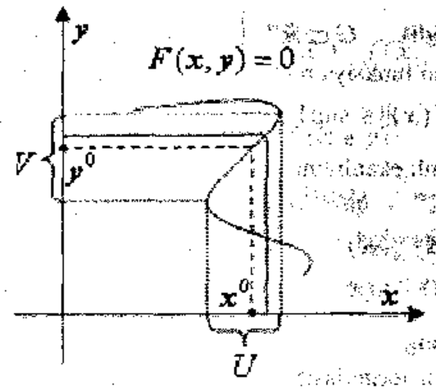
1^0. $x^0 \in \mathbb{R}^m$ *va* $y^0 \in \mathbb{R}^n$ *uchun* $F(x^0, y^0) = 0$;

2^0. $F(x, y)$ *funksiya* $(x^0, y^0) \in \mathbb{R}^{m+n}$ *nuqtaning biror atrofida* C^1 *sinfga tegishli;*

3^0. $(x^0, y^0) \in \mathbb{R}^{m+n}$ *nuqtada*

$$\det \frac{\partial F}{\partial y} \overset{def}{=} \begin{vmatrix} \frac{\partial F_1}{\partial y_1} & \dots & \frac{\partial F_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial y_1} & \dots & \frac{\partial F_n}{\partial y_n} \end{vmatrix} \neq 0$$

shartlar bajarilsin. U holda $x^0 \in \mathbb{R}^m$ *nuqtaning shunday* $U \subset \mathbb{R}^m$ *va* $y^0 \in \mathbb{R}^n$ *nuqtaning shunday* $V \subset \mathbb{R}^n$ *atroflari mavjudki, har qanday* $x \in U$ *uchun* (III.1.14) *(yoki (III.1.15)) tenglamalar sistemasi* V *da yagona* $y = f(x) \in V$ *yechimga ega va bu yerdagi* $f(x)$ *vektor-funksiya* $C^1(U, V)$ *sinfga tegishli bo'ladi.* (III.1- rasm).



III.1- rasm.

Oshkormas funksiya to'g'risidagi teoremaning xususiy holi bo'lgan teskari funksiya to'g'risidagi teoremani ham keltiraylik.

Teorema (teskari funksiya to'g'risidagi). *Aytaylik,* G – $\mathbb{R}^n$ *fazodagi ochiq to'plam,* $f : G \to \mathbb{R}^n$ *uzluksiz differensiallanuvchi vektor-funksiya va* $\det f'(a) \neq 0$, $a \in G$, *bo'lsin. U holda* a *nuqtani o'z ichiga olgan shunday* $U \subset G$ *ochiq to'plam va* $b = f(a)$ *nuqtani o'z ichiga olgan shunday* $V \subset \mathbb{R}^n$ *ochiq to'plamlar mavjudki,* $f : U \to V$ *funksiya* $f^{-1} : V \to U$ *uzluksiz differensiallanuvchi teskari funksiyaga ega va* $\forall\, y \in V$ *uchun*

$$(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1} \ (f^{-1}(b) = a).$$

Masalalar

1. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ matritsa uchun $\|A\|$ va $\|A\|^*$ normalarni hisoblang.

2. Teskarilanuvchi $A \in \mathbb{M}_{n\times n}(\mathbb{R})$ matritsa uchun

$$\|A^{-1}\|^* \geq \frac{1}{\|A\|^*}$$

ekanligini tekshiring.

3. Aytaylik, $G \subset \mathbb{R}^m$ – qavariq soha, $f : G \to \mathbb{R}^n$ differensiallanuvchi funksiya bo'lsin. $x \in G$ va $y \in G$ nuqtalar uchun

$$\| f(y) - f(x) \| \le \sup_{0<s<1} \| f'(x + s(y-x)) \| \cdot \| y - x \|$$

tengsizlikning o'rinli ekanligini isbotlang.

4. $G \subset \mathbb{R}^m$ – qavariq soha, $f : G \to \mathbb{R}^n$ differensiallanuvchi funksiya va $A \in \mathbb{M}_{n\times m}(\mathbb{R})$ bo'lsin. $x \in G$, $x + h \in G$ uchun ushbu

$$\| f(x+h) - f(x) - Ah \| \le \sup_{0<s<1} \| f'_x(x+sh) - A \| \cdot \| h \|$$

tengsizlikni isbotlang.

5. (Lagranj formulasi). $G \subset \mathbb{R}^m$ – ochiq to'plam va $f : G \to \mathbb{R}$ differensiallanuvchi funksiya berilgan bo'lsin. Agar $x \in G$ va $x + h \in G$ nuqtalarni tutashtiruvchi $\{x + sh \mid 0 \le s \le 1\}$ kesma G da joylashsa, u holda shunday $u \in (0;1)$ son mavjudki, uning uchun

$$f(x+h) - f(x) = f'(x + u\,h)\cdot h,$$

ya'ni

$$f(x_1 + h_1, \ldots, x_n + h_n) - f(x_1, \ldots, x_n) = \sum_{i=1}^{n} f'_{x_i}(x + uh) h_i$$

Lagranj formulasi o'rinli ekanligini ko'rsating.

6. $f(t, x) = \|x\|$, $x \in \mathbb{R}^n$, funksiya x bo'yicha $\mathbb{R}^n$ da Lipshits shartini qanoatlantirishini ko'rsating. Bu funksiya differensiallanuvchimi?

7. Agar $f : E \to \mathbb{R}^n$ $(E \subset \mathbb{R}^{1+m})$ vektor-funksiyaning $f_i : E \to \mathbb{R}$, $i = \overline{1,n}$, komponentalari E da Lipshits shartini qanoatlantirsa, u holda f ning o'zi ham E da Lipshits shartini qanoatlantirishini isbotlang. Teskari tasdiqning ham o'rinliligini ko'rsating.

8. $f(x) = \|x\|^2$, $x \in \mathbb{R}^n$, funksiya har qanday $K \subset \mathbb{R}^n$ kompaktda Lipshits shartini qanoatlantirishini ko'rsating.

$$f(t, x_1, x_2) = \left(\sqrt{|x_1 x_2|}; \frac{x_1 + x_2}{1 + t^2} \right)^T$$ funksiya $|t| < 1$, $|x_1| < 1$, $|x_2| < 1$ to'plamda x_1, x_2 bo'yicha Lipshits shartini qanoatlantiradimi? $|t| < 1$, $\varepsilon < |x_1| < 1$, $\varepsilon < |x_2| < 1$ $(0 < \varepsilon < 1)$ to'plamda-chi?

9. $E \subset \mathbb{R}^n$ to'plamda $f : E \to \mathbb{R}^n$ funksiya Lipshits shartini

qanoatlantirsin:

$$\exists L > 0 \ \forall \{x_1, x_2\} \subset E \ \ |f(x_1) - f(x_2)| \le L\,|x_1 - x_2|.$$

Ixtiyoriy $x \in \mathbb{R}^n$ uchun

$$\tilde{f}(x) = \inf_{y \in E} \{ f(y) + L\,|x - y| \}$$

deb, yangi funksiyani aniqlaylik. E to'plamda $\tilde{f} = f$ va $\tilde{f}$ funksiya $\mathbb{R}^n$ fazoda Lipshits shartini qanoatlantirishini ko'rsating ($\tilde{f}$ funksiya f ning E dan $\mathbb{R}^n$ gacha Lipshits sharti saqlangan holda davom ettirilishidir).

III.2. Umumiy ko'rinishdagi differensial tenglamalar sistemasini birinchi tartibli tenglamalar sistemasiga keltirish

Soddalik uchun ikkita $x = x(t)$ va $y = y(t)$ t skalyar argumentning noma'lum haqiqiy funksiyalariga nisbatan ushbu

$$\begin{cases} F(t, x, x', \ldots, x^{(m)}, y, y', \ldots, y^{(n)}) = 0 \\ G(t, x, x', \ldots, x^{(m)}, y, y', \ldots, y^{(n)}) = 0 \end{cases} \quad (m, n \in \mathbb{N}) \qquad \text{(III.2.1)}$$

differensial tenglamalar sistemasini qaraylik. Bu yerda F va G funksiyalari $m + n + 3$ dona haqiqiy o'zgaruvchilarning haqiqiy funksiyalari; ular $\mathbb{R}^{m+n+3}$ fazoning biror D sohasida aniqlangan va uzluksiz deb faraz qilinadi; t-erkli o'zgaruvchi, x va y lar t ning noma'lum funksiyalari. Agar x funksiyaning (III.2.1) sistemada qatnashgan hosilalarining maksimal tartibi m bo'lsa, u holda (III.2.1) sistema x ga nisbatan m- tartibli differensial tenglamalar sistemasi deyiladi. Differensial tenglamalar sistemasining y ga nisbatan tartibi shunga o'xshash aniqlanadi. Agar (III.2.1) sistema x ga nisbatan m- tartibli, y ga nisbatan esa n-tartibli bo'lsa, u holda $m + n$ soni (III.2.1) sistemaning tartibi deyiladi.

Masalan, ushbu

$$\begin{cases} x''' \cos t - tx^2 + yx' + y'' \sin t - (x')^4 = 0 \\ x''' \sin t + x'' + y'' \cos t - ty^3 - 1 = 0 \end{cases} \qquad \text{(III.2.2)}$$

sistema 5-tartibli bo'lib, u x ga nisbatan 3- y ga nisbatan esa 2- tartiblidir.

Agar $I \subset \mathbb{R}$ oraliq va $x = \varphi(t)$, $y = \psi(t)$ funksiyalar uchun

1) $\{\varphi(t), \psi(t)\} \subset C^1(I)$

2) $\begin{cases} F(t, x(t), x'(t), ..., x^{m}(t), y(t), y'(t), ..., y^{(n)}(t)) \equiv 0, \ \forall t \in I \\ G(t, x(t), x'(t), ..., x^{(m)}(t), y(t), y'(t), ..., y^{(n)}(t)) \equiv 0, \ \forall t \in I \end{cases}$

shartlar bajarilsa, u holda $x = \varphi(t)$, $y = \psi(t)$ funksiyalar (III.2.1) sistemaning I **oraliqda aniqlangan yechimi** deyiladi.

Agar (III.2.1) sistemani $x^{(m)}$, va $y^{(n)}$ hosilalarga nisbatan yechish mumkin bo'lsa, u holda uni

$$\begin{cases} x^{(m)} = f(t, x, x', ..., x^{(m-1)}, y, y', ..., y^{(n-1)}) \\ y^{(n)} = g(t, x, x', ..., x^{(n-1)}, y, y', ..., y^{(m-1)}) \end{cases} \qquad (III.2.3)$$

ko'rinishda yozish mumkin. (III.2.3) sistema yuqori hosilalarga nisbatan yechilgan deb ataladi.

Yuqorida misol sifatida keltirilgan (III.2.2) sistema quyidagicha yuqori hosilalarga nisbatan yechilgan ko'rinishga keltiriladi:

$$\begin{cases} x''' = (tx^2 + (x')^4 - yx')\cos t + (1 + ty^3 - x'')\sin t \\ y'' = (tx^2 + (x')^4 - yx')\sin t + (x'' - ty^3 - 1)\cos t \end{cases} \qquad (III.2.4)$$

$(n+m)$ – tartibli (III.2.1) sistema $(n+m)$ dona birinchi tartibli differensial tenglamalar sistemasiga keltirish mumkin. Buning uchun quyidagi belgilashlarni kiritaylik:

$$\begin{cases} x = x_1, x' = x_2, ..., x^{(m-1)} = x_m \\ y = x_{m+1}, \ y' = x_{m+2}, ..., y^{(n-1)} = x_{m+n} \end{cases} \qquad (III.2.5)$$

Bu belgilashlar natijasida (III.2.1) sistema o'rniga

$$\begin{cases} x_1' - x_2 = 0 \\ x_2' - x_3 = 0 \\ .... \\ x_{m-1}' - x_m = 0 \\ F(t, x_1, ..., x_m, x_m', x_{m+1}, ..., x_{n+m}, x_{m+n}') = 0 \\ x_{m+1}' - x_{m+2} = 0 \\ .... \\ x_{m+n-1}' - x_{m+n} = 0 \\ G(t, x_1, ..., x_m, x_m', x_{m+1}, ..., x_{n+m}, x_{m+n}') = 0 \end{cases} \qquad (III.2.6)$$

sistemaga kelamiz.

(III.2.5) belgilashlar natijasida (III.2.3) sistemadan esa ushbu

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ .... \\ x_{m-1}' = x_m \\ x_m' = f(t, x_1, ..., x_m, x_{m+1}, ...x_{m+n}) \\ x_{m+1}' = x_{m+2} \\ .... \\ x_{m+n-1}' = x_{m+n} \\ x_{m+n}' = g(t, x_1, ..., x_m, x_{m+1}, ...x_{m+n}) \end{cases} \qquad (III.2.7)$$

sistemani hosil qilamiz.

Shunday qilib, agar $x = \varphi(t)$ va $y = \psi(t)$ funksiyalar (III.2.1) (yoki (III.2.3)) sistemaning yechimi bo'lsa, u holda

$x_1 = \varphi(t), \ x_2 = \varphi'(t), ..., x_m = \varphi^{(m-1)}(t),$

$$x_{m+1} = \psi(t), \ x_{m+2} = \psi'(t), ..., x_{m+n} = \psi^{(n-1)}(t)$$

funksiyalar (III.2.6) (mos ravishda (III.2.7)) sistemaning yechimi bo'ladi. Aksincha, agar

$x_1 = x_1(t), x_2 = x_2(t), ..., x_m = x_m(t), x_{m+1} = x_{m+1}(t), ..., x_{m+n} = x_{m+n}(t)$

funksiyalar (III.2.6) (yoki (III.2.7)) sistemaning yechimi bo'lsa, u holda $x = x_1(t)$ va $y = x_{m+1}(t)$ funksiyalar (III.2.1) (mos ravishda (III.2.7)) sistemaning yechimi bo'ladi (tekshirib ko'ring).

Shunday qilib, (III.2.1) differensial tenglamalar sistemasi birinchi tartibli differensial tenglamalar sistemasini yechishga keltirildi. Yuqori hosilaga nisbatan yechilgan differensial tenglamalar sistemasi esa quyidagi **normal sistema** deb ataluvchi sistemani yechishga keltiriladi:

$$\begin{cases} x_1' = f_1(t, x_1, ..., x_k) \\ x_2' = f_2(t, x_1, ..., x_k) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_k' = f_k(t, x_1, ..., x_k) \end{cases} \qquad \text{(III.2.8)}$$

Shu munosabat bilan, biz asosan (III.2.8) ko'rinishdagi normal sistemalarni o'rganamiz.

Quyidagi birinchi tartibli differensial tenglamalar sistemasi berilgan bo'lsin:

$$\begin{cases} F_1(t, x_1, ..., x_n, x_1', ..., x_n') = 0 \\ F_2(t, x_1, ..., x_n, x_1', ..., x_n') = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ F_n(t, x_1, ..., x_n, x_1', ..., x_n') = 0 \end{cases} \qquad \text{(III.2.9)}$$

Ba'zi ma'lum shartlar bajarilganda bu sistemani yechishni bitta n-tartibli differensial tenglamani yechishga keltirish mumkin. Dastlab (III.2.9) ni $x_1', x_2', ..., x_n'$ hosilalarga nisbatan yechamiz (buning mumkinligi faraz qilinadi). Natijada ushbu

$$\begin{cases} x_1' = f_1(t, x_1, ..., x_n) \\ x_2' = f_2(t, x_1, ..., x_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_n' = f_n(t, x_1, ..., x_n) \end{cases} \qquad \text{(III.2.10)}$$

sistemani hosil qilamiz. Endi x_1 ga nisbatan bitta n-tartibli differensial tenglama hosil qilish uchun differensiallash va yo'qotish usulidan foydalanish mumkin. Bu usulga ko'ra (III.2.10) sistemadagi 1-tenglikni $n-1$ marta ketma-ket differensiallaymiz (buning mumkinligi faraz qilinadi) va bunda har bir qadamda (II.2.10) dagi tengliklardan foydalanib noma'lum funksiyalar hosilalarini yo'qotib boramiz:

$$x_1'' = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x_1} \cdot x_1' + ... + \frac{\partial f_1}{\partial x_n} \cdot x_n' = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial x_1} f_1 + ... + \frac{\partial f_1}{\partial x_n} f_n \equiv$$
$$\equiv g_2(t, x_1, ... x_n)$$

$$x_1''' = \frac{\partial g_2}{\partial t} + \frac{\partial g_2}{\partial x_1} f_1 + ... + \frac{\partial g_2}{\partial x_n} f_n \equiv g_3(t, x_1, ..., x_n)$$

....

$$x_1^{(n-1)} = \frac{\partial g_{n-1}}{\partial t} + \frac{\partial g_{n-1}}{\partial x_1} f_1 + ... + \frac{\partial g_{n-1}}{\partial x_n} f_n \equiv g_n(t, x_1, ..., x_n)$$

Endi $x_2, x_3, ..., x_n$ o'zgaruvchilarni yo'qotish uchun quyidagi sistemani tuzamiz:

$$\begin{cases} x_1' = g_1(t, x_1, ..., x_n) \qquad (f_1 = g_1) \\ x_1'' = g_2(t, x_1, ..., x_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_1^{(n)} = g_n(t, x_1, ..., x_n) \end{cases} \qquad \text{(III.2.11)}$$

Bu sistemaning dastlabki $n-1$ ta tenglamasidan

$$x_2 = x_2(t,\ x_1,\ x_1',...,x_1^{(n-1)})$$

$$..................................... \qquad (III.2.12)$$

$$x_n = x_n(t,\ x_1,\ x_1',...,x_1^{(n-1)})$$

larni topib (buning mumkinligi faraz qilinadi), oxirgi tenglamaning o'ng tomoniga qo'yamiz. Natijada n- tartibli yuqori hosilaga nisbatan yechilgan tenglamaga kelamiz:

$$x_1^{(n)} = g_n(t, x_1,\ x_2(t,\ x_1,\ x_1',...,x_1^{(n-1)}),....,x_n(t,\ x_1,\ x_1',...,x_1^{(n-1)})) \equiv$$
$$\equiv g(t,\ x_1,\ x_1',...,x_1^{(n-1)}). \qquad (III.3.13)$$

Ba'zi shartlar bajarilganda (III.3.13) tenglamadan topilgan $x_1 = x_1(t)$ yechim va unga ko'ra (III.2.12) tengliklar yordamida aniqlangan $x_2 = x_2(t),...,x_n = x_n(t)$ funksiyalar (III.2.10) normal sistemaning yechimini tashkil etishini ko'rsatish mumkin. Biz bunda to'xtalmaymiz.

Biz yuqorida ikki noma'lum funksiya qatnashgan differensial tenglamalar sistemasi (III.2.1) uchun yechim tushunchasini kiritdik, uni birinchi tartibli differensial tenglamalar sistemasiga keltirish bilan shug'ullandik. Ixtiyoriy chekli sondagi $x_1, x_2, ..., x_n$ noma'lum funksiyalarga nisbatan ushbu

$$\begin{cases} F_1(t, x_1, x_1', ..., x_1^{(m_1)}, x_2, x_2', ..., x_2^{(m_2)}, ..., x_n, x_n', ..., x_n^{(m_n)}) = 0 \\ .... \\ F_n(t, x_1, x_1', ..., x_1^{(m_1)}, x_2, x_2', ..., x_2^{(m_2)}, ..., x_n, x_n', ..., x_n^{(m_n)}) = 0 \end{cases}$$

sistema uchun ham yechim tushunchasi yuqoridagiga o'xshash kiritiladi. Bu sistemani ham normal sistemani yechishga keltirish mumkin (ba'zi shartlar bajarilganda).

Izoh. Ba'zi hollarda n- tartibli normal sistema bir dona n- tartibli tenglamaga keltirilmaydi.

Masalan, ushbu

$$\begin{cases} x' = x \\ y' = y \end{cases}$$

ikkinchi tartibli tenglamalari ajralgan normal sistema, tushunarliki,

bir dona ikkinchi tartibli differensial tenglamaga keltirilmaydi. Bu sistemaning yechimi $x = c_1e^t$, $y = c_2e^t$ (c_1, c_2 – const).

Yuqori tartibli differensial tenglamalar sistemasidan umumiy holda bitta noma'lum funksiyaga nisbatan bitta yuqori tartibli differensial tenglama hosil qilish mumkin. Bunda ham differensiallash va yo'qotish usulidan foydalaniladi. Ba'zi hollarda esa yo'qotish jarayonida oliy algebraning rezultantlar metodini ishlatish mumkin.

Misol. Ushbu

$$\begin{cases} x' - xy - y = 0 \\ y' - x^2 + 2y^2 + x = 0 \end{cases}$$

sistemadagi $y = y(t)$ noma'lum funksiya qanoatlantiruvchi bitta differensial tenglamani topaylik.

⊶ Berilgan sistemadagi ikkinchi tenglamani differensiallaymiz va bunda hosil bo'luvchi x' hosilani birinchi tenglamadan $x' = xy + y$ ekanligini topib, yo'qotamiz:

$$y'' - 2xx' + 4yy' + x' = 0\ ,\ y'' - 2x(xy + y) + 4yy' + xy + y = 0,$$

$$y'' + 4yy' - yx - 2yx^2 = 0.$$

Endi berilgan sistemaning ikkinchi tenglamasi va hosil qilingan tenglamadan tuzilgan

$$\begin{cases} y' + 2y^2 + x - x^2 = 0 \\ y'' + 4yy' + y - yx - 2yx^2 = 0 \end{cases}$$

sistemadan x noma'lumni yo'qotish kerak. Bu ishni radikallarsiz bajarish mumkin. Buning uchun oxirgi sistemaning tenglamalarini x ga ko'paytirib, $1, x, x^2, x^3$ noma'lumlarga nisbatan quyidagi chiziqli bir jinsli algebraik sistemani tuzaylik:

$$\begin{pmatrix} y' + 2y^2 & 1 & -1 & 0 \\ 0 & y' + 2y^2 & 1 & -1 \\ y'' + 4yy' + y & -y & -2y & 0 \\ 0 & y'' + 4yy' + y & -y & -2y \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Bu sistema notrivial yechimga ega bo'lganligi sababli (masalan, yechimning birinchi komponentasi noldan farqli: $1 \neq 0$) uning determinanti nolga teng bo'lishi kerak:

$$\begin{vmatrix} y'+2y^2 & 1 & -1 & 0 \\ 0 & y'+2y^2 & 1 & -1 \\ y''+4yy'+y & -y & -2y & 0 \\ 0 & y''+4yy'+y & -y & -2y \end{vmatrix} = 0.$$

Bu yerdagi determinantni hisoblab, $y = y(t)$ noma'lum funksiyaga nisbatan quyidagi ikkinchi tartibli differensial tenglamani hosil qilamiz:

$$y''^2 + y(4y' - 8y^2 - 1)y'' + 4y^2y'^2 - 11y^2y' + 2(8y^2 - 7)y^4 = 0.$$

Masalalar

1. Ushbu

$$\begin{cases} x' - xy - y = 0 \\ y' - x^2 + y^2 + 2x = 0 \end{cases}$$

sistemadan $y = y(t)$ noma'lum funksiya qanoatlantiruvchi bitta oddiy differensial tenglama hosil qiling.

2. Ushbu

$$\begin{cases} x' - xy - y^2 + x^3 = 0 & (1) \\ y' + y^2 - xy - x^2 = 0 & (2) \end{cases}$$

sistemadan $x = x(t)$ noma'lum funksiya uchun bir dona differensial tenglama tuzing.

III.3. Mavjudlik va yagonalik teoremasi

Vektor ko'rinishda yozilgan differensial tenglamalar sistemasi uchun quyidagi Koshi masalasini qaraylik:

$$\begin{cases} \boldsymbol{x}' = \boldsymbol{f}(t, \boldsymbol{x}) & \text{(III.3.1)} \\ \boldsymbol{x}|_{t_0} = \boldsymbol{x}^0 & \text{(III.3.2)} \end{cases}$$

Bu yerda $\boldsymbol{x} = \boldsymbol{x}(t) - n \times 1$ o'lchamli noma'lum vektor-funksiya, $\boldsymbol{f}(t, \boldsymbol{x})$ vektor-funksiya $D \subset \mathbb{R}^{1+n}$ sohada aniqlangan va uzluksiz, $\boldsymbol{f}(t, \boldsymbol{x}) \in C(D, \mathbb{R}^n)$, va $(t_0, \boldsymbol{x}^0) \in D$. Bu Koshi masalasini yechish (III.3.1) sistemaning biror I, $t_0 \in I$, oraliqda aniqlangan va (III.3.2) boshlang'ich shartlarni qanoatlantiruvchi yechimini topish demakdir.

(III.3.1), (III.3.2) Koshi masalasining skalyar ko'rinishi

$$\begin{cases} x_1' = f_1(t, x_1, x_2, \ldots, x_n) \\ x_2' = f_2(t, x_1, x_2, \ldots, x_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_n' = f_n(t, x_1, x_2, \ldots, x_n) \\ x_1|_{t_0} = x_1^0,\ x_2|_{t_0} = x_2^0,\ \ldots,\ x_n|_{t_0} = x_n^0 \end{cases}$$

(III.3.1), (III.3.2) Koshi masalasiga quyidagi vektor ko'rinishda yozilgan integral tenglamalar sistemasini mos qo'yaylik:

$$\boldsymbol{x}(t) = \boldsymbol{x}^0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{x}(s))ds \qquad \text{(III.3.3)}$$

Agar I oraliqda aniqlangan $\boldsymbol{\varphi}: I \to \mathbb{R}^n$ vektor-funksiya uchun

1) $\boldsymbol{\varphi} \in C(I, \mathbb{R}^n) - \boldsymbol{\varphi}$ vektor-funksiya I da uzluksiz

2) $\forall t \in I$ uchun $\boldsymbol{\varphi}(t) = \boldsymbol{x}^0 + \int_{t_0}^{t} \boldsymbol{f}(s, \boldsymbol{\varphi}(s))ds$,

ya'ni $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ vektor-funksiya I da (III.3.3) ni qanoatlantiradi shartlar bajarilsa, u holda $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ vektor-funksiya (III.3.3) integral tenglamalar sistemasining I oraliqda yechimi deyiladi.

Ekvivalentlik lemmasi. $t_0 \in I$, $(t_0, \boldsymbol{x}^0) \in D$ *bo'lsin.* $\boldsymbol{x} = \boldsymbol{\varphi}(t)$ *vektor-funksiya (III.3.1), (III.3.2) Koshi masalasining yechimi bo'lishi uchun uning (III.3.3) integral tenglama yechimi bo'lishi yetarli va zarurdir.*

Isboti bevosita tekshirish yo'li bilan amalga oshiriladi.

Endi (III.3.1), (III.3.2) Koshi masalasi lokal (t_0 nuqtaning biror atrofida aniqlangan) yechimining mavjudligi va yagonaligi to‘g‘risidagi teorema (MYaT)ni keltiramiz. U Koshi-Pikar-Lindelyof teoremasi deb ham yuritiladi.

Teorema (Koshi-Pikar-Lindelyof, MYaT). *Aytaylik,*
$S=\{(t,x)^T\in\mathbb{R}^{1+n}\mid |t-t_0|\le a\ ,\ \|x-x^0\|\le b\}\qquad (a>0,b>0)$ *silindr,* $f(t,x)\in C(S,\mathbb{R}^n)$ *va u* S *da* $x=(x_1,x_2,\ldots,x_n)$ *bo‘yicha Lipshits shartini qanoatlantirsin. S kompakt bo‘lgani uchun S da uzluksiz* f *vektor-funksiya chegaralangan :*

$$\exists M>0\ \forall (t,x)^T\in S\ \ \|f(t,x)\|\le M \qquad \text{(III.3.4)}$$

$h=\min\left\{a,\frac{b}{M}\right\}$ *deylik. U holda (III.3.1), (III.3.2) Koshi masalasining* $t\in[t_0-h;t_0+h]$ *segmentda aniqlangan yechimi mavjud va bu yechim yagonadir.*

🗝 $n=1$ holidagidek ish tutamiz.

(III.3.1), (III.3.2) Koshi masalasining o‘rniga unga ekvivalent bo‘lgan (III.3.3) integral tenglamani yechamiz. Yechimning mavjudligini ketma-ket yaqinlashishlar metodi yordamida isbotlaymiz. $|t-t_0|\le h$ segmentda ketma-ket yaqinlashishlar deb ataluvchi $x^0(t),x^1(t),\ldots,x^k(t),\ldots$ vektor-funksiyalar ketma-ketligini quyidagicha (rekurrent usulda) aniqlaylik:

$$x^0(t)=x^0,$$

$$x^k(t)=x^0+\int_{t_0}^{t}f(s,x^{k-1}(s))ds,\ k\in\mathbb{N}\ . \qquad \text{(III.3.5}_k\text{)}$$

Bu yerdagi barcha integrallar mavjud bo‘lishi uchun $|t-t_0|\le h$ bo‘lganda har qanday $k=0,1,2,\ldots$ uchun $(t,x^k(t))^T\in S$ bo‘lishini ko‘rsatish kerak. $k=0$ bo‘lganda bu tushunarli. $k=1$ da (III.3.5$_1$) formuladan $t\in[t_0-h,t_0+h]$ lar uchun quyidagi baholashlarni bajaramiz:

$$\|x^1(t)-x^0\|=\left\|\int_{t_0}^{t}f(s,x^0(s))ds\right\|\le\left|\int_{t_0}^{t}\|f(s,x^0)\|ds\right|\le M\left|\int_{t_0}^{t}1ds\right|=$$

$$=M\,|t-t_0|\le Mh\le M\frac{b}{M}=b.$$

Demak, $|t-t_0|\le h$ bo‘lganda $(t,x^1(t))^T\in S$ va $x^2(t)$ yaqinlashish aniqlangan. Endi matematik induksiyani qo‘llaymiz. Faraz qilaylik, $|t-t_0|\le h$ bo‘lganda $(t,x^k(t))^T\in S$ bo‘lsin. Biz $|t-t_0|\le h$ bo‘lganda $(t,x^{k+1}(t))^T\in S$ ekanligini ko‘rsatishimiz kerak. Farazimizga ko‘ra

$$x^{k+1}(t)=x^0+\int_{t_0}^{t}f(s,x^k(s))ds$$

funksiya $|t-t_0|\le h$ oraliqda aniqlangan. Demak,

$$\|x^{k+1}(t)-x^0\|=\left\|\int_{t_0}^{t}f(s,x^k(s))ds\right\|\le\left|\int_{t_0}^{t}\|f(s,x^k(s))\|ds\right|\le$$

$$\le M\,|t-t_0|\le Mh\le b,$$

ya’ni $|t-t_0|\le h$ bo‘lganda $(t,x^{k+1}(t))^T\in S$. Shunday qilib, o‘sha t lar uchun (III.3.5$_k$) dagi barcha integrallar mavjud hamda $x^k(t)$ larning hammasi uzluksiz funksiyalardan iborat bo‘ladi. (Aslida x^k lar C^1 sinfga tegishli. Nega?).

Qurilgan $x^k(t)$ yaqinlashishlar $[t_0-h;t_0+h]$ segmentda tekis yaqinlashuvchi bo‘ladi. Buni isbotlash uchun ushbu

$$x^0(t)+\left(x^1(t)-x^0(t)\right)+\left(x^2(t)-x^1(t)\right)+\ldots+$$
$$+\left(x^{k+1}(t)-x^k(t)\right)+\ldots \qquad \text{(III.3.6)}$$

vektor-funksiyalardan tuzilgan funksional qatorning tekis yaqinlashuvchi ekanligini ko‘rsatish kifoya. Buni ko‘rsatish uchun esa funksional qatorning tekis yaqinlashishi to‘g‘risidagi Veyershtrass alomatidan foydalanamiz. Buning uchun (III.3.6) qator hadlarini normasi bo'yicha yuqoridan baholaymiz. (III.3.5$_1$) formuladan

$$\left\| x^1(t) - x^0(t) \right\| = \left\| \int_{t_0}^{t} f(s, x^0) ds \right\| \le M \left| t - t_0 \right| . \qquad \text{(III.3.7}_1\text{)}$$

Agar L bilan Lipshits doiymisini belgilasak, ya'ni ixtiyoriy $(t,x) \in S$ va $(t,\tilde{x}) \in S$ nuqtalar uchun $\left\| f(t,x) - f(t,\tilde{x}) \right\| \le L \left\| x - \tilde{x} \right\|$ bo'lsa, u holda matematik induksiya prinsipi yordamida ko'rsatish mumkinki, $\forall k \in \mathbb{N}$ uchun

$$\left\| x^k(t) - x^{k-1}(t) \right\| \le ML^{k-1} \frac{\left| t - t_0 \right|^k}{k!}, t \in [t_0 - h; t_0 + h], \qquad \text{(III.3.7}_k\text{)}$$

tengsizliklar o'rinli bo'ladi. (III.3.7$_k$) tengsizlikdan tekis yaqinlashish to'g'risidagi Veyershtrass alomatiga ko'ra (III.3.6) qatorning $\left| t - t_0 \right| \le h$ bo'lganda tekis yaqinlashishi $n = 1$ holidagiga o'xshash kelib chiqadi. Shunday qilib, $x^k(t)$ funksional ketma-ketlik tekis yaqinlashuvchi. Uning limitini $\varphi(t)$ bilan belgilaylik :

$$\lim_{k \to \infty} x^k(t) = \varphi(t), \ t \in [t_0 - h, t_0 + h]. \qquad \text{(III.3.8)}$$

Uzluksiz funksiyalarning tekis limiti sifatida $\varphi(t)$ funksiya uzluksiz bo'ladi.

$\varphi(t)$ ning $[t_0 - h; t_0 + h]$ da (III.3.3) integral tenglama yechimi ekanligi $n = 1$ holidagidek isbotlanadi.

Endi $[t_0 - h; t_0 + h]$ segmentda aniqlangan (III.3.3) ning boshqa yechimi yo'qligini ko'rsataylik. $x = \psi(t)$ uning $[t_0 - h; t_0 + h]$ segmentda aniqlangan ixtiyoriy yechimi bo'lsin:

$$\psi(t) = x^0 + \int_{t_0}^{t} f(s, \psi(s)) ds, \quad t \in [t_0 - h; t_0 + h]$$

Yuqorida qurilgan $x^k(t)$ ketma-ket yaqinlashishlar bilan $\psi(t)$ orasidagi farqni baholaymiz. Ravshanki,

$$\left\| \psi(t) - x^0(t) \right\| = \left\| \psi(t) - x^0 \right\| = \left\| \int_{t_0}^{t} f(s, \psi(s)) ds \right\| \le M \left| t - t_0 \right| .$$

Matematik induksiya prinsipi yordamida $\forall k \in \mathbb{N}$ uchun

$$\left\| \psi(t) - x^k(t) \right\| \le ML^k \frac{\left| t - t_0 \right|^{k+1}}{(k+1)!}, \ t \in [t_0 - h, t_0 + h], \qquad \text{(III.3.9)}$$

ekanligini ko'ramiz. (III.3.9) da $k \to \infty$ deb, (III.3.1) ga ko'ra $\left\| \psi(t) - \varphi(t) \right\| \le 0$ ekanligini hosil qilamiz. Oxirgi tengsizlik $\forall t \in [t_0 - h, t_0 + h]$ uchun $\psi(t) = \varphi(t)$ bo'lishini ko'rsatadi. Teoremaning yagonalik qismi ham isbotlandi. ♁

Eslatma. $\psi(t) = \varphi(t), \left| t - t_0 \right| \le h$, bo'lgani uchun (III.3.9) dan ketma-ket yaqinlashishlarning xatoligini baholovchi tengsizlikni hosil qilamiz:

$$\left\| \varphi(t) - x^k(t) \right\| \le ML^k \frac{\left| t - t_0 \right|^{k+1}}{(k+1)!}, \ \left| t - t_0 \right| \le h .$$

Teorema. *Aytaylik,* $G \subset \mathbb{R}^{1+n}$ – *soha,* $f : G \to \mathbb{R}^n$ *uzluksiz va u* G *da lokal Lipshits shartini qanoatlantirsin. U holda* G *sohaning ixtiyoriy* (t_0, x^0) *nuqtasidan* (III.3.1) *tenglamaning integral chizig'i o'tadi. Bunda* (t_0, x^0) *nuqta orqali o'tuvchi ixtiyoriy ikki yechim ularning umumiy aniqlanish oralig'ida ustma-ust tushadi.*

⊸ $(t_0, x^0) \in G$ nuqtadan o'tuvchi integral chiziqning mavjudligini isbotlaylik. (t_0, x^0) nuqta uchun $a > 0$ va $b > 0$ sonlarini shunday kichik tanlaylikki, ularga ko'ra qurilgan ushbu

$$S = \left\{ (t, x)^T \in \mathbb{R}^{1+n} \middle| \ \left| t - t_0 \right| \le a \ , \left\| x - x^0 \right\| \le b \right\}$$

silindr G da joylashsin: $S \subset G$. Agar kerak bo'lsa, a va b larni kichraytirib, $f(t, x)$ funksiya S da x bo'yicha Lipshits shartini qanoatlantiradi deb, hisoblaymiz. (Aslida bunga hojat yo'q, chunki 26- betdagi teoremaga ko'ra G dagi har qanday kompaktda f

funksiya x bo'yicha Lipshits shartini qanoatlantiradi (albatta har bir kompaktda o'zining Lipshits konstantasi bilan). Endi shu S ga tatbiq etilgan Koshi-Pikar-Lindelyof teoremasidan $(t_0, x^0) \in G$ nuqta orqali o'tuvchi integral chiziqning mavjudligi ravshan.

Faraz qilaylik, I_1 oraliqda aniqlangan $\varphi_1(t)$ va I_2 oraliqda aniqlangan $\varphi_2(t)$ yechimlar $(t_0, x^0) \in G$ nuqta orqali o'tsin, $\varphi_1(t_0) = \varphi_2(t_0) = x^0$ $(t_0 \in I_1 \cap I_2)$. Biz $I_1 \cap I_2$ oraliqda $\varphi_1(t) = \varphi_2(t)$ ekanligini ko'rsatishimiz kerak. t_0 ning o'ng tomonidagi $t \in I_1 \cap I_2$ nuqtalarda $\varphi_1(t) = \varphi_2(t)$ ekanligini isbotlaymiz. t_0 ning chap tomoni uchun isbot shunga o'xshash bajariladi.

Agar t_0 dan o'ngdagi biror $\tilde{t} \in I_1 \cap I_2$ nuqtada $\varphi_1(\tilde{t}) \neq \varphi_2(\tilde{t})$ bo'lsa, $\tau = \inf\{s \mid \forall t \in (s, \tilde{t}]\ \varphi_1(t) = \varphi_2(t)\}$ deymiz. Ravshanki, $t_0 < \tau < \tilde{t}$ va $\varphi_1(\tau) = \varphi_2(\tau)$, chunki, agar $\varphi_1(\tau) \neq \varphi_2(\tau)$ bo'lganda edi, $\varphi_1(t)$ va $\varphi_2(t)$ uzluksiz bo'lgani uchun τ nuqtaning chap tomonidagi unga yaqin t nuqtalarda ham $\varphi_1(t) \neq \varphi_2(t)$ bo'lardi. Bu esa τ ning inf ekanliliga zid.

Shunday qilib, $\varphi_1(\tau) = \varphi_2(\tau) = \tilde{x}$, lekin τ ning o'ng tomonidagi τ ga yetarlicha yaqin barcha t lar uchun $\varphi_1(t) \neq \varphi_2(t)$. Bu $(\tau, \tilde{x}) \in G$ nuqtadan ikkita integral chiziq chiqqanligini anglatadi.

Bu esa shu nuqta uchun tatbiq etilgan MYaT ning yechimning yagonaligi to'g'risidagi xulosasiga zid. ♁

Umumiy holda $(t_0, x_0) \in G$ nuqta orqali o'tuvchi yechimning MYaT (Koshi-Pikar-Lindelyof) teoremasi ta'minlovchi mavjudlik segmenti $[t_0 - h, t_0 + h]$ ning uzunligi $2h$ shu (t_0, x_0) nuqtaga bog'liq bo'ladi. Ya'ni turli $(t_0, x_0) \in G$ nuqtalar uchun bu segmentning uzunligi umumiy holda har xil. Shu munosabat bilan quyidagi teoremani keltiramiz.

Teorema. *$G \subset \mathbb{R}^{1+n}$ – soha, $f(t,x) \in C(G, \mathbb{R}^n)$ va f funksiya G da (x bo'yicha) lokal Lipshits shartini qanoatlantirsin. U holda G da yotuvchi ixtiyoriy K kompakt uchun shunday $\delta = \delta(K)$, $\delta > 0$, soni topiladiki, K ning ixtiyoriy (t_0, x^0) nuqtasidan kamida $[t_0 - \delta, t_0 + \delta]$ segmentda aniqlangan yechim o'tadi (bu yerda δ soni $(t_0, x^0) \in K$ nuqtaga bog'liq emas, u K ga bog'liq xolos).*

⊸ $K \subset G$ ixtiyoriy kompakt bo'lsin. K dan G ning chegarasi ∂G gacha bo'lgan masofani d deylik: $d = \mathrm{dist}(K, \partial G)$. K to'plam G sohada yotuvchi kompakt bo'lgani uchun $K \cap \partial G \neq \varnothing$ va, demak, $d > 0$ bo'ladi. $G_0 = \{x \in G \mid \mathrm{dist}(x, K) < d/2\}$ deylik. Ravshanki, G_0 – ochiq va $K \subset G_0 \subset \bar{G}_0 \subset G$. f funksiya G da uzluksiz bo'lgani uchun u G ning qismi bo'lmish $\bar{G}_0$ kompaktda ham uzluksiz hamda chegaralangan:

$$\|f(t,x)\| \leq M, \ (t,x) \in \bar{G}_0$$

Ixtiyoriy $(t_0, x^0) \in K$ nuqta uchun $S = \{(t,x) \in \mathbb{R}^n \mid |t - t_0| \leq d/2, \| x - x^0 \| \leq d/2\}$ silindrni qaraylik. Tushunarliki, $S \subset \bar{G}_0$. Demak, S da $\|f(t,x)\| \leq M$ va $f(t,x)$ funksiya x bo'yicha Lipshits shartini qanoatlantiradi. Endi Koshi-Pikar-Lindelyof teoremasidan ravshanki, $\delta = \min\{\frac{d}{2}, \frac{d}{2M}\} > 0$ soni hamma $(t_0, x^0) \in K$ nuqtalar uchun umumiy $h = \delta$ bo'lib xizmat qiladi. ♁

Misol 1. Ushbu

$$\begin{cases} x' = 2x + y, \\ y' = x + 2y, \\ x(0) = 1,\ y(0) = -1; \end{cases} \qquad (x = x^1, y = x^2)$$

Koshi masalasini yechishga ketma-ket yaqinlashishlar metodini tatbiq etaylik.

🔑 Ixtiyoriy

$S=\{(t,x,y)\in\mathbb{R}^3 \mid |t|<a,\ (x-1)^2+(y+1)^2<b^2\}\ (a>0,b>0)$

silindrda MYaTning shartlari bajariladi: $f_1(x,y)=2x+y$, $f_2(x,y)=x+2y$ funksiyalari S da uzluksiz va x,y o'zgaruvchilari bo'yicha Lipshits shartini qanoatlantiradi.

Berilgan masalaga ekvivalent integral tenglamalar sistemasini yozamiz:

$$x(t)=1+\int_0^t[2x(s)+y(s)]ds$$

$$y(t)=-1+\int_0^t[x(s)+2y(s)]ds$$

Nolinchi yaqinlashish: $x^0(t)=1$, $y^0(t)=-1$.

Birinchi yaqinlashish:

$$x^1(t)=1+\int_0^t[2x^0(s)+y^0(s)]ds=1+\int_0^t[2-1]ds=1+t,$$

$$y^1(t)=-1+\int_0^t[x^0(s)+2y^0(s)]ds=-1+\int_0^t[1-2]ds=-1-t.$$

Ikkinchi yaqinlashish:

$$x^2(t)=1+\int_0^t[2x^1(s)+y^1(s)]ds=1+\int_0^t[2(1+s)-1-s]ds=1+t+\frac{t^2}{2},$$

$$y^2(t)=-1+\int_0^t[x^1(s)+2y^1(s)]ds=-1+\int_0^t[1+s+2(-1-s)]ds=$$

$$=-1-t-\frac{t^2}{2}.$$

....

k – yaqinlashish ham shunga o'xshash topiladi. Undan esa

$$\lim_{k\to\infty}x^k(t)=e^t\ ,\ \lim_{k\to\infty}y^k(t)=-e^t$$

ekanligi hosil bo'ladi. Bevosita tekshirib ko'rish mumkinki, $x=e^t$, $y=-e^t$ funksiyalar berilgan Koshi masalasining yechimidir. ✍

Misol 2. Ushbu

$$\begin{cases}x'=x^2\sin y+ty^2\\ y'=x^3y\end{cases}$$

sistemaning o'ng tomoni $f_1(t,x,y)=x+2yx^2\sin y+ty^2$, $f_2(t,x,y)=x^3y$ barcha $(t,x,y)\in\mathbb{R}^3$ nuqtalarda uzluksiz va ixtiyoriy kompaktda chegaralangan

$$\frac{\partial f_1}{\partial x}=2x\sin y,\ \frac{\partial f_1}{\partial y}=x^2\cos y+2ty,\ \frac{\partial f_2}{\partial x}=3x^2y,\ \frac{\partial f_2}{\partial y}=x^3$$

xususiy hosilalarga ega, ya'ni f_1 va f_2 lar $\mathbb{R}^3$ da lokal Lipshits shartini qanoatlantiradi. Demak, ixtiyoriy $(t,x,y)\in\mathbb{R}^3$ nuqtadan bu sistemaning yagona integral chizig'i o'tadi. ✍

Teorema (yechimning global mavjudligi to'g'risida). *Faraz qilaylik,* (III.3.3) *sistemaning o'ng tomonidagi* $\boldsymbol{f(t,x)}$ *funksiya* $\forall t\in[a;b]$, $\forall \boldsymbol{x}\in\mathbb{R}^n$ *bo'lganda aniqlangan va uzluksiz, hamda* $\boldsymbol{x}$ *vektor o'zgaruvchi bo'yicha Lipshits shartini qanoatlantirsin. U holda* $\forall t_0\in[a;b]$ *va* $\forall x^0\in R^n$ *uchun* (K) *masalasining birato'la* $[a;b]$ *da aniqlangan yechimi mavjud va yagonadir.*

🔑 Yana (III.3.5) ketma-ket yaqinlashishlarni tuzaylik. $(5_1),...,(5_k),...$ integrallar endi $\forall t\in[a;b]$ uchun ma'noga ega, chunki $f(s,\boldsymbol{x}(s))$ vektor funksiya $\forall s\in[a;b]$ uchun aniqlangan. Teoremaning shartiga ko'ra

$$\|f(t,\boldsymbol{x})-f(t,\boldsymbol{x}^0)\|\le L\|\boldsymbol{x}-\boldsymbol{x}^0\|.$$

Bundan

$$\|f(t,x)\| \le \|f(t,x^0)\| + L\|x - x^0\|$$

yoki

$$\|f(t,x)\| \le A + L\|x - x^0\|,\ A \overset{def}{=} \max_{t\in[a;b]}\|f(t,x^0)\|. \qquad \text{(III.3.10)}$$

Endi (III.3.10) baholashdan foydalanib, (III.3.5) ketma-ket yaqinlashishlar uchun quyidagilarni hosil qilamiz.

$$\|x^1(t) - x^0(t)\| \le A\cdot|t - t_0|$$

$$\|x^2(t) - x^1(t)\| \le \left\|\int_{t_0}^{t}\left[f(s,x^1(s)) - f(s,x^0(s))\right]ds\right\| \le$$

$$\le \left|L\int_{t_0}^{t}\|x^1(s) - x^0(s)\|ds\right| \le L\cdot A\frac{|t-t_0|^2}{2!},$$

....

$$\|x^k(t) - x^{k-1}(t)\| \le \left|L\int_{t_0}^{t}\|x^{k-1}(s) - x^{k-2}(s)\|ds\right| \le L^{k-1}A\frac{|t-t_0|^k}{k!},$$

....

Hosil qilingan bu baholashlarda $t\in[a;b]$, $t_0\in[a;b]$ bo'lishi kerak. Demak, (III.3.6) funksional qator $t\in[a;b]$ da tekis yaqinlashuvchi. Qolgan fikr yuritishlar Koshi-Pikar-Lindelyof teoremasining isbotidagi kabidir. ♁

MYaTdan yuqori tartibli hosilaga nisbatan yechilgan skalyar noma'lum funksiya uchun Koshi masalasining yechimi borligi va yagonaligi to'g'risidagi teoremani keltirib chiqaraylik.

Skalyar t o'zgaruvchining skalyar no'malum funksiyasi $y = y(t)$ ga nisbatan ushbu

$$y^{(n)} = g(t, y, y', \ldots, y^{(n-1)}) \qquad \text{(III.3.11)}$$

$n-$ tartibli yuqori hosilaga nisbatan yechilgan differensial tenglamani qaraylik. Quyidagi boshlang'ich shartlarni qo'yaylik:

$$y\big|_{t_0} = y_0,\ y'\big|_{t_0} = y'_0, \ldots,\ y^{(n-1)}\big|_{t_0} = y_0^{(n-1)} \qquad \text{(III.3.12)}$$

(III.3.11) tenglamani

$$y = x_1,\ y' = x_2, \ldots,\ y^{(n-1)} = x_n \qquad \text{(III.3.13)}$$

o'zgaruvchilarga nisbatan birinchi tartibli differensial tenglamalar sistemasiga keltiramiz:

$$\begin{cases} x'_1 = x_2 \\ x'_2 = x_3 \\ \vdots \\ x'_{n-1} = x_n \\ x'_n = g(t, x_1, \ldots, x_n) \end{cases} \qquad \text{(III.3.14)}$$

Bu sistemaning vektor ko'rinishi

$$x' = f(t,x),\ f(t,x) = \left(x_2, x_3, \ldots, x_n, g(t, x_1, \ldots, x_n)\right)^T. \qquad \text{(III.3.15)}$$

(III.3.12) boshlang'ich shartlar

$$x_1\big|_{t_0} = y_0,\ x_2\big|_{t_0} = y'_0, \ldots,\ x_n\big|_{t_0} = y_0^{(n-1)} \qquad \text{(III.3.16)}$$

ko'rinishga mos keladi. Uni

$$x\big|_{t_0} = x^0\ \ (x_1^0 = y_0,\ x_2^0 = y'_0 \ldots,\ x_n^0 = y_0^{(n-1)}) \qquad \text{(III.3.17)}$$

vektor ko'rinishida yozamiz.

Shunday qilib, (III.3.15), (III.3.17) Koshi masalasi hosil bo'ldi. Agar (t_0, x^0) nuqtaning biror atrofida $g(t,x)$ haqiqiy funksiya uzluksiz va uning $\dfrac{\partial g}{\partial x_1}, \ldots, \dfrac{\partial g}{\partial x_n}$ xususiy hosilalari chegaralangan bo'lsa, u holda (III.3.15) dagi $f(t,x)$ vektor funksiya x vektor o'zgaruvchi bo'yicha shu atrofda Lipshits shartini qanoatlantiradi. Demak, bu holda normal sistema uchun Koshi–Pikar–Lindelyof teoremasini qo'llab, (III.3.11), (III.3.13) Koshi masalasi yechimining mavjudligi va yagonaligini ko'ramiz.

Agar (III.3.1) sistemaning o'ng tomonidan faqat uzluksizlik talab qilinsa, quyidagi mavjudlik teoremasi o'rinli bo'ladi.

Teorema (Peano). *Agar* $D \subset \mathbb{R}^{1+n}$ *sohada*

$f(t,x)\in C(D,\mathbb{R}^n)$ *bo'lsa, ixtiyoriy* $(t_0,x^0)\in D$ *uchun* (III.3.1), (III.3.2) *Koshi masalasi kamida bitta yechimga ega bo'ladi.*

Agar (III.3.1) sistemaning o'ng tomonidan $f(t,x)\in C(D,\mathbb{R}^n)$ dan boshqa shart talab qilinmasa, yechim yagona bo'lmasligi mumkin. Buni M. A. Lavrent'yev misoli asoslaydi.

Masalalar

1. $\mathbb{R}^n$ **fazoda** quyidagi qismiy tartibni kiritaylik:

$$x\le y \Leftrightarrow x^j\le y^j,\ j=\overline{1,n};$$

$$x<y \Leftrightarrow x^j<y^j,\ j=\overline{1,n}.$$

Berilgan $f:(a,b]\to\mathbb{R}^n$ funksiya uchun Dini hosilalari

$$D^-f(t)=\overline{\lim_{h\to 0-}}\frac{f(t+h)-f(t)}{h}\quad \text{(yuqori chap hosila)}$$

$$D_-f(t)=\underline{\lim_{h\to 0-}}\frac{f(t+h)-f(t)}{h}\quad \text{(quyi chap hosila)}$$

formulalar bilan aniqlanadi. Quyidagi tasdiqlarni isbotlang:

Faraz qilaylik, $f\in C([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$ va $\forall t\in[a,b]$ uchun $x\le y$, $x^j=y^j$ ekanligidan $f^j(t,x)\le f^j(t,y)$ tengsizlik kelib chiqsin ($j=\overline{1,n}$). Bundan tashqari, $\varphi(t)$ uchun $\varphi'(t)=f(t,\varphi(t))$, $t\in[a,b]$, bo'lsin. U holda, agar $u\in C([a,b],\mathbb{R}^n)$ va

$$\begin{cases} D^-u(t)>f(t,u(t)),\ a<t\le b,\\ u(a)>\varphi(a)\end{cases}$$

bo'lsa, $u(t)>\varphi(t)$, $t\in[a,b]$, baholash o'rinli; agar $v\in C([a,b],\mathbb{R}^n)$ va

$$\begin{cases} D_-v(t)<f(t,v(t)),\ a<t\le b,\\ v(a)<\varphi(a)\end{cases}$$

bo'lsa esa, $u(t)<\varphi(t)$, $t\in[a,b]$, tengsizlik o'rinli bo'ladi.

2. Faraz qilaylik, $f\in C(\mathbb{R},\mathbb{R})$ hamda $x=x(t)$ funksiya $x'=f(x)$ tenglamaning $t\in[a,b]$ segmentda aniqlangan yechimi bo'lsin. Agar $x(a)=x(b)$ bo'lsa, $x(t)=\text{const}$ ekanligini isbotlang. Bu tasdiq $f\in C(\mathbb{R}^n,\mathbb{R}^n)$, $n>1$, holida o'rinli emas. Lekin, agar $f\in C(\mathbb{R}^n,\mathbb{R}^n)$ funksiya biror $\varphi\in C^1(\mathbb{R}^n,\mathbb{R})$ skalyar funksiyaning gradiyentidan iborat, ya'ni $f=\operatorname{grad}\varphi$ bo'lsa, yuqorida keltirilgan tasdiq o'rinli bo'ladi. Shularni isbotlang.

3. Faraz qilaylik, $f\in C([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$ va

$$\big(f(t,x)-f(t,y)\big)\cdot(x-y)\le\theta\frac{\|x-y\|^2}{t-a},\ a<t\le b,\ \{x,y\}\subset\mathbb{R}^n,\ 0<\theta<1$$

shart o'rinli bo'lsin. U holda

$$\begin{cases} x'=f(t,x)\\ x|_{t_0}=x^0\quad (t_0\in[a,b],\ x^0\in\mathbb{R}^n)\end{cases}$$

boshlang'ich masalaning ko'pi bilan bitta yechimi borligini isbotlang.

4. $D\subset\mathbb{R}^{n+1}$ – soha, $(t_0,x^0)\in D$ va $f\in C(D;\mathbb{R}^n)$ funksiya quyidagi shartni qanoatlantirsin

$$\forall t\ge t_0\ \forall(t,x)\in D\ \ \forall(t,y)\in D$$

$$(f(t,x)-f(t,y))\cdot(x-y)\le 2|x-y|\cdot\varphi(|x-y|)$$

bu yerda $\varphi\in C([0;+\infty);\mathbb{R}_+)$ o'suvchi funksiya, $\varphi(0)=0$ va

$$\int_0^r\frac{ds}{\varphi(s)}=+\infty\ (r>0).$$

U holda

$$\begin{cases} x'=f(t,x)\\ x|_{t_0}=x^0\end{cases}\qquad (K)$$

Koshi masalasi $t\ge t_0$ da yagona yechimga ega bo'lishini ko'rsating. Shu tasdiqqa o'xshash tasdiq t_0 dan chap tomonda, ya'ni $t\le t_0$ da ham o'rinli.

Yechimning yagonaligi quyidagini anglatadi: agar (K) masalaning grafiklari D da joylashgan ikkita yechimi bo'lsa, u holda bu yechimlar ularning umumiy aniqlanish sohasida o'zaro teng bo'ladi.

5. $D\subset\mathbb{R}^{n+1}$ – soha, $f\in C(D;\mathbb{R}^n)$ funksiya uchun

$\forall (t,x)\in D\ \forall (t,y)\in D\ |f(t,x)-f(t,y)|\le \varphi(|x-y|)$.

bo'lsin; bu yerda $\varphi\in C([0;+\infty);\mathbb{R}_+)$ – o'suvchi funksiya, $\varphi(0)=0$ va

$$\int_0^r \frac{ds}{\varphi(s)}=+\infty,\ (r>0).$$

U holda $\forall (t_0,x^0)\in D$ uchun

$$\begin{cases} x'=f(t,x)\\ x|_{t_0}=x^0\end{cases}$$

Koshi masalasi yagona yechimga ega bo'ladi. Shuni isbotlang.

III.4. Davomsiz yechim

Ushbu

$$x'=f(t,x) \qquad \text{(III.4.1)}$$

sistemani qaraylik, bu paragrafda $f(t,x)$ vektor-funksiya $D\subset\mathbb{R}^{1+n}$ sohada uzluksiz ($f\in C(D,\mathbb{R}^n)$) va D da joylashgan har qanday kompaktda x vektor o'zgaruvchi bo'yicha Lipshits shartini qanoatlantiradi deb faraz qilinadi. Bu farazimizga ko'ra $\forall (t_0,\mathbf{x}^0)\in D$ uchun ushbu

$$\begin{cases} x'=f(t,x)\\ x|_{t_0}=x^0\end{cases} \qquad \text{(III.4.2)}$$

Koshi masalasi biror $[t_0,\ t_0+h_0]$ $(h_0>0)$ segmentda aniqlangan yagona $x=\varphi(t)$ yechimga ega.

Agar $x=\varphi(t)$ funksiya (III.4.1) differensial tenglamaning $I=[a;b]$ oraliqda, $x=\psi(t)$ funksiya esa uning $J=[a;b^*]$, $b\le b^*$, yoki $J=[a;b^*)$, $b<b^*$, oraliqda aniqlangan yechimi bo'lib, ular I da ustma-ust ham tushsa, u holda $x=\psi(t)$ yechim $x=\varphi(t)$ yechimning I dan J gacha o'ngga davomi

(davom ettirilishi) deb ataladi.

Yechimning boshqa tur oraliqlardan o'ngga hamda chapga davomi shunga o'xshash aniqlanadi.

Yechimning o'ngga davom ettirishni amalga oshirishdan avval yechimlarni yelimlash (biriktirish) bilan bog'liq bo'lgan bir jumlani keltiramiz.

Jumla 1. *Agar $x=\varphi(t)$ funksiya (III.4.1) differensial tenglamaning $[t_0,t_1]$ segmentda, $x=\varphi_1(t)$ esa uning $[t_1,t_2]$ segmentda aniqlangan yechimlari bo'lib, $\varphi(t_1)=\varphi_1(t_1)$ shart ham bajarilsa, u holda bu yechimlarning yelimlanishi (biriktirilishi) bo'lgan*

$$\psi(t)=\begin{cases}\varphi(t),\ \text{agar } t\in[t_0,t_1] \text{ bo'lsa}\\ \varphi_1(t),\ \text{agar } t\in[t_1,t_2] \text{ bo'lsa}\end{cases}$$

funksiya (III.4.1) differensial tenglamaning $[t_0,t_2]$ segmentda aniqlangan yechimini beradi, ya'ni $x=\psi(t)$ yechim $x=\varphi(t)$ yechimning $[t_0,t_1]$ segmentdan $[t_0,t_2]$ segmentgacha davomidan iborat.

🗝 Berilganga ko'ra

$$\psi(t)\in C^1([t_0,t_1]);\ \psi'(t)=f(t,\psi(t)),\ t\in[t_0,t_1];$$
$$\psi(t)\in C^1([t_1,t_2]);\ \psi'(t)=f(t,\psi(t)),\ t\in[t_1,t_2];$$
$$\psi(t_1-0)=\varphi(t_1)=\varphi_1(t_1)=\psi(t_1+0).$$

Shuning uchun

$$\psi'(t_1-0)=\varphi'(t_1)=f(t_1,\varphi(t_1))=f(t_1,\varphi_1(t_1))=\varphi_1'(t_1)=\psi'(t_1+0).$$

Demak, $\psi'(t_1)$ mavjud, $\psi(t)\in C^1([t_1,t_2])$ va $\psi(t)$ funksiya (III.4.1) differensial tenglamani $[t_0,t_2]$ segmentda qanoatlantiradi. ✍

Yechimning o'ng uchi bo'lmish

$(t_1;x^1)\overset{def}{=}(t_0+h_0,\varphi(t_0+h_0))\in D$ nuqtaga ko'ra

$$\begin{cases} x' = f(t,x) \\ x\big|_{t_1} = x^1 \end{cases}$$

Koshi masalasini yechib, $[t_1, t_1 + h_1]$ $(h_1 > 0)$ segmentda aniqlangan yagona $x = \varphi_1(t)$ yechimni topamiz. Yuqoridagi ikki yechimni yelimlanishi (biriktirilishi)dan ushbu

$$\varphi^*(t) = \begin{cases} \varphi(t), \text{ agar } t \in [t_0, t_1] \text{ bo'lsa}, \\ \varphi_1(t), \text{ agar } t \in [t_1, t_1 + h_1] \text{ bo'lsa} \end{cases}$$

$$(t_1 = t_0 + h_0, \varphi(t_1) = \varphi_1(t_1) = x^1)$$

funksiyani quramiz. Jumla 1 ga ko'ra bu $\varphi^*(t)$ funksiya (III.4.2) masalaning $[t_0, t_1] \cup [t_1, t_1 + h_1] = [t_0, t_1 + h_1]$ segmentda aniqlangan yechimidir. U $[t_0, t_0 + h_0]$ da aniqlangan $x = \varphi(t)$ yechimning $[t_0, t_1 + h_1]$ segmentgacha (o'ngga) davomidir. Bu yechimni (funksiyani) yana $x = \varphi(t)$ bilan belgilaymiz; endi bu $x = \varphi(t)$ funksiya (III.4.2) masalaning $[t_0, t_1 + h_1]$ segmentda aniqlangan yechimidir. Yechimning bu davomi bir qiymatli aniqlanadi. Endi bu yechimni yana o'ngga davom ettiramiz va hokazo.

Yechimning chapga davomi yuqoridagiga o'xshash amalga oshiriladi.

Endi **davomsiz yechim** tushunchasini kiritamiz.

Monoton kengayib (o'sib) D ga intiluvchi K_j kompaktlar ketma-ketligini qaraylik:

$$K_1 \subset K_2 \subset ... \subset K_j \subset ..., \bigcup_{j=1}^{\infty} K_j = D. \qquad \text{(III.4.3)}$$

Masalan,

$$K_j = \left\{ (t,x) \in D \middle| \ dist((t,x), \partial D) \geq \frac{1}{j}, |t| \leq j, \|x\| \leq j \right\}$$

deyish mumkin.

Berilgan $(t_0, x^0) \in D$ nuqta K_{j_1} da yotsin, $(t_0, x^0) \in K_{j_1}$.

III.3- paragraf 45- betdagi teoremaga ko'ra) (III.4.2) masala yechimini o'ngga o'zgarmas qadam uzunligi bilan davom ettirib, chekli qadamdan so'ng $(t_1, \varphi(t_1)) \notin K_{j_1}$ $(t_1 > t_0)$ nuqtani hosil qilamiz (yechim $t = t_1$ da K_{j_1} kompaktdan tashqarida). Aytaylik, $(t_1, \varphi(t_1)) \in K_{j_2}$ bo'lsin ($j_2 < j_1$ bo'lishi ham mumkin; buning ahamiyati yo'q). Endi yechimni t_1 dan o'ngga K_{j_2} dan chiqqunga qadar davom ettiramiz va hokazo. Natijada monoton o'suvchi $t_1, t_2, ..., t_m, ...$ ketma-ketlikni hosil qilamiz. Demak, chekli yoki cheksiz $T = \lim_{j \to \infty} t_j$ mavjud va $[t_0, t_1] \cup [t_1, t_2] \cup ... \cup [t_{m-1}, t_m] \cup ... = [t_0, T)$ bo'ladi. Davom ettirish natijasida biz (III.4.2) masalaning $[t_0, T)$ oraliqda aniqlangan $x = \varphi(t)$ yechimini hosil qilamiz. Bu yechim grafigi D da joylashgan har qanday kompaktdan chiqib ketadi. Haqiqatan ham, agar $\left\{ (t, \varphi(t)) \in \mathbb{R}^{1+n} \mid t_0 \leq t < T \right\}$ grafik biror $K \subset D$ kompaktda joylashgan bo'lganda edi, u holda bu grafik K ni qoplagan biror K_j da yotardi; bunday bo'lishi mumkin emas, chunki t_0 dan boshlangan yechim o'ngga davom ettirilishi natijasida chekli qadamdan so'ng o'sha K_j dan tashqariga chiqishi kerak edi.

Yechim t_0 dan chapga ham shu yo'sinda davom ettiriladi. Natijada (τ, T) intervalda aniqlangan $x = \varphi(t)$ yechim hosil bo'ladi ($\tau = -\infty$ yoki/va $T = +\infty$ bo'lishi mumkin). Bu yechim (K) masalaning (D sohadagi) **davomsiz yechimi** deyiladi.

Endi $\varphi(t)$ davomsiz yechim (III.4.3) dagi K_j $(j \in \mathbb{N})$ kompaktlarning tanlanishiga bog'liq emasligini ko'rsataylik.

Faraz qilaylik, $\tilde{K}_j$ kompaktlar ham monoton o'sib D ga intiluvchi, ya'ni

$$\tilde{K}_1 \subset \tilde{K}_2 \subset ... \subset \tilde{K}_j \subset ..., \bigcup_{j=1}^{\infty} \tilde{K}_j = D.$$

xususiyatga ega va ularga ko'ra qurilgan (III.4.2) Koshi masalasining davomsiz yechimi $(\tilde{\tau};\tilde{T})$ intervalda aniqlangan $\tilde{\varphi}(t)$ funksiyadan iborat bo'lsin.

Jumla 2. *$\varphi(t)$ va $\tilde{\varphi}(t)$ davomsiz yechimlar ustma-ust tushadi, ya'ni $(\tau;T)=(\tilde{\tau};\tilde{T})$ va $\forall t\in(\tau;T)=(\tilde{\tau};\tilde{T})$ uchun $\varphi(t)=\tilde{\varphi}(t)$.*

Ikkala $\varphi(t)$ va $\tilde{\varphi}(t)$ davomsiz yechim ham bitta (III.4.2) Koshi masalasining yechimi bo'lgani uchun yechimning yagonalik xossasiga ko'ra ular aniqlanish sohalarining tengligidan ($(\tau;T)=(\tilde{\tau};\tilde{T})$) bu yechimlarning tengligi, ya'ni jumlaning isboti kelib chiqadi. Demak, $\tau=\tilde{\tau}$ va $T=\tilde{T}$ ekanligini isbotlashimiz kifoya. Biz $T=\tilde{T}$ tenglikni ko'rsatamiz, $\tau=\tilde{\tau}$ ekanligi shunga o'xshash isbotlanadi.

Faraz qilaylik, $T\neq\tilde{T}$ bo'lsin. Aniqlik uchun $T<\tilde{T}$ deylik. Tushunarliki, $T-$chekli son va $\forall t\in[t_0,T)$ uchun $\varphi(t)=\tilde{\varphi}(t)$. Demak, $\lim\limits_{t\to T-0}\varphi(t)=\tilde{\varphi}(T)$ limit mavjud va $(T,\tilde{\varphi}(T)\in D$. Ravshanki, K_j $(j\in\mathbb{N})$ kompaktlarning birortasi, masalan K_{j_0}, ushbu $\{(t,\varphi(t))\in\mathbb{R}^{1+n}\mid t_0\le t<T\}$ grafikni qoplaydi: $K_{j_0}\supset\{(t,\varphi(t))\in\mathbb{R}^{1+n}\mid t_0\le t<T\}$. Lekin $x=\phi(t)$ davomsiz yechimning qurilishiga ko'ra uning biror $t\in[t_0,T)$ dagi qiymati K_{j_0} kompaktdan tashqarida bo'lishi kerak edi. Hosil bo'lgan ziddiyat farazimizning noto'g'ri va, demak, $T=\tilde{T}$ ekanligini isbotlaydi.

Shunday qilib, (III.4.2) Koshi masalasining davomsiz yechimi bir qiymatli aniqlangan.
Davomsiz yechimning xususiyatini quyidagi *teorema* ochadi.

Teorema. *Faraz qilaylik, $f(t,x)$ vektor-funksiya $D\subset R^{n+1}$ sohada uzluksiz ($f\in C(D,\mathbb{R}^n)$) va D da joylashgan*

har qanday kompaktda x bo'yicha Lipshits shartini qanoatlantirsin hamda $(t_0,x^0)\in D$ uchun qo'yilgan (III.4.2) Koshi masalasining $x=\varphi(t)$ davomsiz yechimi $(\tau;T)$ intervalda aniqlangan bo'lsin. U holda ixtiyoriy $K\subset D$ kompat uchun shunday chekli $\tau_1\in(\tau;T)$ va $T_1\in(\tau;T)$ lar mavjudki, har qanday $\tilde{t}\in(\tau;\tau_1)$ uchun $(\tilde{t};\varphi(\tilde{t}))\notin K$ va har qanday $t\in(T_1,T)$ uchun $(t;\varphi(t))\notin K$ bo'ladi.

Ixtiyoriy $K\subset D$ kompakt berilgan bo'lsin. Teoremani T_1 uchun isbotlaymiz. τ_1 uchun isbot shunga o'xshash bo'ladi. (III.4.3) munosabatlarga ko'ra K_j $(j\in\mathbb{N})$ kompaktlar orasida K kompaktni qoplovchi K_{j_0} mavjud, $K_{j_0}\supset K$. Agar barcha $t\in(\tau;T)$ lar uchun $\varphi(t)\notin K_{j_0}$ bo'lsa, teorema isbot bo'ldi; chunki bu holda, masalan, $T_1=t_0$ olish mumkin. Endi faraz qilaylik, shunday $t_*\in(\tau,T)$ mavjud bo'lsinki, uning uchun $\varphi(t_*)\in K_{j_0}$ bo'lsin. U holda ushbu $\Delta\stackrel{def}{=}\{t\in(\tau,T)|(t,\varphi(t)\in K_{j_0}\}$ to'plam bo'shmas ($t_*\in\Delta$) va chegaralangan (chunki K_{j_0}-kompakt). Demak, chekli $\sup\Delta$ mavjud. $T_1=\sup\Delta$ deylik. U holda supremum ta'rifiga ko'ra barcha $t\in(T_1,T)$ lar uchun $(t;\phi(t))\notin K_{j_0}$ va, demak, $(t;\phi(t))\notin K$ ham bo'ladi.

Biz yuqorida (III.4.2) Koshi masalasining davomsiz yechimini qurdik. (III.IV.1) differensial tenglamaning biror yechimini davom ettirish va davomsiz yechim tushunchasi yuqoridagidan bevosita kelib chiqadi, chunki (III.IV.1) tenglamaning I oraliqda aniqlangan $x=\varphi(t)$ yechimi ushbu

$$\begin{cases}x'=f(t,x)\\ x|_{t_0}=\varphi(t_0)\quad(t_0\in I)\end{cases}$$

Koshi masalasining yechimi demakdir. Bu masala yechimining I dan tashqariga davomi (u $t_0 \in I$ ga bog'liq emas) I oraliqda aniqlangan $x = \varphi(t)$ yechimning davomidir.

Yuqoridagi tekshirishlardan quyidagi teorema bevosita kelib chiqadi.

Teorema. *Faraz qilaylik,* $f(t,x)$ *vektor-funksiya* $D \subset \mathbb{R}^{1+n}$ *sohada uzluksiz* ($f \in C(D,\mathbb{R}^n)$) *va* D *da joylashgan har qanday kompaktda* x *bo'yicha Lipshits shartini qanoatlantirsin va* (III.4.1) *differensial tenglamaning* $x = \varphi(t)$, $\varphi : [t_0;T) \to D$, *yechimi berilgan bo'lsin. U holda bu yechimning o'ngga davom ettirilishi mumkin bo'lishi uchun bu yechim grafigi* $\{(t,\varphi(t)) \in \mathbb{R}^{1+n} \mid t_0 \le t < T\}$ *ning* D *sohada joylashgan biror kompaktda yotishi yetarli va zarurdir.*

Masalalar

1. Ushbu

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{y}{t} \\ \dfrac{dy}{dt} = -\dfrac{x}{t} \\ x(1) = 1,\ y(1) = 0 \end{cases}$$

masalaning davomsiz yechimi $(0,+\infty)$ intervalda aniqlangan ekanligini ko'rsating.

2. Faraz qilaylik, $x' = f(x)$ tenglamaning o'ng tomoni $x \in G$ larda ($G \subset \mathbb{R}^n$ – soha) aniqlangan va tenglama uchun yechimning mavjudlik va yagonalik xossasi o'rinli bo'lsin. Agar bu tenglamaning $[t_0;t_1]$ ($t_0 < t_1$) segmentda aniqlangan $x = \varphi(t)$ yechimi uchun $\varphi(t_0) = \varphi(t_1)$ bo'lsa, bu yechim $t \in (-\infty;+\infty)$ oraliqqa davom ettirilishi mumkinligini ko'rsating.

3. Ushbu $y' = y^2 + x^2$, $y(0) = 0$, masala $[0;2,6)$ oraliqda aniqlangan yechimga ega emas (yechim $[0;2,6)$ gacha davom etmas) ligini isbotlang.

4. $\{f,g\} \subset C(\mathbb{R},\mathbb{R})$ va $G(x) = \int_0^x g(s)ds$ funksiyalar uchun $\exists m > 0\ \forall x \in \mathbb{R}\ G(x) \ge mx^2$ va $\forall y \in \mathbb{R}\ yf(y) \ge 0$ shartlar o'rinli bo'lsin. Ushbu

$$\begin{cases} x'' + f(x') + g(x) = 0 \\ x(0) = x_0,\ x'(0) = v_0\ \ (\{x_0, v_0\} \subset \mathbb{R}) \end{cases}$$

boshlang'ich masalaning yechimi o'ngga $[0,+\infty)$ gacha davom etishini ko'tsating.

III.5. Muhim integral tengsizliklar

Differensial va integral tenglamalar yechimlarini baholashda qo'l keluvchi muhim integral tengsizliklar bilan bog'liq tasdiqlarning ba'zilari bilan tanishamiz.

Teorema 1 (Bihari tipidagi tengsizlik). *Aytaylik, biror* $\alpha \ge 0$ *son, uzluksiz nomanfiy* $u : [x_0,b) \to \mathbb{R}_+$, $k : [x_0,b) \to \mathbb{R}_+$ *funksiyalar hamda uzluksiz va (keng ma'noda) o'suvchi uzluksiz musbat* $g : \mathbb{R}_+ \to (0,+\infty)$ *funksiya uchun*

$$u(x) \le \alpha + \int_{x_0}^{x} k(s)g(u(s))ds,\ x \in [x_0,b), \quad \text{(III.5.1)}$$

tengsizlik qanoatlansin. Ushbu

$$G_\alpha(t) = \int_\alpha^t \frac{d\sigma}{g(\sigma)},\ t \in \mathbb{R}_+, \quad \text{(III.5.2)}$$

funksiyani aniqlaylik va uning teskarisini G_α^{-1} *bilan belgilaylik. U holda* $G_\alpha^{-1}\left(\int_{x_0}^{x} k(s)ds\right)$ *ma'noga ega bo'lgan barcha* $x \in [x_0,b)$ *lar uchun*

$$u(x) \le G_\alpha^{-1}\left(\int_{x_0}^{x} k(s)ds\right) \quad \text{(III.5.3)}$$

baholash o'rinli bo'ladi.

⊶ Dastlab $\alpha > 0$ deb faraz qilamiz. Ushbu

$$v(x) = \alpha + \int_{x_0}^{x} k(s)g(u(s))ds,\ x \in [x_0, b),$$

funksiyani aniqlaylik. Ravshanki, $v(x_0) = \alpha,\ v \in C^1([x_0, b), \mathbb{R}_+)$ va $u(x) \le v(x),\ x \in [x_0, b)$. Demak, g funksiya o'suvchi (keng ma'noda) va $k \ge 0$ bo'lganligi uchun

$$v'(x) = k(x)g(u(x)) \le k(x)g(v(x)),\ x \in [x_0, b),$$

tengsizlik o'rinli. Bundan $g > 0$ bo'lgani uchun

$$\frac{v'(t)}{g(v(t))} \le k(t),\ t \in [x_0, b),$$

tengsizlik kelib chiqadi. Bu tengsizlikni $t = x_0$ dan $t = x,\ x \in [x_0, b)$, gacha integrallaymiz:

$$G_\alpha(v(x)) \le \int_{x_0}^{x} k(t)dt,\ x \in [x_0, b). \quad \text{(III.5.4)}$$

Tushunarliki, G_α funksiya qat'iy o'suvchi $\left(\dfrac{dG_\alpha(t)}{dt} = \dfrac{1}{g(t)} > 0,\ t \in \mathbb{R}_+\right)$ va uning qiymatlar to'plami

$$[c, d],\ c \overset{def}{=} \int_{\alpha}^{0} \frac{d\sigma}{g(\sigma)}\ (-\infty \le c < 0),\ d \overset{def}{=} \int_{\alpha}^{+\infty} \frac{d\sigma}{g(\sigma)}\ (0 < d \le +\infty),$$

oraliqdan iborat. Demak, G_α^{-1} teskari funksiya $[c, d]$ oraliqda aniqlangan va qat'iy o'suvchi bo'ladi. Ravshanki, (III.5.4) va $u(x) \le v(x)$ tengsizliklardan $G_\alpha^{-1}\left(\int_{x_0}^{x} k(s)ds\right)$ ma'noga ega bo'lgan barcha $x \in [x_0, b)$ lar uchun

$$u(x) \le v(x) \le G_\alpha^{-1}\left(\int_{x_0}^{x} k(s)ds\right).$$

(III.5.3) tengsizlik kelib chiqadi.

Endi $\alpha = 0$ bo'lsin. U holda (III.5.1) tengsizlik ixtiyoriy $\alpha > 0$ uchun ham o'rinli. Yuqorida isbotlangan (III.5.4) tensizlikda $\alpha \to 0+$ deb limitga o'tamiz va teoremaning isbotini tugatamiz. ♘

Isbotlangan teoremadan foydalanib Gronuoll-Bellman tipidagi tengsizliklarni hosil qilish mumkin.

Teorema 2 (Gronuoll-Bellman tipidagi tengsizlik). *Aytaylik, biror* $\alpha \ge 0, \beta \ge 0, \gamma > 0$ *sonlar va uzluksiz nomanfiy* $u : [x_0, b) \to \mathbb{R}_+,\ u \in C([x_0, b), \mathbb{R}_+)$, *funksiya uchun ushbu*

$$u(x) \le \alpha + \beta(x - x_0) + \gamma \int_{x_0}^{x} u(s)ds,\ x \in [x_0, b), \quad \text{(III.5.5)}$$

tengsizlik bajarilsin. U holda

$$u(x) \le \alpha e^{\gamma(x-x_0)} + \frac{\beta}{\gamma}\left(e^{\gamma(x-x_0)} - 1\right),\ x \in [x_0, b), \quad \text{(III.5.6)}$$

baholash o'rinlidir.

⊶ **I.** Teorema 1 dan foydalanib isbotlash. Teoremaning shartlarida, ravshanki, ixtiyoriy $\varepsilon > 0$ uchun

$$u(x) \le \alpha + \beta(x - x_0) + \gamma \int_{x_0}^{x} u(s)ds + \varepsilon\gamma(x - x_0),\ x \in [x_0, b),$$

ya'ni

$$u(x) \le \alpha + \gamma \int_{x_0}^{x} \left(u(s) + \beta/\gamma + \varepsilon\right)ds,\ x \in [x_0, b),$$

tengsizlik o'rinli bo'ladi. Agar $k(s) = \gamma,\ g(\sigma) = \sigma + \beta/\gamma + \varepsilon$ desak, teorema 1 ning barcha shartlari bajariladi. Bu holda

$$G_\alpha(t)=\int_\alpha^t \frac{d\sigma}{g(\sigma)}=\int_\alpha^t \frac{d\sigma}{\sigma+\beta/\gamma+\varepsilon}=\ln\frac{t+\beta/\gamma+\varepsilon}{\alpha+\beta/\gamma+\varepsilon},$$

$$G_\alpha^{-1}(y)=(\alpha+\beta/\gamma+\varepsilon)e^{y}-\beta/\gamma-\varepsilon \quad (y\in\mathbb{R}),$$

$$\int_{x_0}^{x} k(s)ds=\beta(x-x_0)$$

va teorema 1ga ko'ra quyidagi tengsizlikni topamiz:

$$u(x)\le(\alpha+\beta/\gamma+\varepsilon)e^{\gamma(x-x_0)}-\beta/\gamma-\varepsilon,\ x\in[x_0,b).$$

Oxirgi tengsizlikda ε ni $0+$ga intiltirib limitga o'tamiz va (III.5.6) tensizlikni hosil qilamiz.

II. Bevosita (teorema 1dan mustaqil) isbotlash. Ushbu

$$v(x)=\alpha+\beta(x-x_0)+\gamma\int_{x_0}^{x}u(s)ds,\ x\in[x_0,b),$$

yordamchi funksiyani qaraylik. U holda $u(x)\le v(x)$. Endi $v(x)$ ni yuqoridan baholaymiz. Ravshanki, $v'(x)=\beta+\gamma u(x)\le\beta+\gamma v(x)$. Demak, $v'(x)-\gamma v(x)\le\beta$. Bu tengsizlikning har ikkala tomonini $e^{-\gamma x}$ ga ko'paytiramiz: $\left(v(x)e^{-\gamma x}\right)'\le\beta e^{-\gamma x}$. Oxirgi tengsizlikni x_0 dan $x\in[x_0,b)$ gacha integrallaymiz:

$$v(x)e^{-\gamma x}-v(x_0)e^{-\gamma x_0}\le\frac{\beta}{\gamma}\left(e^{-\gamma x_0}-e^{-\gamma x}\right).$$

Bundan $v(x_0)=\alpha$ ekanligini hisobga olib, (III.5.6) tengsizlikni hosil qilamiz. ✍

Natija (Gronuoll-Bellman tipidagi tengsizlik). *Agar $u\in C((a,b),\mathbb{R}_+)$ funksiya va $\alpha\ge0,\ \beta\ge0,\ \gamma>0,\ x_0\in(a,b)$ sonlar uchun*

$$u(x)\le\alpha+\beta|x-x_0|+\gamma\left|\int_{x_0}^{x}u(s)ds\right|,\quad x\in(a,b),\qquad \text{(III.5.7)}$$

tengsizlik o'rinli bo'lsa, u holda

$$u(x)\le\alpha e^{\gamma|x-x_0|}+\frac{\beta}{\gamma}\left(e^{\gamma|x-x_0|}-1\right),\ x\in(a,b),\qquad \text{(III.5.8)}$$

baholash ham o'rinlidir.

$x\ge x_0$ holi yuqorida isbotlandi. $x\le x_0$ bo'lganda $z-x_0=x_0-x$ deb, $v(z)=u(2x_0-z)=u(x)$ funksiyani kiritamiz. U holda $z\ge x_0$ va

$$\left|\int_{x_0}^{x}u(s)ds\right|=\int_{x_0}^{z}v(t)dt,\quad |x-x_0|=z-x_0,$$

bo'ladi va isbotlanadigan tengsizlik yana yuqorida isbotlangandan kelib chiqadi. ✍

Gronuoll-Bellman tipidagi tengsizliklardan foydalanib yechimning mavjudlik oralig'ini baholash mumkin.

Masalalar

1. Ushbu $y'=y^2+x^2,\ y(0)=0$, masala yechimi aniqlangan oraliqni baholang.
2. Ushbu $y'=3x^2+y^3,\ y(1)=1$, masala yechimi o'ngga va chapga qayergacha davom etadi?

III.6. Yechimning boshlang'ich ma'lumot va parametrlarga uzluksiz bog'liqligi

Dastlab skalyar x noma'lum funksiya uchun

$$\begin{cases}\dfrac{dx}{dt}+p(t,\mu)x=q(t,\mu)\\ x(\tau)=\xi\end{cases}$$

chiziqli boshlang'ich masalani qaraylik; bu yerda μ – haqiqiy parametr, $\mu\in(\mu_1,\mu_2),\ \tau\in I$ va $\{p,q\}\subset C(I\times(\mu_1,\mu_2))$. Tushunarliki, bu masalaning yechimi na faqat t ga, balki u boshlang'ich ma'lumot τ, ξ va μ parametrlarga bog'liq. U, ma'lumki,

$$x(t;\tau,\xi,\mu)=\xi\cdot e^{-\int_\tau^t p(s,\mu)ds}+$$

$$+e^{-\int_\tau^t p(s,\mu)ds}\cdot\int_\tau^t q(r,\mu)\cdot e^{\int_\tau^r p(s,\mu)ds}dr$$

formula bilan kvadraturalarda ifodalanadi. Bu formula ko'rinishidan ravshanki, qo'yilgan shartlarda yechim $C(I\times I\times\mathbb{R}\times(\mu_1,\mu_2))$ sinfga tegishli; xususan, yechim boshlang'ich ma'lumotlar va parametrga uzluksiz bog'liq. Umumiy holda differensial tenglama chiziqli emas va qo'yilgan masala yechimning oshkor formulasi mavjud bo'lmaganligi uchun uning boshlang'ich ma'lumotlar va parametrlarga uzluksiz bog'liqligini o'rganish oson emas.

Endi ushbu

$$\begin{cases} x'=f(t,x,\mu) \\ x|_{t_0}=x^0 \end{cases} \qquad \text{(III.6.1)}$$

$\mu=(\mu_1,\mu_2,\dots,\mu_m)\in M$ $(M\subset\mathbb{R}^m-\text{soha})$ μ parametr(lar)ga bo'gliq bo'lgan Koshi masalasini qaraylik. Faraz qilaylik, $f(t,x,\mu)$ vektor-funksiya $(t,x)\in D$, $\mu\in M$ bo'lganda aniqlangan va (barcha argumentlari bo'yicha) uzluksiz ($f\in C(D\times M,\mathbb{R}^n)$) hamda $D\times M$ sohada x vektor o'zgaruvchi bo'yicha lokal Lipshits shartini qanoatlantirsin, ya'ni har qanday $(t,x,\mu)\in D\times M$ nuqtaning yetarlicha kichik atrofi uchun shunday $L>0$ son mavjudki, shu atrofdagi barcha (t,x^1,μ) va (t,x^2,μ) nuqtalar uchun

$$\|f(t,x^2,\mu)-f(t,x^1,\mu)\|\le L\|x^2-x^1\|$$

tengsizlik o'rinli. Oxirgi shart bajarilishi uchun, masalan, ixtiyoriy $(t,x,\mu)\in D\times M$ nuqtaning biror atrofida $\left|\dfrac{\partial f_i}{\partial x_j}\right|\le\text{const}$ bo'lishi yetarli. Qo'yilgan shartlarda har qanday $(t_0,x^0,\mu)\in D\times M$ uchun

(III.6.1) masala yagona davomsiz $x=\varphi(t;t_0,x^0,\mu)$, $t\in I$, yechimga ega. Bu davomsiz yechimning aniqlanish intervali, tushunarliki, tayinlangan (t_0,x^0,μ) qiymatlarga bog'liq bo'ladi, $I=I(t_0,x^0,\mu)$. Demak, $x=\varphi(t;t_0,x^0,\mu)$ yechim $(t;t_0,x^0,\mu)\in I\times D\times M\subset\mathbb{R}^{2+n+m}$ sohada aniqlangan. Agar (t_0,x^0) tayinlangan bo'lsa, u holda $x=\varphi(t;t_0,x^0,\mu)$ yozuv o'rniga $x=\varphi(t;\mu)$ yozuvni ishlatamiz.

Matematik modellar tuzilganda odatda t_0 va x^0 boshlang'ich ma'lumotlar hamda o'ng tomondagi $f(t,x,\mu)$ vektor-funksiya kichik xatolikka ega bo'lgan har xil o'lchashlar yoki hisoblashlar yordamida topilgan bo'ladi. Shuning uchun "bu kichik xatoliklar hisobiga yechim chekli (katta) qiymatga o'zgarib ketmaydimi?" degan savolga javob berish muhim amaliy ahamiyatga ega. Albatta, amaliyotga tatbiq nuqtai nazaridan kichik xatoliklar yechimni ko'p o'zgartirmasligi, ya'ni boshlang'ich ma'lumotlar va parametrlarning kichik o'zgarishi yechimning kichik o'zgarishiga olib kelishi kerak.

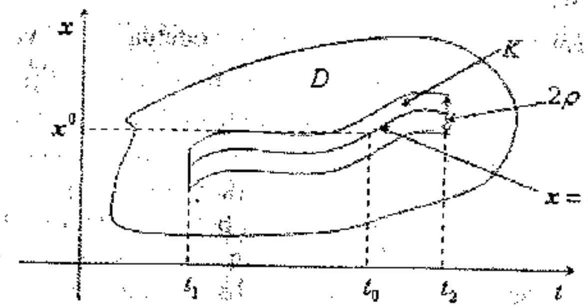
Biz bu bandda yechimning boshlang'ich ma'lumotlar va parametrlarga uzluksiz bog'liqligini o'rganamiz. Bu bog'lanishning silliqligini VII.1- bandda tekshiramiz.

Dastlab yechimning parametrlarga uzluksiz bog'liqligini ifodalovchi teoremani isbotlaymiz, so'ngra esa yechimning boshlang'ich ma'lumotlarga uzluksiz bog'liqligini ana shu teoremadan keltirib chiqaramiz.

Teorema (*yechimning parametrlarga uzluksiz bog'liqligi*). *Faraz qilaylik, $f(t,x,\mu)\in C(D\times M,\mathbb{R}^n)$ $((t,x)\in D,\mu\in M)$ bo'lsin va u $D\times M$ sohada x vektor o'zgaruvchi bo'yicha lokal Lipshits shartini qanoatlantirsin hamda $\mu=\mu^0$ bo'lganda (III.6.1) masala $t\in[t_1,t_2]$ $(t_0\in[t_1,t_2],(t_0,x^0,\mu^0)\in D\times M)$ segmentda aniqlangan $x=\varphi(t;\mu^0)$ yechimga ega bo'lsin. U holda shunday yetarlicha kichik $\delta>0$ son mavjudki, $\|\mu-\mu^0\|<\delta$ bo'lganda*

$x=\varphi(t;\mu)$ $(=\varphi(t;t_0,x^0,\mu))$ *yechim barcha* $t\in[t_1,t_2]$ *larda aniqlangan va* $(t;\mu)$ *o'zgaruvchilar bo'yicha uzluksiz vektor-funksiyadan iborat bo'ladi,* $\varphi(t;\mu)\in C([t_1,t_2]\times B_\delta(\mu^0))$.

Tushunarliki, berilgan yechim grafigi bo'lmish $\{(t,x)\in\mathbb{R}^{1+n} \mid t_1\le t\le t_2,\ x=\varphi(t;\mu^0)\}$ to'plam $\mathbb{R}^{1+n}$ fazoda yopiq va chegaralangan hamda u D sohada joylashgan (III.2-rasm).



III.2- rasm

Demak, bu grafik va ∂D orasidagi masofa qat'iy musbat va biror yetarlicha kichik $\rho>0$ uchun ushbu

$K=\{(t,x)\in\mathbb{R}^{1+n} \mid t_1\le t\le t_2,\ \|x-\varphi(t;\mu^0)\|\le\rho\}$ nay kompakt to'plamdan iborat bo'ladi va u ham D sohada yotadi.

Yetarlicha kichik $\sigma>0$ uchun

$\overline{B}_\sigma(\mu^0)=\{\mu\in\mathbb{R}^m \mid \|\mu-\mu^0\|\le\sigma\}\subset M$ bo'ladi. $K\times\overline{B}_\sigma(\mu^0)$ to'plam $D\times M$ sohada yotuvchi kompakt bo'lgani uchun bu kompaktda $f(t,x,\mu)$ vektor-funksiya x vektor o'zgaruvchi bo'yicha Lipshits shartini qanoatlantiradi, ya'ni shunday $L>0$ son mavjudki, ixtiyoriy $(t,x^1,\mu)\in K\times\overline{B}_\sigma(\mu^0)$ va $(t,x^2,\mu)\in K\times\overline{B}_\sigma(\mu^0)$ nuqtalar uchun

$$\|f(t,x^2,\mu)-f(t,x^1,\mu)\|\le L\|x^2-x^1\|$$

tengsizlik bajariladi. Ushbu $f(t,\varphi(t;\mu^0),\mu)$ vektor-funksiya $(t,\mu)\in[t_1,t_2]\times\overline{B}_\sigma(\mu^0)$ kompaktda uzluksiz, demak, tekis uzluksiz ham. Shuning uchun ixtioriy $\varepsilon>0$, $\varepsilon\le\rho$, songa ko'ra shunday $\delta=\delta(\varepsilon)>0$, $\delta\le\sigma$, topiladiki, $\|\mu-\mu^0\|<\delta$ shartni qanoatlantiruvchi barcha μ lar va ixtiyoriy $t\in[t_1,t_2]$ uchun $\|f(t,\varphi(t;\mu^0),\mu)-f(t,\varphi(t;\mu^0),\mu^0)\|<\varepsilon$ bo'ladi. Ixtiyoriy μ, $\|\mu-\mu^0\|<\delta$, uchun $x=\varphi(t;\mu)$ yechim biror $t\in I\subset[t_1,t_2]$ oraliqda aniqlangan. Bu yerdagi $\delta>0$ sonini kichraytirib, mos $x=\varphi(t;\mu)$, $\mu\in B_\delta(\mu^0)$, yechimlarni $t\in[t_1,t_2]$ oraliqqacha davom ettirish mumkinligini ko'rsatamiz. Yechimning grafigi K nayda yotgan t paytlar uchun (grafik K dan chiqib ketmagunga qadar)

$$\|f(t,\varphi(t;\mu),\mu)-f(t,\varphi(t;\mu^0),\mu^0)\|\le$$
$$\le\|f(t,\varphi(t;\mu),\mu)-f(t,\varphi(t;\mu^0),\mu)\|+$$
$$+\|f(t,\varphi(t;\mu),\mu)-f(t,\varphi(t;\mu^0),\mu^0)\|\le$$
$$\le L\|\varphi(t;\mu)-\varphi(t;\mu^0)\|+\varepsilon$$

va, demak,

$$\|\varphi(t;\mu)-\varphi(t;\mu^0)\|=\left\|\int_{t_0}^{t}f(s,\varphi(s;\mu),\mu)ds-\int_{t_0}^{t}f(s,\varphi(s;\mu^0),\mu^0)ds\right\|\le$$
$$\le\left|\int_{t_0}^{t}\|f(s,\varphi(s;\mu),\mu)-f(s,\varphi(s;\mu^0),\mu^0))\|ds\right|\le$$
$$\le\left|\int_{t_0}^{t}(L\|\varphi(s;\mu)-\varphi(s;\mu^0)\|+\varepsilon)ds\right|\le$$
$$\le\varepsilon|t-t_0|+L\left|\int_{t_0}^{t}\|\varphi(s;\mu)-\varphi(s;\mu^0)\|ds\right|.$$

Gronuoll-Bellman tengsizligiga ko'ra ((III.5.7) formula)

$$\| \varphi(t;\mu)-\varphi(t;\mu^0) \| \le \frac{\varepsilon}{L}(e^{L|t-t_0|}-1).$$

Endi $\varepsilon>0$ sonni shunday kichik tanlaylikki, uning uchun

$$\frac{\varepsilon}{L}\sup_{t\in[t_1,t_2]}(e^{L|t-t_0|}-1)<\rho$$

bo'lsin. Shu $\varepsilon>0$ songa ko'ra topilgan $\delta>0$, $\delta\le\sigma$, sonni tayinlaylik. U holda $\|\mu-\mu^0\|<\delta$ shartni qanoatlantiruvchi ixtiyoriy $\mu\in B_\delta(\mu^0)$ va ixtiyoriy $t\in[t_1,t_2]$ uchun $\|\varphi(t;\mu)-\varphi(t;\mu^0)\|<\rho$ bo'ladi, ya'ni $x=\varphi(t;\mu)$ yechim $t\in[t_1,t_2]$ paytlarda K nayning yon sirtiga yetib borolmaydi va, demak, $[t_1,t_2]$ gacha davom etadi. Endi $x=\varphi(t;\mu)$ vektor-funksiyaning aytilgan t va μ lar bo'yicha uzluksiz ekanligini ko'rsatishimiz qoldi. Analizdan ma'lumki, agar bu funksiya $\mu\in B_\delta(\mu^0)$ tayinlanganda t bo'yicha $[t_1,t_2]$ da uzluksiz (bizda esa uning t bo'yicha hosilasi ham uzluksiz) va μ bo'yicha $t\in[t_1,t_2]$ ga nisbatan tekis uzluksiz bo'lsa, u holda $x=\varphi(t;\mu)$ vektor-funksiya $(t;\mu)\in[t_1;t_2]\times B_\delta(\mu^0)$ o'zgaruvchilar majmuasi bo'yicha uzluksiz bo'ladi. Demak, biz quyidagini ko'rsatishimiz kifoya: har qanaday $\tilde\mu\in B_\delta(\mu^0)$ uchun yetarlicha kichik ixtiyoriy $\eta>0$ son berilganda ham shunday $\omega>0$ topiladiki, $\|\mu-\tilde\mu\|<\omega$ shartni qanoatlantiruvchi barcha $\mu\in B_\delta(\mu^0)$ lar va barcha $t\in[t_1,t_2]$ lar uchun $\|\varphi(t;\mu)-\varphi(t;\tilde\mu)\|<\eta$ tengsizlik o'rinli bo'ladi. Ixtiyoriy $\tilde\mu\in B_\delta(\mu^0)$ parametrni tayinlab, $(t,\mu)\in[t_1,t_2]\times\overline{B}_\sigma(\mu^0)$ ning ushbu $f(t,\varphi(t;\tilde\mu),\mu)$ vektor-funksiyasini qaraylik. Yuqoridagiga o'xshash fikr yuritib, $\|\varphi(t;\mu)-\varphi(t;\tilde\mu)\|$ ni baholaymiz. $f(t,\varphi(t;\tilde\mu),\mu)$ funksiya $[t_1,t_2]\times\overline{B}_\sigma(\mu^0)$ kompaktda uzluksiz b'lgani uchun u shu yerda tekis uzluksiz hamdir. Demak, ixtiyoriy $\theta>0$ son berilganda ham shunday $\omega=\omega(\theta,\tilde\mu)>0$ topiladiki, $\|\mu-\tilde\mu\|<\omega$ shartni qanoatlantiruvchi barcha $\mu\in\overline{B}_\delta(\mu^0)$ lar va ixtiyoriy $t\in[t_1,t_2]$ uchun $\|f(t,\varphi(t;\tilde\mu),\mu)-f(t,\varphi(t;\tilde\mu),\mu^0)\|<\theta$ bo'ladi. Endi yuqoridagi baholashlarga o'xshash ushbu

$$\|\varphi(t;\mu)-\varphi(t;\tilde\mu)\|\le\frac{\theta}{L}(e^{L|t-t_0|}-1),\ t\in[t_1,t_2],$$

tengsizlikni topamiz. Agar $\theta>0$ sonni

$$\frac{\theta}{L}\sup_{t\in[t_1,t_2]}(e^{L|t-t_0|}-1)<\eta$$

shartdan tanlab, unga mos $\omega=\omega(\theta,\tilde\mu)>0$ sonni topsak, u holda $\|\mu-\tilde\mu\|<\omega$ ekanligidan barcha $t\in[t_1,t_2]$ lar uchun

$$\|\varphi(t;\mu)-\varphi(t;\tilde\mu)\|\le\frac{\theta}{L}(e^{L|t-t_0|}-1)\le\frac{\theta}{L}\sup_{t\in[t_1,t_2]}(e^{L|t-t_0|}-1)<\eta$$

ekanligi kelib chiqadi. ♁

Biz teoremaning shartlarida quyidagi tenglikning o'rinli ekanligini isbotladik:

$$\varphi(t;\mu)=\varphi(t;\tilde\mu)+r_0(t;\mu^0),$$

bunda $r_0(t;\mu)$ qoldiq (farq) μ o'zgaruvchi μ^0 ga intilganda $t\in[t_1,t_2]$ ga nisbatan tekis nolga intiladi, ya'ni $r_0(t;\mu)\underset{\mu\to\mu^0}{\overset{t\in[t_1,t_2]}{\rightrightarrows}}0$

($r_0(t;\mu)$ qoldiq tekis $o(\mu-\mu^0)$ dan iborat).

Demak, μ^0 ga yaqin μ larda $\varphi(t;\mu)\approx\varphi(t;\mu^0)$ desak, bunda qilingan xato $r_0(t;\mu)=o(\mu-\mu^0)$, bo'ladi, ya'ni $\varphi(t;\mu)=\varphi(t;\mu^0)+o(\mu-\mu^0)$, $\mu\to\mu^0$.

Keltirilgan tenglikda yechimlar ayirmasining chegaralangan va tayin $[t_1,t_2]$ segmentda qaralayotganligi (baholanayotganligi) muhimdir. Buni quyidagi misollar asoslaydi.

Misol 1. Ushbu

$$x'=(x+\mu)^2\quad(\mu>0)$$

skalyar tenglamani qaraylik; bunda μ – kichik musbat parametr. Uning $\mu=\mu_0=0$ bo'lganda $x(0)=0$ boshlang'ich shartni qanoatlantiruvchi yechimi $x=\varphi(t;0)=0,\ 0\le t<+\infty$. Lekin qaralayotgan tenglamaning o'sha $x(0)=0$ boshlang'ich shartni qanoatlantiruvchi yechimi, topish qiyin emaski,

$$x=\varphi(t;\mu)=\frac{\mu}{1-\mu t}-\mu$$

formula bilan aniqlanadi. Bu yechim $0\le t<1/\mu$ ($\mu>0$) oralig'ida aniqlangan va μ parametr nolga intilganda bu oraliq cheksiz kengayadi va $x=\varphi(t;\mu)$ yechim $x=\varphi(t;0)$ yechimga shu oraliqda tekis intilmaydi; aslida

$$\sup_{0\le t<1/\mu}|\varphi(t;\mu)-\varphi(t;0)|=\sup_{0\le t<1/\mu}\frac{\mu^2 t}{1-\mu t}=+\infty.$$ ♘

Misol 2. Ushbu

$$\begin{cases} x'=y \\ y'=-\mu y-\omega^2 x \end{cases} \quad (\mu\ge 0,\ \omega>0)$$

sistemani qaraylik. U elastik prujinaga berkitilgan moddiy nuqtaning harakatiga tezlikka proporsional qarshilik kuchi bilan to'sqinlik qiluvchi muhitdagi harakatini ifodalaydi:

$$x''=-\mu x'-\omega^2 x. \qquad (*)$$

Qarshilik yo'qolganda $\mu=0$ va garmonik ossilyator tenglamasi hosil bo'ladi:

$$\begin{cases} x'=y \\ y'=-\omega^2 x \end{cases} \quad \text{yoki} \quad x''=-\omega^2 x;$$

uning umumiy yechimi $x=\varphi(t;0)=A\cos(\omega t+\alpha_0)$ ($A,\alpha_0-const$).

Qarshilik kichik, aniqrog'i $0<\mu<2\omega$ bo'lganda qaralayotgan (*) tenglamaning (sistemaning) umumiy yechimi

$$x=\varphi(t;\mu)=\tilde{A}e^{-t\mu/2}\cos(\tilde{\omega}t+\tilde{\alpha}_0),\ \tilde{\omega}=\frac{\sqrt{4\omega^2-\mu}}{2}\ (\tilde{A},\tilde{\alpha}_0-\text{const})$$

formula bilan beriladi. Ixtiyoriy chegaralangan vaqt oralig'ida bir xil boshlang'ich shartli $x=\varphi(t;\mu)$ va $x=\varphi(t;0)$ yechimlar kichik μ larda yaqin bo'ladi. Lekin cheksiz vaqtlar uchun ular yaqin bo'lmaydi, chunki $\varphi(t;\mu)\xrightarrow[t\to+\infty]{}0$, $\varphi(t;0)$ esa (o'zgarmas amplitudali, so'nmas) garmonik tebranishlarni ifodalaydi. ♘

Yechimning boshlang'ich ma'lumotlarga bog'liqligini o'rganish maqsadida Koshi masalasini ushbu

$$\begin{cases} x'=f(t,x,\mu) \\ x|_\tau=\xi \end{cases} \qquad (III.6.2)$$

ko'rinishda yozib olaylik, bunda $f(t,x,\mu)$ vektor-funksiya yuqorida aytilgan shartlarni qanoatlantiradi deb faraz qilinadi. Bu masalaning yechimi $x=\varphi(t;\tau,\xi,\mu)$ kabi yoziladi. Yangi $s=t-\tau,\ y=x-\xi$ o'zgaruvchilarga o'tamiz. Natijada ushbu

$$\begin{cases} \dfrac{dy}{ds}=f(s+\tau,y+\xi,\mu) \\ y|_0=0 \end{cases}$$

yangi masalani hosil qilamiz. Bu masalada endi (τ,ξ,μ) o'zgaruvchilar parametrlar rolini o'ynaydi:

$$\begin{cases} \dfrac{dy}{ds}=g(s,y,\tau,\xi,\mu) \\ y|_0=0 \end{cases} \qquad (III.6.3)$$

(bunda $g(s,y,\tau,\xi,\mu)=f(s+\tau,y+\xi,\mu)$)

Yechimni $y=\psi(s,\tau,\xi,\mu)$ bilan belgilaymiz. Bunda ravshanki, eski $x=\varphi(t;\tau,\xi,\mu)$ va yangi $y=\psi(s,\tau,\xi,\mu)$ yechimlar orasida $\varphi(t;\tau,\xi,\mu)=\xi+\psi(t-\tau,\tau,\xi,\mu)$ bog'lanish o'rinli. Isbotlangan teoremani (III.6.3) masalaga, ya'ni $y=\psi(s,\tau,\xi,\mu)$ yechimga qo'llab, oxirgi munosabatga ko'ra quyidagi teoremani hosil qilamiz.

Teorema (yechimning boshlang'ich ma'lumotlar va parametrlarga uzluksiz bog'liqligi). *Faraz qilaylik, $f\in C(D\times M,\mathbb{R}^n)$ bo'lsin va u $D\times M$ sohada x vektor o'zgaruvchi bo'yicha lokal Lipshits shartini qanoatlantirsin hamda*

$\mu = \mu^0$ *bo'lganda* (III.6.1) *masala* $t \in [t_1, t_2]$ $(t_0 \in [t_1, t_2], (t_0, x^0, \mu^0) \in D \times M)$ *segmentda aniqlangan* $x = \varphi(t; t_0, x^0, \mu^0)$ *yechimga ega bo'lsin. U holda shunday yetarlicha kichik* $\delta > 0$ *soni mavjudki, ushbu* $|\tau - t_0| < \delta$, $\|\xi - x^0\| < \delta$ *va* $\|\mu - \mu^0\| < \delta$ *shartlar bajarilganda* $x = \varphi(t; \tau, \xi, \mu)$ *yechim barcha* $t \in [t_1, t_2]$ *larda aniqlangan va* $(t; \tau, \xi, \mu)$ *(t yechimning argumenti,* (τ, ξ) *boshlang'ich ma'lumotlar,* μ *parametrlar) o'zgaruvchilar bo'yicha uzluksiz vektor-funksiyadan iborat bo'ladi, ya'ni*

$$\varphi(t; \tau, \xi, \mu) \in C([t_1, t_2] \times (t_0 - \delta, t_0 + \delta) \times B_\delta(x^0) \times B_\delta(\mu^0))$$

Masalalar

1. Quyidagi teoremani isbotlang.

Teorema (yechimning boshlang'ich qiymatlarga uzluksiz bog'liqligi). *Aytaylik,* $f(t, x)$ *vektor-funksiya* $(t, x) \in D$ $(D \subset \mathbb{R}^{1+n})$ *sohada uzluksiz va* x *bo'yicha Lipshits shartini qanoatlantirsin hamda* $(t_0, \xi^0) \in D$ *uchun ushbu*

$$\begin{cases} x' = f(t, x) \\ x|_{t_0} = \xi^0 \end{cases}$$

boshlang'ich masalaning $x = \varphi(t; \xi^0)$ *yechimi* $t \in [t_1; t_2]$ *oraliqda aniqlangan bo'lsin. U holda shunday* $\delta > 0$ *soni mavjudki,* $\|\xi - \xi^0\| < \delta$ *bo'lganda*

$$\begin{cases} x' = f(t, x) \\ x|_{t_0} = \xi \end{cases}$$

masalaning $x = \varphi(t; \xi)$ *yechimi ham* $t \in [t_1; t_2]$ *oraliqda aniqlangan va* ξ *boshlang'ich qiymat bo'yicha Lipshits shartini qanoatlantiradi, ya'ni shunday* $L > 0$ *soni mavjudki,* $t \in [t_1; t_2]$, $\|\xi' - \xi^0\| < \delta$, $\|\xi'' - \xi^0\| < \delta$ *ekanligidan* $\|\varphi(t; \xi') - \varphi(t; \xi'')\| \le L\|\xi' - \xi''\|$ *tengsizlik kelib chiqadi.*

IV BOB. CHIZIQLI NORMAL SISTEMALAR

IV.1. Chiziqli differensial tenglamalar normal sistemasining umumiy xossalari

1. Biz bu yerda ushbu

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + g_1(t) \\ \dfrac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + g_2(t) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + g_n(t) \end{cases}$$

n- tartibli chiziqli normal sistema yechimlarining umumiy xossalarini o'rganamiz. Qulaylik uchun bu sistemani vektorli ko'rinishda yozamiz:

$$x' = A(t)x + g(t), \qquad \text{(IV.1.1)}$$

bunda

$$A(t) = \|a_{ij}(t)\| \in C(I; \mathbb{M}_{n\times n}(\mathbb{R})),$$

$$g(t) = (g_1(t), g_2(t), \ldots, g_n(t))^T \in C(I, \mathbb{R}^n),$$

$$x = (x_1, x_2, \ldots, x_n)^T, \ x = x(t), \ t \in I,$$

deb hisoblanadi. Bu shartlarda, ma'lumki, ushbu

$$\begin{cases} x' = A(t)x + g(t) \\ x|_{t_0} = x^0 \quad (t_0 \in I, x^0 \in \mathbb{R}^n) \end{cases}$$

Koshi masalasi birato'la I oraliqda aniqlangan yagona yechimga ega.

Jumla 1 (Superpozitsiya prinsipi). *Agar* $x = x^1(t)$ *vektor-funksiya* $x' = A(t)x + g^1(t)$ *sistemaning,* $x = x^2(t)$ *vektor-funksiya esa* $x' = A(t)x + g^2(t)$ *sistemaning* *yechimlari bo'lsa, u holda* $x = \lambda_1 x^1(t) + \lambda_2 x^2(t)$ (λ_1, λ_2 – const) *vektor-funksiya ushbu* $x' = A(t)x + \lambda_1 g^1(t) + \lambda_2 g^2(t)$ *sistemaning yechimi bo'ladi.*

Isboti oson. Berilganga ko'ra

$$\frac{dx^1}{dt} = A(t)x^1 + g^1(t) \text{ va } \frac{dx^2}{dt} = A(t)x^2 + g^2(t).$$

Bu ayniyatlarning birinchisini λ_1 ga, ikkinchisini esa λ_2 ga ko'paytirib hadma-had qo'shsak, jumla isbot bo'ladi:

$$\frac{d(\lambda_1 x^1 + \lambda_2 x^1)}{dt} = A(t)(\lambda_1 x^1 + \lambda_2 x^1) + \lambda_1 g^1(t) + \lambda_2 g^2(t).$$

(IV.1.1) ga mos bir jinsli chiziqli sistema deb

$$x' = A(t)x \qquad \text{(IV.1.2)}$$

differensial tenglamalar sistemasiga aytiladi.

Jumla 2. *(IV.1.1) sistemaning barcha yechimlari (umumiy yechimi) uning biror (xususiy) yechimiga mos bir jinsli sistema (IV.1.2) ning barcha yechimlarini (umumiy yechimini) qo'shishdan hosil bo'ladi.*

$x = x_{xus}(t)$ vektor-funksiya (IV.1.1) sistemaning biror tayin yechimi bo'lsin, $\frac{dx_{xus}}{dt} = A(t)x_{xus} + g(t)$. Uning ixtiyoriy $x = \psi(t)$ yechimini olaylik, $\frac{d\psi}{dt} = A(t)\psi + g(t)$.

Superpozitsiya prinsipidan ravshanki, $x = x_{b.j.}(t) \overset{def}{=} \psi(t) - x_{xus}(t)$ vektor-funksiya (IV.1.2) bir jinsli sistemaning yechimi: $\frac{dx_{b.j.}}{dt} = A(t)x_{b.j.}$. Demak, $\psi(t) = x_{xus}(t) + x_{b.j.}(t)$, ya'ni (IV.1.1) sistemaning ixtiyoriy $x = \psi(t)$ yechimi uning $x = x_{xus}(t)$ xususiy yechimiga mos bir jinsli sistema (IV.1.2) ning $x = x_{b.j.}(t)$ yechimini qo'shishdan hosil bo'lgan. Ikkinchi tomondan, yana superpozitsiya prinsipidan ravshanki, (IV.1.1) sistemaning yechimiga mos bir jinsli

sistema (IV.1.2) ning yechimini qo'shib, yana (IV.1.1) sistemaning yechimini hosil qilamiz.

2. Yechimning yagonalik xossasidan kelib chiquvchi quyidagi natijani alohida e'tirof etaylik:

agar $x = x(t)$ vektor-funksiya (IV.1.2) bir jinsli sistemaning I oraliqda yechimi va biror $t_0 \in I$ nuqtada $x(t_0) = 0$ bo'lsa, u holda I oraliqda yechim aynan nolga teng, ya'ni $x(t) \equiv 0$, bo'ladi.

n- tartibli bir jinsli sistema (IV.1.2) ning barcha yechimlari to'plamini V_n bilan belgilaylik:

$$V_n = \left\{ x(t) \in C^1(I;\mathbb{R}^n) \,\middle|\, x'(t) \equiv A(t)x(t),\ t \in I \right\}.$$

Bu V_n to'plamda elementni (vektor-funksiyani) elementga (vektor-funksiyaga) qo'shish va elementni (vektor-funksiyani) songa ko'paytirish amallari odatdagidek, ya'ni nuqtama-nuqta aniqlanadi. Agar V_n to'plam bu amallarga nisbatan yopiq bo'lsa, tushunarliki, u chiziqli (vektor) fazoni tashkil etadi. Bu holda u $C^1(I;\mathbb{R}^n)$ chiziqli (vektor) fazonining qismfazosi ham bo'ladi.

Teorema. *Bir jinsli sistema* (IV.1.2) *ning ixtiyoriy ikki* $x = x^1(t)$ *va* $x = x^2(t)$ *yechimining ixtiyoriy* $x = \lambda_1 x^1(t) + \lambda_2 x^2(t)$ (λ_1, λ_2 – *const*) *chiziqli kombinatsiyasi ham shu sistemaning yechimidir. Demak,* V_n *chiziqli fazo.*

Berilganga ko'ra $\frac{dx^1}{dt} = A(t)x^1$ va $\frac{dx^2}{dt} = A(t)x^2$.

Superpozitsiya prinsipiga ko'ra har qanday λ_1 va λ_2 sonlar uchun $x = \lambda_1 x^1(t) + \lambda_2 x^2(t)$ vektor-funksiya ham (IV.1.2) ning yechimi. Demak,

$$\{x^1, x^2\} \subset V_n \Rightarrow \lambda_1 x^1 + \lambda_2 x^2 \in V_n.$$

V_n fazoda nol-vektor $x(t) \equiv 0$ trivial yechimdan iborat. $x(t) \in V_n$ vektorning qarama-qarshisi $(-1)x(t) = -x(t) \in V_n$ vektordir.

Masalalar

1. $V_n = \{x(t) \in C^1(I;\mathbb{R}^n) \mid x'(t) \equiv A(t)x(t),\ t \in I\}$ ($A(t) \in C(I;\mathbf{M}_{n\times n}(\mathbb{R}))$) ning chiziqli fazo ekanligini bevosita va qat'iy isbotlang.

2. V_n chiziqli fazoni $\mathbb{R}^n$ chiziqli fazoga akslantirishni quyidagicha kiritamiz.

$t_0 \in I$ nuqtani tayinlab, har bir $x(t) \in V_n$ yechimga uning $x(t_0) \in \mathbb{R}^n$ qiymatini mos qo'yaylik: $V_n \to \mathbb{R}^n$, $x(t) \to x(t_0)$.

Bu akslantirishning chiziqli fazolar izomorfizmi ekanligini (in'yektivligini, syuryektivligini va chiziqli amallarni saqlashini) ko'rsating. Izomorf chiziqli fazolarning o'lchamlari teng bo'lishidan $\dim V_n = \dim \mathbb{R}^n = n$ ekanligini asoslang.

IV.2. Chiziqli erkli va chiziqli bog'langan vektor-funksiyalar. Vronskian

$x^1(t), x^2(t), \ldots, x^m(t)$ vektor-funksiyalarning chiziqli kombinatsiyasi deb ushbu

$$\lambda_1 x^1(t) + \lambda_2 x^2(t) + \cdots + \lambda_m x^m(t)$$

ifodaga aytiladi. Bu yerdagi $\lambda_1, \lambda_2, \ldots, \lambda_m$ sonlar chiziqli kombinatsiyaning koeffitsientlari deb ataladi. Koeffitsientlar $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$ bo'lganda trivial chiziqli kombinatsiya hosil bo'ladi. Ravshanki, trivial chiziqli kombinatsiya nol-vektordan iborat.

Agar $x^1(t), x^2(t), \ldots, x^m(t)$, $t \in I$, vektor-funksiyalarning biror notrivial chiziqli kombinatsiyasi I oraliqda nol-vektorga teng, ya'ni kamida bittasi noldan farqli bo'lgan $\lambda_1, \lambda_2, \ldots, \lambda_m$ ($|\lambda_1| + |\lambda_2| + \cdots + |\lambda_m| \neq 0$) sonlar mavjud bo'lib, ular uchun

$$\lambda_1 x^1(t) + \lambda_2 x^2(t) + \cdots + \lambda_m x^m(t) \equiv 0,\ t \in I,$$

ayniyat o'rinli bo'lsa, u holda bu $x^1(t), x^2(t), \ldots, x^m(t)$ vektor-funksiyalar I oraliqda **chiziqli bog'langan** deyiladi. Aks holda, ya'ni berilgan vektor-funksiyalarning faqat trivial chiziqli

kombinatsiyasigina nol-vektordan iborat bo'lsa, ular **chiziqli erkli** (chiziqli bog'lanmagan) vektor-funksiyalar deb ataladi. Demak, $x^1(t), x^2(t), \ldots, x^m(t)$ funksiyalarning I oraliqda chiziqli erkliligi ushbu

$$\lambda_1 x^1(t) + \lambda_2 x^2(t) + \cdots + \lambda_m x^m(t) \equiv 0\ (t \in I) \Rightarrow$$
$$\Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$$

implikatsiyaning rostligini anglatadi.

Misol. Ushbu

$$x^1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix},\ x^2(t) = \begin{pmatrix} t \\ \sqrt{t-1} \end{pmatrix}$$

vektor-funksiyalarni tayinlangan ixriyoriy $I \subset [1;+\infty)$ oraliqda chiziqli erklilikka tekshiraylik.

Faraz qilaylik, biror λ_1 va λ_2 sonlar uchun

$$\lambda_1 \begin{pmatrix} 1 \\ t \end{pmatrix} + \lambda_2 \begin{pmatrix} t \\ \sqrt{t-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ t \in I,$$

bo'lsin. Bu vektor tenglikning skalyar ko'rinishi

$$\begin{cases} \lambda_1 \cdot 1 + \lambda_2 t = 0 \\ \lambda_1 t + \lambda_2 \sqrt{t-1} = 0 \end{cases},\ t \in I.$$

Bu yerdagi birinchi ayniyatning o'zidan $\lambda_1 = \lambda_2 = 0$ ekanligini topamiz. Demak, berilgan funksiyalarning trivial chiziqli kombinatsiyasigina I da aynan nolga teng. Shuning uchun ular ixtiyoriy $I \subset [1,+\infty)$ oraliqda chiziqli erkli.

Ushbu

$$x^1(t) = \begin{pmatrix} 1 \\ t \end{pmatrix},\ x^2(t) = \begin{pmatrix} 2 \\ 2t \end{pmatrix}$$

funksiyalar esa $(-\infty,+\infty)$ oralig'ida chiziqli bog'langan, chunki ularning quyidagi notrivial chiziqli kombinatsiyasi nol-vektorga teng:

$$(-2)\cdot x^1(t)+1\cdot x^2(t)=-2\begin{pmatrix}1\\t\end{pmatrix}+\begin{pmatrix}2\\2t\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$ ✍

Agar biror $\lambda_1,\lambda_2,\dots,\lambda_m$ sonlar uchun

$$x(t)=\lambda_1 x^1(t)+\lambda_2 x^2(t)+\dots+\lambda_m x^m(t)$$

tenglik o'rinli bo'lsa, $x(t)$ vektor-funksiya $x^1(t),x^2(t),\dots,x^m(t)$ vektor-funksiyalar orqali chiziqli ifodalangan deyiladi.

Teorema 1. $x^1(t),x^2(t),\dots,x^m(t)$ *vektor-funksiyalar* I *oraliqda chiziqli bog'langan bo'lishi uchun ularning birortasi qolganlari orqali* I *da chiziqli ifodalanishi yetarli va zarurdir.*

🗝 **Yetarliligi.** $x^1(t),x^2(t),\dots,x^m(t)$ vektor-funksiyalarning birortasi, masalan, $x^1(t)$ qolganlari $x^2(t),\dots,x^m(t)$ orqali chiziqli ifodalansin, ya'ni

$x^1(t)=\lambda_2 x^2(t)+\dots+\lambda_m x^m(t)$ bo'lsin. Bundan quyidagi notrivial chiziqli kombinatsiyaning nol-vektorga tengligi kelib chiqadi:

$$(-1)x^1(t)+\lambda_2 x^2(t)+\dots+\lambda_m x^m(t)=0.$$

Zarurligi. $x^1(t),x^2(t),\dots,x^m(t)$ vektor-funksiyalar chiziqli bog'langan, ya'ni ularning biror notrivial chiziqli kombinatsiyasi nol-vektordan iborat bo'lsin:

$$\lambda_1 x^1(t)+\lambda_2 x^2(t)+\dots+\lambda_m x^m(t)=0.$$

Bu yerda koeffitsientlarning kamida bittasi noldan farqli. Aniqlik uchun $\lambda_1\neq 0$ deylik. U holda $x^1(t)=-\dfrac{\lambda_2}{\lambda_1}x^2(t)-\dots-\dfrac{\lambda_m}{\lambda_1}x^m(t)$,

ya'ni $x^1(t)$ qolganlari orqali chiziqli ifodalangan. ✍

$n\times 1$ o'lchamli

$$x^1(t)=\begin{pmatrix}x_1^1(t)\\x_2^1(t)\\\vdots\\x_n^1(t)\end{pmatrix},\ x^2(t)=\begin{pmatrix}x_1^2(t)\\x_2^2(t)\\\vdots\\x_n^2(t)\end{pmatrix},\dots,x^n(t)=\begin{pmatrix}x_1^n(t)\\x_2^n(t)\\\vdots\\x_n^n(t)\end{pmatrix}$$

vektor-funksiyalarning **vronskiani (Vronskiy determinanti)** deb ushbu

$$W[x^1,x^2,\dots,x^n]=\begin{vmatrix}x_1^1(t)&x_1^2(t)&\cdots&x_1^n(t)\\x_2^1(t)&x_2^2(t)&\cdots&x_2^n(t)\\\vdots&\vdots&\ddots&\vdots\\x_n^1(t)&x_n^2(t)&\cdots&x_n^n(t)\end{vmatrix}\qquad \text{(IV.13)}$$

determinantga aytiladi.

Teorema 2. *Agar* $x^1(t),x^2(t),\dots,x^n(t)$ *vektor-funksiyalar* I *oraliqda chiziqli bog'langan bo'lsa, ularning vronskiani* I *da aynan nolga teng.*

🗝 Berilganga ko'ra kamida bittasi noldan farqli $\lambda_1,\lambda_2,\dots,\lambda_n$ sonlar ($|\lambda_1|+|\lambda_2|+\dots+|\lambda_m|\neq 0$) uchun

$$\lambda_1\begin{pmatrix}x_1^1(t)\\x_2^1(t)\\\vdots\\x_n^1(t)\end{pmatrix}+\lambda_2\begin{pmatrix}x_1^2(t)\\x_2^2(t)\\\vdots\\x_n^2(t)\end{pmatrix}+\dots+\lambda_n\begin{pmatrix}x_1^n(t)\\x_2^n(t)\\\vdots\\x_n^n(t)\end{pmatrix}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix},\quad t\in I,$$

ayniyat o'rinli. Bu tenglik $W[x^1,x^2,\dots,x^n]$ vronskian ustunlari orasida ixtiyoriy $t\in I$ nuqtada chiziqli bog'lanish mavjudligini anglatadi. Algebradan ma'lum teoremaga ko'ra, bu determinant ixtiyoriy $t\in I$ nuqtada nolga teng. ✍

Natija. *Agar* $x^1(t),x^2(t),\dots,x^n(t)$ *vektor-funksiyalarning vronskiani* I *oraliqning biror nuqtasida noldan farqli bo'lsa, bu vektor-funksiyalar* I *da chiziqli erkli.*

🗝 Haqiqatan ham, agar berilgan funksiyalar chiziqli bog'liq bo'lganda edi, u holda isbotlangan teoremaga ko'ra vronskian aynan nolga teng bo'lardi; bu esa berilganga zid. ♁

Umumiy holda vronskianning nolga tengligidan mos vektor-funksiyalarning chiziqli bog'langanligi kelib chiqmaydi. Lekin (IV.1.2) sistemaning yechimi bo'lgan funksiyalar uchun – kelib chiqadi. Buni quyidagi teorema asoslaydi.

Teorema 3. *n- tartibli bir jinsli sistema* (IV.1.2)*ning n dona $x^1(t), x^2(t), \dots, x^n(t)$, $t \in I$, yechimlari berilgan va ularning vronskiani $W(t)$ bo'lsin. Quyidagi alternativa o'rinli:*

yo $W(t)$ biror nuqtada ham nolga aylanmaydi va bu holda yechimlar chiziqli erkli; yoki $W(t)$ aynan nolga teng va bu holda yechimlar chiziqli bog'langan.

⊶ $W(t)$ biror nuqtada ham nolga aylanmasin. U holda yuqoridagi natijaga ko'ra berilgan $x^1(t), x^2(t), \dots, x^n(t)$, $t \in I$, yechimlar chiziqli erkli.

Endi faraz qilaylik, $x^1(t), x^2(t), \dots, x^n(t)$, $t \in I$, yechimlarning $W(t)$ vronskiani biror $t = t_0 \in I$ nuqtada nolga teng bo'lsin. Demak, ushbu

$$\begin{cases} \lambda_1 x_1^1(t_0) + \lambda_2 x_1^2(t_0) + \dots + \lambda_n x_1^n(t_0) = 0 \\ \lambda_1 x_2^1(t_0) + \lambda_2 x_2^2(t_0) + \dots + \lambda_n x_2^n(t_0) = 0 \\ \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \lambda_1 x_n^1(t_0) + \lambda_2 x_n^2(t_0) + \dots + \lambda_n x_n^n(t_0) = 0 \end{cases}$$

chiziqli bir jinsli algebraik sistema biror notrivial $\lambda_1, \lambda_2, \dots, \lambda_n$ $(|\lambda_1| + |\lambda_2| + \dots + |\lambda_n| \neq 0)$ yechimga ega. Ana shu notrivial yechimga ko'ra $x(t) = \lambda_1 x^1(t) + \lambda_2 x^2(t) + \dots + \lambda_n x^n(t)$ vektor-funksiyani tuzaylik. Yechimlarning chiziqli kombinatsiyasi sifatida $x(t)$ ham (IV.1.2) bir jinsli sistemaning yechimi. $\lambda_1, \lambda_2, \dots, \lambda_n$ larning tanlanishiga ko'ra $x(t_0) = 0$. Yechimning yagonalik xossasiga ko'ra $x(t) \equiv 0$, ya'ni

$\lambda_1 x^1(t) + \lambda_2 x^2(t) + \dots + \lambda_n x^n(t) \equiv 0$, $t \in I$. Bu ayniyat $|\lambda_1| + |\lambda_2| + \dots + |\lambda_n| \neq 0$ bo'lgani uchun $x^1(t), x^2(t), \dots, x^n(t)$, $t \in I$, yechimlarning chiziqli b og'langanligini isbotlaydi. Demak, ularning vronskiani aynan nolga teng. ♁

Shunday qilib, yechimlarning chiziqli erkli yoki chiziqli bog'langan ekanligini vronskian to'laligicha aniqlashga imkon beradi.

Masalalar

1. I oraliqda aniqlangan vektor-funksiyalar berilgan bo'lsin. Agar bu funksiyalar

a) biror $I' \subset I$ oraliqda chiziqli erkli bo'lsa, ular I oraliqda ham chiziqli erkli bo'ladi;

b) I oraliqda chiziqli bog'langan bo'lsa, ular ixtiyoriy $\tilde{I} \subset I$ oraliqda ham chiziqli bog'langan bo'ladi.

Shu tasdiqlarni isbotlang.

2. Ushbu

$$H(t) = \begin{cases} 0, \text{ agar } t < 0 \text{ bo'lsa} \\ 1, \text{ agar } t \geq 0 \text{ bo'lsa} \end{cases}$$

funksiya yordamida aniqlangan

$$x^1(t) = \begin{pmatrix} H(t) \\ H(t) \end{pmatrix}, \; x^2(t) = \begin{pmatrix} 1 - H(t) \\ 1 - H(t) \end{pmatrix}$$

vektor-funksiyalarni chiziqli erklilikka tekshiring.

IV.3. Fundamental matritsa. Chiziqli bir jinsli normal sistema umumiy yechimining tuzilishi

n - tartibli (IV.1.2) bir jinsli sistemaning n dona chiziqli erkli yechimlari bazis yechimlar yoki **yechimlarning fundamental sistemasi (fundamental sistema)** deb ataladi.

$x = \varphi^1(t), x = \varphi^2(t), \dots, x = \varphi^n(t)$ bazis yechimlarning koordinatalarini ustunlar bo'ylab yozishdan hosil bo'lgan ushbu

$$\Phi(t)=[\varphi^1(t),\varphi^2(t),\dots,\varphi^n(t)]=\begin{pmatrix}\varphi_1^1(t) & \varphi_1^2(t) & \cdots & \varphi_1^n(t)\\ \varphi_2^1(t) & \varphi_2^2(t) & \cdots & \varphi_2^n(t)\\ \vdots & \vdots & \ddots & \vdots\\ \varphi_n^1(t) & \varphi_n^2(t) & \cdots & \varphi_n^n(t)\end{pmatrix}$$

matritsa **fundamental matritsa** deb ataladi. Ravshanki, fundamental matritsaning determinanti mos yechimlarning vronskianidan iborat. $\det\Phi(t)=W(t)\neq 0$ bo'lgani uchun fundamental matritsa teskarilanuvchi, ya'ni $\Phi^{-1}(t)$ teskari matritsa mavjud: $\Phi^{-1}(t)\Phi(t)=\Phi(t)\Phi^{-1}(t)=E$, $E-n\times n$ o'lchamli birlik matritsa.

Agar $x=\varphi^1(t), x=\varphi^2(t),\dots,x=\varphi^n(t)$ $(t\in I)$ yechimlardan tuzilgan $\Phi(t)=[\varphi^1(t),\varphi^2(t),\dots,\varphi^n(t)]$ matritsa biror $t=t_0$ nuqtada teskarilanuvchi, ya'ni $\det\Phi(t_0)\neq 0$ bo'lsa, u holda barcha $t\in I$ nuqtalarda ham $\det\Phi(t)=W(t)\neq 0$ va $x=\varphi^1(t), x=\varphi^2(t),\dots,x=\varphi^n(t)$ yechimlar chiziqli erkli, berilgan $\Phi(t)$ matritsa esa fundamental matritsadan iborat bo'ladi.

Bazis yechimlarning (fundamental matritsaning) mavjudligini va umumiy yechimning ko'rinishini quyidagi teorema ifodalaydi.

Теорема. (IV.1.2) *bir jinsli sistema bazis yechimlarga ega va uning umumiy yechimi biror fundamental sistemasining ixtiyoriy chiziqli kombinatsiyasi sifatida ifodalanadi, ya'ni yechimlar fazosining o'lchami sistemaning tartibiga teng:* $\dim V_n=n$.

⊶ $\mathbb{R}^n$ fazoning standart bazisini odatdagidek $e^1=(1,0,0,\dots,0)^T$, $e^2=(0,1,0,\dots,0)^T,\dots,e^n=(0,0,\dots,0,1)^T$ bilan belgilab, ushbu

$$\begin{cases}x'=A(t)x\\ x(t_0)=e^j\end{cases},\quad j=\overline{1,n},$$

n dona Koshi masalasini qaraylik. Bu masalalarning har biri I oraliqda aniqlangan yagona yechimga ega. Yechimlarni mos ravishda $x=\varphi^1(t), x=\varphi^2(t),\dots,x=\varphi^n(t)$ bilan belgilaylik. Bu yechimlar chiziqli erkli, chunki ularning vronskiani t_0 nuqtada noldan farqli (birga teng). Shunday qilib, topilgan yechimlar (IV.1.2) sistemaning bazis yechimlaridir.

Endi faraz qilaylik, (IV.1.2) sistemaning $x=\varphi^1(t), x=\varphi^2(t),\dots,x=\varphi^n(t)$ fundamental sistemasi berilgan bo'lsin; bu yechimlar yuqorida qurilgan yechimlardan iborat bo'lishi shart emas. (IV.1.2) sistemaning umumiy yechimi $x=c_1\varphi^1(t)+c_2\varphi^2(t)+\dots+c_n\varphi^n(t)$ formula bilan ifodalanishini ko'rsatishimiz kerak; bunda $c_1,c_2,\dots,c_n$ – ixtiyoriy o'zgarmaslar. Birinchidan, bu formula o'zgarmaslarning ixtiyoriy qiymatida (IV.1.2) sistemaning yechimi (yechimlarning chiziqli kombinatsiyasi sifatida). Ikkinchidan, (IV.1.2) ning har qanday yechimi shu ko'rinishda ekanligini ko'rsatish kerak. (IV.1.2) ning ixtiyoriy $x=x(t)$ yechimi berilgan bo'lsin. Biror $t_0\in I$ nuqtani tayinlab, $x(t_0)\in\mathbb{R}^n$ vektorni $\varphi^1(t_0),\varphi^2(t_0),\dots,\varphi^n(t_0)$ vektorlarning chiziqli kombinatsiyasi ko'rinishida ifodalaylik:

$$x(t_0)=c_1\varphi^1(t_0)+c_2\varphi^2(t_0)+\dots+c_n\varphi^n(t_0).$$

Bu yerdagi $c_1,c_2,\dots,c_n$ sonlar bir qiymatli aniqlanadi, chunki $x=\varphi^1(t), x=\varphi^2(t),\dots,x=\varphi^n(t)$ chiziqli erkli yechimlarning vronskiani noldan farqli:

$$x(t_0)=\Phi(t_0)c,\ c=(c_1,c_2,\dots,c_n)^T\Rightarrow c=\Phi^{-1}(t_0)x(t_0).$$

Shu $c_1,c_2,\dots,c_n$ sonlarga ko'ra ushbu $\tilde{x}(t)=c_1\varphi^1(t)+c_2\varphi^2(t)+\dots+c_n\varphi^n(t)$ funksiyani tuzaylik. U (IV.1.2) ning yechimi (yechimlarning chiziqli kombinatsiyasi sifatida) va $\tilde{x}(t_0)=x(t_0)$ ($c_1,c_2,\dots,c_n$ larning tanlanishiga ko'ra). Shuning uchun yechimning yagonalik xossasidan $\tilde{x}(t)\equiv x(t)$, ya'ni $x(t)=c_1\varphi^1(t)+c_2\varphi^2(t)+\dots+c_n\varphi^n(t)$ kelib chiqadi. Shunday qilib, (IV.1.2) ning ixtiyoriy $x=x(t)$ yechimi berilgan bazis yechimlarning chiziqli kombinatsiyasi ko'rinishida bir qiymatli ifodalandi:

$$x = \Phi(t)c\ , \qquad \text{(IV.1.4)}$$

bunda

$\Phi(t) = [\varphi^1(t), \varphi^2(t), \ldots, \varphi^n(t)]$, $c = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n$. ♘

Demak, (IV.1.2) sistemaning barcha yechimlarini (umumiy yechimini) topish uchun uning n dona chiziqli erkli yechimlarini, ya'ni fundamental matritsasini topish yetarli.

Bu yerda shuni e'tirof etish kerakki, umumiy holda bazis yechimlarni qurish algoritmi (usuli) mavjud emas. Biz ularning mavjudligini isbotladik xolos.

IV.4. Fundamental matritsa xossalari

Jumla 1. *Fundamental matritsa* $\Phi = \Phi(t)$ *ushbu*

$$\Phi' = A(t)\Phi \qquad \text{(IV.4.1)}$$

matritsali differensial tenglamani qanoatlantiradi.

🗝 Isboti matritsalarni ko'paytirishning xossalaridan osongina kelib chiqadi:

$$\Phi'(t) = [(\varphi^1)'(t), (\varphi^2)'(t), \ldots, (\varphi^n)'(t)] =$$
$$= [A(t)\varphi^1(t), A(t)\varphi^2(t), \ldots, A(t)\varphi^n(t)] =$$
$$= A(t)[\varphi^1(t), \varphi^2(t), \ldots, \varphi^n(t)] =$$
$$= A(t)\Phi(t). \ ♘$$

Jumla 2. *Ushbu*

$$\begin{cases} \Phi' = A(t)\Phi \\ \Phi(t_0) = \Phi_0, \det\Phi_0 \neq 0 \end{cases}$$

matritsali Koshi masalasining yechimi (IV.1.2) sistemaning fundamental matritsasidir.

🗝 $\Phi = \Phi(t) = [\varphi^1(t), \varphi^2(t), \ldots, \varphi^n(t)]$ matritsa $\Phi' = A(t)\Phi$ tenglamani qanoatlantirgani uchun uning ustunlari (IV.1.2) sistemaninig yechimlaridan iborat bo'ladi. $\det\Phi(t_0) \neq 0$ bo'lgani uchun esa $\varphi^1(t), \varphi^2(t), \ldots, \varphi^n(t)$ lar chiziqli erkli yechimlarni tashkil etadi. ♘

Natija. *Agar* $\det C \neq 0$ *bo'lsa,* $\Phi(t)$ *bilan birgalikda* $\Phi(t)C$ *ham fundamental matritsadir.*

Teorema. (IV.1.2) *bir jinsli sistemaning ixtiyoriy ikki fundamental matritsasidan biri ikkinchisini biror teskarilanuvchi o'zgarmas matritsaga o'ngdan ko'paytirishdan hosil bo'ladi.*

🗝 $\Phi = \Phi(t)$ va $\tilde{\Phi} = \tilde{\Phi}(t)$ fundamental matritsalar berilgan bo'lsin:

$$\Phi' = A(t)\Phi,\ \det\Phi(t) \neq 0 \text{ va } \tilde{\Phi}' = A(t)\tilde{\Phi},\ \det\tilde{\Phi}(t) \neq 0. \qquad \text{(IV.4.2)}$$

Biz biror teskarilanuvchi o'zgarmas C matritsa uchun $\tilde{\Phi}(t) = \Phi(t)C$ bo'lishini ko'rsatishimiz kerak. Quyidagilarga egamiz:

$$\Phi(t)\Phi^{-1}(t) = E \Rightarrow \frac{d\left(\Phi(t)\Phi^{-1}(t)\right)}{dt} = 0 \Rightarrow$$
$$\Rightarrow \frac{d\Phi(t)}{dt}\Phi^{-1}(t) + \Phi(t)\frac{d\Phi^{-1}(t)}{dt} = 0.$$

Oxirgi tenglikdan teskari matritsa hosilasi uchun quyidagi formula kelib chiqadi:

$$\frac{d\Phi^{-1}(t)}{dt} = -\Phi^{-1}(t)\frac{d\Phi(t)}{dt}\Phi^{-1}(t). \qquad \text{(IV.4.3)}$$

Endi $\Phi^{-1}(t)\tilde{\Phi}(t)$ matritsaning hosilasi nol-matritsadan iborat ekanligini (IV.4.3) va (IV.4.2) formulalardan foydalanib, ko'rsatamiz:

$$\frac{d\left(\Phi^{-1}(t)\tilde{\Phi}(t)\right)}{dt} = \frac{d\Phi^{-1}(t)}{dt}\tilde{\Phi}(t) + \Phi^{-1}(t)\frac{d\tilde{\Phi}(t)}{dt} =$$
$$= -\Phi^{-1}(t)\frac{d\Phi(t)}{dt}\Phi^{-1}(t)\tilde{\Phi}(t) + \Phi^{-1}(t)\frac{d\tilde{\Phi}(t)}{dt} =$$

$$= -\Phi^{-1}(t)A(t)\underbrace{\Phi(t)\Phi^{-1}(t)}_{=E}\tilde{\Phi}(t) + \Phi^{-1}(t)A(t)\tilde{\Phi}(t) =$$

$$= 0.$$

Demak, $\Phi^{-1}(t)\tilde{\Phi}(t)$ matritsa o'zgarmas, ya'ni ixtiyoriy $t \in I$ uchun

$$\Phi^{-1}(t)\tilde{\Phi}(t) = C, \qquad \text{(IV.4.4)}$$

bunda

$$C = \Phi^{-1}(t_0)\tilde{\Phi}(t_0), \ \det C = \det\tilde{\Phi}(t_0)/\det\Phi(t_0) \neq 0, \ t_0 \in I.$$

Nihoyat, (IV.4.4) formuladan $\tilde{\Phi}(t) = \Phi(t)C$ ekanligini topamiz. ♦

Endi ushbu

$$\begin{cases} x' = A(t)x \\ x|_{t_0} = x^0, \ (t_0 \in I, x^0 \in \mathbb{R}^n) \end{cases} \qquad \text{(IV.4.5)}$$

Koshi masalasini yechaylik. Umumiy yechim formulasi (IV.1.4) $x = \Phi(t)c$ ga ko'ra boshlang'ich shart qanoantlanishi uchun $x^0 = \Phi(t_0)c$, ya'ni $c = \Phi^{-1}(t_0)x^0$ bo'lishi kerakligini topamiz. Demak, (IV.4.5) masala yechimi $x = \Phi(t)\Phi^{-1}(t_0)x^0$ ko'rinishda bo'ladi.

Ushbu

$$\Phi(t, t_0) = \Phi(t)\Phi^{-1}(t_0) \qquad \text{(IV.4.6)}$$

$t = t_0$ nuqtada normalangan ($\Phi(t_0, t_0) = \Phi(t_0)\Phi^{-1}(t_0) = E$) fundamental matritsani kiritib, (IV.4.5) Koshi masalasining yechimini

$$x = \Phi(t, t_0)x^0 \qquad \text{(IV.4.7)}$$

ko'rinishda ifodalaymiz.

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$$x_1' = \frac{2}{t}x_1 + \frac{5}{t}x_2, \ x_2' = \frac{1}{t}x_1 - \frac{2}{t}x_2 \quad (t > 0)$$

sistemaning barcha yechimlarini topaylik.

⊢ Berilgan sistemaning darajali funksiya sifatidagi yechimlari mavjudligi uning shaklidan ko'rinib turibdi. Yechimni

$x_1 = \alpha t^k, x_2 = \beta t^k$ ko'rinishda izlaymiz. Bu funksiyalarni sistemaga qo'yib, quyidagi munosabatlarga kelamiz:

$$\begin{cases} \alpha k = 2\alpha + 5\beta = 0 \\ \beta k = \alpha - 2\beta \end{cases} \Leftrightarrow \begin{cases} (2-k)\alpha + 5\beta = 0 \\ \alpha - (2+k)\beta = 0 \end{cases} \qquad \text{(IV.4.8)}$$

α, β larga nisbatan bu chiziqli bir jinsli algebraik tenglamalar sistemasi notrivial yechimga ega bo'lishi uchun uning determinanti nolga teng bo'lishi kerak:

$$\begin{vmatrix} 2-k & 5 \\ 1 & -2-k \end{vmatrix} = 0 \Leftrightarrow k = \pm 3.$$

Yuqoridagi (IV.4.8) chiziqli sistemadan $k = 3$ va $k = -3$ larga mos α, β larni aniqlaymiz:

$$k = 3: \ \alpha = 5, \beta = 1,$$

$$k = -3: \ \alpha = 1, \beta = -1.$$

Shunday qilib, biz berilgan sistemaning

$$x^1(t) = \begin{pmatrix} 5t^3 \\ t^3 \end{pmatrix} \ \text{va} \ x^2(t) = \begin{pmatrix} \frac{1}{t^3} \\ -\frac{1}{t^3} \end{pmatrix}$$

yechimlarini topdik. Ular chiziqli erkli, chunki mos vronskian

$$W[x^1(t), x^2(t)] = \begin{vmatrix} 5t^3 & \frac{1}{t^3} \\ t^3 & -\frac{1}{t^3} \end{vmatrix} = -6 \neq 0.$$

Demak, qaralayotgan sistemaning umumiy yechimi topilgan yechimlarning chiziqli kombinatsiyasidan iborat:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 5t^3 \\ t^3 \end{pmatrix} + c_2 \begin{pmatrix} \frac{1}{t^3} \\ -\frac{1}{t^3} \end{pmatrix}$$

yoki skalyar ko'rinishda

$$\begin{cases} x_1(t) = 5c_1t^3 + \dfrac{c_2}{t^3} \\ x_2(t) = c_1t^3 - \dfrac{c_2}{t^3} \end{cases} \qquad c_1,\, c_2 - \text{const}.$$

Teorema (Liuvill formulasi). *Agar n ta vektor-funksiya n- tartibli (IV.1.2) bir jinsli sistemaning I oraliqda yechimi bo'lsa, u holda ularning $W(t)$ vronskiani uchun I da ushbu*

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t} trA(s)ds\right) (t_0 \in I) \qquad \text{(IV.4.9)}$$

Liuvill formulasi o'rinli; bunda $trA(s) = \sum_{j=1}^{n} a_{jj}(s)$ miqdor $A(s)$ matritsaning izi.

(IV.1.2) sistemaning $x^1(t), x^2(t), \ldots, x^n(t)$ yechimlari vronskiani

$$W(t) = \begin{vmatrix} x_1^1(t) & x_1^2(t) & \cdots & x_1^n(t) \\ x_2^1(t) & x_2^2(t) & \cdots & x_2^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_n^1(t) & x_n^2(t) & \cdots & x_n^n(t) \end{vmatrix}$$

ning hosilasini hisoblaylik. Determinantni differensiallash qoidasiga ko'ra

$$W'(t) = W_1(t) + W_2(t) + \cdots + W_n(t), \qquad \text{(IV.4.10)}$$

bunda $W_j(t)$ determinant $W(t)$ dan uning j- satridagi elementlarini ularning hosilasi bilan almashtirishdan hosil bo'lgan. $W_1(t)$ ni hisoblaymiz ($\dot{x}_i^j(t) \equiv \dfrac{dx_i^j(t)}{dt}$):

$$W_1(t) = \begin{vmatrix} \dot{x}_1^1(t) & \dot{x}_1^2(t) & \cdots & \dot{x}_1^n(t) \\ x_2^1(t) & x_2^2(t) & \cdots & x_2^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_n^1(t) & x_n^2(t) & \cdots & x_n^n(t) \end{vmatrix}. \qquad \text{(IV.4.11)}$$

$x^1(t), x^2(t), \ldots, x^n(t)$ funksiyalar (IV.1.2) sistemaning yechimi bo'lgani uchun

$$\dot{x}_1^1(t) = \sum_{k=1}^{n} a_{1k}(t)x_k^1(t),\ \dot{x}_1^2(t) = \sum_{k=1}^{n} a_{1k}(t)x_k^2(t), \ldots,$$
$$\dot{x}_1^n(t) = \sum_{k=1}^{n} a_{1k}(t)x_k^n(t). \qquad \text{(IV.4.12)}$$

Bu formulalarni hisobga olib, (IV.4.11) determinantning 2-satrini $(-a_{12}(t))$ ga, 3-satrini $(-a_{13}(t))$ ga va h.k., n-satrini $(-a_{1n}(t))$ ga ko'paytirib 1-satrga qo'shamiz. Bunda determinantning qiymati o'zgarmaydi va natijada

$$W_1(t) = \begin{vmatrix} a_{11}(t)x_1^1(t) & a_{11}(t)x_1^2(t) & \cdots & a_{11}(t)x_1^n(t) \\ x_2^1(t) & x_2^2(t) & \cdots & x_2^n(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_n^1(t) & x_n^2(t) & \cdots & x_n^n(t) \end{vmatrix} = a_{11}(t)W(t)$$

munosabatga kelamiz. Shunga o'xshash almashtirishlarni bajarib, qolgan $W_j(t)$ determinantlarni ham hisoblaymiz:

$$W_2(t) = a_{22}(t)W(t), \ldots, W_n(t) = a_{nn}(t)W(t).$$

Hisoblangan $W_j(t)$ larni (IV.4.10) formulaga qo'yib,

$$W'(t) = \left(a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t)\right)W(t),$$

ya'ni

$$W'(t) = trA(t) \cdot W(t)$$

ekanligini topamiz. Oxirgi tenglik $W(t)$ ga nisbatan birinchi tartibli chiziqli differensial tenglamadir. Undan (IV.4.9) Liuvill formulasi ravshan. ♁

Liuvill formulasidan bizga ma'lum bo'lgan quyidagi tasdiq o'z-o'zidan kelib chiqadi: agar (IV.1.2) bir jinsli sistema yechimlarining vronskiani biror $t_0 \in I$ nuqtada nolga teng bo'lsa, u barcha $t \in I$ nuqtalarda ham nolga teng.

Bazis yechimlariga ko'ra chiziqli bir jinsli sistemani tiklash

Biz yuqorida (IV.1.2) bir jinsli sistema uchun bazis yechimlarning mavjudligini ko'rsatdik. Endi bazis yechimlariga ko'ra mos bir jinsli sistemani tiklash masalasini qaraymiz.

Teorema. *Aytaylik,* $\{\varphi^1(t), \varphi^2(t), \ldots, \varphi^n(t)\} \subset C^1(I;\mathbb{R}^n)$ *funksiyalarning* $W(t) = W[\varphi^1(t), \varphi^2(t), \ldots, \varphi^n(t)]$ *vronskiani* I *oraliqda nolga aylanmasin. U holda bazis yechimlari shu funksiyalardan iborat bo'lgan* $x' = A(t)x$ *ko'rinishdagi normal sistema mavjud, yagona va u* $x' = \Phi'(t)\Phi^{-1}(t)x$, *bunda* $\Phi(t) = [\varphi^1(t), \varphi^2(t), \ldots, \varphi^n(t)]$. *sistemadan iborat.*

⊶ Dastlab teoremaning yagonalik qismini isbotlaylik. Faraz qilaylik, $x' = A(t)x$ ko'rinishdagi sistema $x = \varphi^j(t)$, $j = 1,2,\ldots,n$, yechimlarga ega bo'lsin. Demak,

$$\dot{\varphi}^j(t) = A(t)\varphi^j(t),\ t \in I,\ j = \overline{1,n}$$

ayniyatlar o'rinli. Ularni bitta matritsaviy ayniyat $\Phi'(t) = A(t)\Phi(t),\ t \in I,$ $(\Phi(t) = [\varphi^1(t), \varphi^2(t), \ldots, \varphi^n(t)])$ ko'rinishida yozib, $A(t)$ matritsaning $A(t) = \Phi'(t)\Phi^{-1}(t),\ t \in I,$ formula bilan bir qiymatli aniqlanishini topamiz.

Endi teoremaning mavjudlik qismini isbotlaymiz. Buning uchun ushbu $x' = \Phi'(t)\Phi^{-1}(t)x$ ciziqli normal sistemaning bazis yechimlari $x = \varphi^j(t)$, $j = 1,2,\ldots,n$, ekanligini ko'rsatamiz. Ixtiyoriy o'zgarmas $c \in \mathbb{R}^n$ vektor uchun $x = \Phi(t)c$ funksiya qaralayotgan $x' = \Phi'(t)\Phi^{-1}(t)x$ sistemaning yechimi:

$$\Phi'(t)c = \Phi'(t)\underbrace{\Phi^{-1}(t)\Phi'(t)}_{=E}c = \Phi'(t)Ec = \Phi'(t)c;$$

c o'rniga $e^j \in \mathbb{R}^n$, $j = 1,2,\ldots,n$, bazis vektorlarni olib, $x = \varphi^j(t)$, $j = 1,2,\ldots,n$, lar yechim ekanligini ko'ramiz. $\det\Phi(t) = W(t)$ noldan farqli bo'lgani bu yechimlar bazis yechimlarni tashkil etadi.

Eslatma. Izlangan normal sistemani quyidagi ko'rinishda ham yozish mumkin:

$$\frac{1}{W(t)}\begin{vmatrix} \dot{x}_i & \dot{\varphi}_i^1(t) & \cdots & \dot{\varphi}_i^j(t) & \cdots & \dot{\varphi}_i^n(t) \\ x_1 & \varphi_1^1(t) & \cdots & \varphi_1^j(t) & \cdots & \varphi_1^n(t) \\ x_2 & \varphi_2^1(t) & \cdots & \varphi_2^j(t) & \cdots & \varphi_2^n(t) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ x_n & \varphi_n^1(t) & \cdots & \varphi_n^j(t) & \cdots & \varphi_n^n(t) \end{vmatrix} = 0,\ i = 1,2,\ldots,n$$

(IV.4,13) ♁

Masalalar

1. Aytaylik, $A(t) \in C(I;\mathbb{M}_{n\times n}(\mathbb{R}))$ matritsa ushbu $A^T(t) = -A(t),\ t \in I,$ shartni qanoatlantirsin. Agar

$$x' = A(t)x$$

sistemaning $\Phi(t)$ fundamental matritsasi biror $t_0 \in I$ nuqtada ortogonal ($\Phi(t_0)\Phi^T(t_0) = E$) bo'lsa, u ixtiyoriy $t \in I$ nuqtada ham ortogonal bo'lishini isbotlang.

2. Faraz qilaylik, $A(t) \in C(I;\mathbb{M}_{n\times n}(\mathbb{R}))$ matritsa har qanday $t \in I$ nuqtada simmetrik, ya'ni $A^T(t) = A(t)$ bo'lsin. Agar

$$x' = A(t)x$$

sistemaning $\Phi(t)$ fundamental matritsasi biror $t_0 \in I$ nuqtada simmetrik bo'lsa, u ixtiyoriy $t \in I$ nuqtada ham simmetrik bo'lishini ko'rsating.

IV.5. Bir jinsli bo'lmagan normal sistemani yechish

Bir jinsli bo'lmagan chiziqli differensial tenglamalar sistemasi (IV.1.1) ga qaytaylik.

IV.1. paragrafda isbotlagan edikki, (IV.1.1) sistemaning umumiy yechimi uning biror (xususiy) yechimiga mos bir jinsli sistema (IV.1.2) ning umumiy yechimini qo'shishdan hosil bo'ladi. (IV.1.1) ning xususiy yechimini esa mos bir jiinsli sistema (IV.1.2) ning fundamental matritsasi $\Phi(t)$ orqali topish mumkin. Buning uchun bir jinsli sistema (IV.1.2) ning umumiy yechimidagi ixtiyoriy o'zgarmaslar $(c_1, c_2, \dots, c_n)^T = \boldsymbol{c}$ ni variatsiyalaymiz (Lagranj metodi) va (IV.1.1) ning xususiy yechimini

$$\boldsymbol{x}(t) = \Phi(t)\boldsymbol{u}(t) \qquad \text{(IV.5.1)}$$

ko'rinishda izlaymiz, bunda $\boldsymbol{u}(t)$ – hozircha noma'lum vektor-funksiya. (IV.5.1) dan (IV.4.1) ga ko'ra

$$\boldsymbol{x}'(t) = \Phi'(t)\boldsymbol{u}(t) + \Phi(t)\boldsymbol{u}'(t) = A(t)\Phi(t)\boldsymbol{u}(t) + \Phi(t)\boldsymbol{u}'(t).$$

Buni va (IV.5.1)ni (IV.1.1) sistemaga qo'yib, $\boldsymbol{u}(t)$ noma'lum vektor-funksiya uchun

$$A(t)\Phi(t)\boldsymbol{u}(t) + \Phi(t)\boldsymbol{u}'(t) = A(t)\Phi(t)\boldsymbol{u}(t) + \boldsymbol{g}(t),$$

ya'ni

$$\boldsymbol{u}'(t) = \Phi^{-1}(t)\boldsymbol{g}(t)$$

tenglamani hosil qilamiz. Oxirgi tenglamaning

$$\boldsymbol{u}(t) = \int_{t_0}^{t} \Phi^{-1}(s)\boldsymbol{g}(s)ds$$

xususiy yechimni olamiz va uni (IV.5.1) ga qo'yib, (IV.1.1)ning izlangan

$$\boldsymbol{x}(t) = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\boldsymbol{g}(s)ds \qquad \text{(IV.5.2)}$$

xususiy yechimini topamiz. Bir jinsli bo'lmagan (IV.1.1) sistemaning (IV.5.2) xususiy yechimiga unga mos bir jinsli sistemaning umumiy yechimni qo'shib, (IV.1.1) sistemaning umumiy yechimi uchun ushbu

$$\boldsymbol{x}(t) = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\boldsymbol{g}(s)ds + \Phi(t)\boldsymbol{c} \qquad \text{(IV.5.3)}$$

formulani topamiz, bunda $\boldsymbol{c} \in \mathbb{R}^n$ – ixtiyoriy o'zgarmas vektor.

Endi (IV.5.3) umumiy yechimdan foydalanib, ushbu

$$\begin{cases} \boldsymbol{x}' = A(t)\boldsymbol{x} + \boldsymbol{g}(t) \\ \boldsymbol{x}|_{t_0} = \boldsymbol{x}^0 \quad (t_0 \in I, \boldsymbol{x}^0 \in \mathbb{R}^n) \end{cases} \qquad \text{(IV.5.4)}$$

boshlang'ich masalaning yechimi

$$\boldsymbol{x}(t) = \Phi(t,t_0)\boldsymbol{x}^0 + \int_{t_0}^{t} \Phi(t,s)\boldsymbol{g}(s)ds \qquad \text{(IV.5.5)}$$

ko'rinishda bo'lishini osongina topamiz. Bu (IV.5.5) formula Koshi formulasi deb ataladi.

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$$\begin{cases} x_1' = \dfrac{2}{t}x_1 + \dfrac{5}{t}x_2 + g_1(t) \\ x_2' = \dfrac{1}{t}x_1 - \dfrac{2}{t}x_2 + g_2(t) \end{cases}, \ \{g_1(t), g_2(t)\} \subset C\big((0;+\infty);\mathbb{R}\big),$$

chiziqli sistemaning $x_1(1) = 1$, $x_2(1) = -1$ boshlang'ich shartlarni qanoatlantiruvchi yechimini topaylik.

⊶ Berilgan sistemaga mos bir jinsli sistemaning bazis yechimlarini yuqorida (IV.4 dagi misolga qarang) topgan edik. Unga ko'ra

$$\Phi(t) = \begin{pmatrix} 5t^3 & t^{-3} \\ t^3 & -t^{-3} \end{pmatrix} \ (t > 0).$$

normalangan fundamental matritsa $\Phi(t,s)$ ni hisoblash uchun $\Phi^{-1}(s)$ teskari matritsani topish kerak. Kerakli hisoblashlarni bajarib,

$$\Phi^{-1}(s) = \frac{1}{6}\begin{pmatrix} s^{-3} & s^{-3} \\ s^3 & -5s^3 \end{pmatrix} \ (s > 0)$$

ekanligini topamiz. Demak ($t > 0, s > 0$),

$$\Phi(t,s) = \Phi(t)\Phi^{-1}(s) = \frac{1}{6}\begin{pmatrix} 5t^3 & t^{-3} \\ t^3 & -t^{-3} \end{pmatrix}\begin{pmatrix} s^{-3} & s^{-3} \\ s^3 & -5s^3 \end{pmatrix} =$$

$$= \frac{1}{6}\begin{pmatrix} 5t^3s^{-3} + t^{-3}s^3 & 5t^3s^{-3} - 5t^{-3}s^3 \\ t^3s^{-3} - t^{-3}s^3 & t^3s^{-3} + 5t^{-3}s^3 \end{pmatrix}.$$

Endi (IV.5.5) Koshi formulasiga ko'ra izlangan yechimni topamiz

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 5t^3 + t^{-3} & 5t^3 - 5t^{-3} \\ t^3 - t^{-3} & t^3 + 5t^{-3} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} +$$

$$+ \frac{1}{6}\int \begin{pmatrix} 5t^3s^{-3} + t^{-3}s^3 & 5t^3s^{-3} - 5t^{-3}s^3 \\ t^3s^{-3} - t^{-3}s^3 & t^3s^{-3} + 5t^{-3}s^3 \end{pmatrix}\begin{pmatrix} g_1(s) \\ g_2(s) \end{pmatrix} ds$$

yoki soddalashtirishlardan keyin uni

$$x_1(t) = \frac{1}{t^3} + \frac{1}{6}\int \left(5t^3\frac{g_1(s) + g_2(s)}{s^3} + \frac{s^3(5g_1(s) - g_2(s))}{5t^3}\right)ds,$$

$$x_2(t) = -\frac{1}{t^3} + \frac{1}{6}\int \left(t^3\frac{g_1(s) + g_2(s)}{s^3} + \frac{s^3(-5g_1(s) + g_2(s))}{5t^3}\right)ds$$

(bunda $t > 0$)

skalyar ko'rinishga keltiramiz.

IV.6. Sistemani komplekslashtirish

Biz yuqorida $x' = A(t)x + g(t)$ sistemani haqiqiy sohada o'rgandik, ya'ni berilgan funksiyalar va yechim haqiqiy edi. Ba'zan bunday sistemalarni kompleks sohada qarashga to'g'ri keladi.

Kompleks sonlar maydoni ustida qurilgan $\mathbb{C}^n = \{ w = (w_1, w_2, \ldots, w_n)^T \mid w_j \in \mathbb{C}, j = \overline{1,n} \}$ chiziqli (vektor) fazoni ushbu $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$ to'g'ri yig'indi sifatida tasvirlaylik, ya'ni $\mathbb{C}^n = \{ w = u + iv \mid \{u, v\} \subset \mathbb{R}^n \}$ deylik; bu yerda

$\mathrm{Re}\, w = u \in \mathbb{R}^n$, $\mathrm{Im}\, w = v \in \mathbb{R}^n$ haqiqiy vektorlar $w \in \mathbb{C}^n$ kompleks vektorning mos ravishda haqiqiy va mavhum qismlari deb ataladi. Bunda $\lambda = \alpha + i\beta$ ($\{\alpha, \beta\} \subset \mathbb{R}\}$) kompleks son va $w = u + iv \in \mathbb{C}^n$ kompleks vektor uchun

$$\lambda w = (\alpha + i\beta)(u + iv) = \alpha u - \beta v + i(\alpha v + \beta v)$$

va $w^1 = u^1 + iv^1 \in \mathbb{C}^n$, $w^2 = u^2 + iv^2 \in \mathbb{C}^n$ kompleks vektorlar uchun

$$w^1 + w^2 = (u^1 + iv^1) + (u^2 + iv^2) = u^1 + u^2 + i(v^1 + v^2) \in \mathbb{C}^n$$

bo'ladi. $\mathbb{C}^n$ fazoning bunday tasvirlanishi $\mathbb{R}^n$ fazoning komplekslashtirilishi deb ataladi. $\mathbb{C}^n$ fazoning $\mathrm{Re}\,\mathbb{C}^n = \mathbb{R}^n \oplus i0$ qismi $\mathbb{R}^n$ fazo bilan tenglashtiriladi. $w = u + iv \in \mathbb{C}^n$ kompleks vektorning qo'shmasi deb $\overline{w} = \overline{u + iv} = u - iv \in \mathbb{C}^n$ kompleks vektorga aytiladi.

Jumla. *$\mathbb{R}^n$ fazoning ixtiyoriy bazisi uning komplekslashtirilishi bo'lgan $\mathbb{C}^n$ fazoning ham bazisidir.*

$b^1, b^2, \ldots, b^n$ vektorlar $\mathbb{R}^n$ fazoning ixtiyoriy bazisi bo'lsin. Ixtiyoriy $w = u + iv \in \mathbb{C}^n$ ($u \in \mathbb{R}^n, v \in \mathbb{R}^n$) vektorni olaylik. $u = \alpha_1 b^1 + \alpha_2 b^2 + \cdots + \alpha_n b^n$ va $v = \beta_1 b^1 + \beta_2 b^2 + \cdots + \beta_n b^n$ bo'lgani uchun

$$w = \alpha_1 b^1 + \alpha_2 b^2 + \cdots + \alpha_n b^n + i(\beta_1 b^1 + \beta_2 b^2 + \cdots + \beta_n b^n) =$$

$$= (\alpha_1 + i\beta_1)b^1 + (\alpha_2 + i\beta_2)b^2 + \cdots + (\alpha_n + i\beta_n)b^n,$$

ya'ni ixtiyoriy $w = u + iv \in \mathbb{C}^n$ vektor $b^1, b^2, \ldots, b^n$ vektorlar orqali chiziqli ifodalanadi. Endi $b^1, b^2, \ldots, b^n$ vektorlarning $\mathbb{C}^n$ fazoda chiziqli erkli ekanligini ko'rsatamiz. Agar $(\alpha_1 + i\beta_1)b^1 + (\alpha_2 + i\beta_2)b^2 + \cdots + (\alpha_n + i\beta_n)b^n = 0 + i0$ bo'lsa, u holda

$$\alpha_1 b^1 + \alpha_2 b^2 + \cdots + \alpha_n b^n + i(\beta_1 b^1 + \beta_2 b^2 + \cdots + \beta_n b^n) = 0 + i0$$

va bundan

$$\left.\begin{array}{l}\alpha_1 b^1 + \alpha_2 b^2 + \cdots + \alpha_n b^n = 0\\ \beta_1 b^1 + \beta_2 b^2 + \cdots + \beta_n b^n = 0\end{array}\right\} \Rightarrow \alpha_1 = \alpha_2 = \cdots \alpha_n = \beta_1 = \beta_2 = \cdots = \beta_n = 0 \cdot$$

Demak, $b^1, b^2, \ldots, b^n$ vektorlar $\mathbb{C}^n$ fazoning ham bazisi. ♠

$n \times n$ o'lchamli A matritsaning elementlari kompleks sonlardan iborat bo'lsa ($A \in \mathbb{M}_{n\times n}(\mathbb{C})$), u holda $A = \operatorname{Re} A + i \operatorname{Im} A$ deb yozish mumkin, bunda $\{\operatorname{Re} A, \operatorname{Im} A\} \subset \mathbb{M}_{n\times n}(\mathbb{R})$ matritsalar haqiqiy elementlardan tuzilgan va $n \times n$ o'lchamli, ular A ning (mos ravishda) haqiqiy va mavhum qismlari deb ataladi.

$t \in I$ haqiqiy o'zgaruvchining kompleks qiymatli vektor-funksiyasi $w : I \to \mathbb{C}^n$ akslantirishni anglatadi. Bu funksiya har bir $t \in I$ haqiqiy songa $w(t) = u(t) + iv(t) \in \mathbb{C}^n$ kompleks vektorni mos keltiradi, bunda $u : I \to \mathbb{R}^n$, $v : I \to \mathbb{R}^n$ – haqiqiy vektor-funksiyalar; ular $w : I \to \mathbb{C}^n$ kompleks vektor-funksiyaning (mos ravishda) haqiqiy va mavhum qismlari deb ataladi. Agar $\operatorname{Re} w(t) = u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T \in \mathbb{R}^n$,

$\operatorname{Im} w(t) = v(t) = (v_1(t), v_2(t), \ldots, v_n(t))^T \in \mathbb{R}^n$ desak, $w : I \to \mathbb{C}^n$ kompleks vektor-funksiyani $2n$ dona $u_j(t), v_j(t)$ $(j = \overline{1,n})$ haqiqiy funksiyalar (koordinata funksiyalari) orqali berish mumkin. $A : I \to \mathbb{M}_{n\times n}(\mathbb{C})$ matritsaviy qiymatli funksiya har bir $t \in I$ haqiqiy songa

$A(t) = \operatorname{Re} A(t) + i \operatorname{Im} A(t) \in \mathbb{M}_{n\times n}(\mathbb{C})$ $(\{\operatorname{Re} A(t), \operatorname{Im} A(t)\} \subset \mathbb{M}_{n\times n}(\mathbb{R}))$ matritsani mos keltiradi. Analizning kompleks vektor-funksiyalar (kompleks matritsaviy qiymatli funksiyalar) uchun limit, uzluksizlik, hosila, integral va h.k. tushunchalari, odatdagidek, ya'ni koordinatalar bo'ylab kiritiladi. Masalan, $w(t) = u(t) + iv(t)$ kompleks vektor-funksiyaning hosilasi

$$w'(t) = u'(t) + iv'(t) = \lim_{h\to 0} \frac{u(t+h) - u(t)}{h} + i \lim_{h\to 0} \frac{v(t+h) - v(t)}{h} =$$

$$= \lim_{h\to 0} \frac{w(t+h) - w(t)}{h}$$

formula yordamida aniqlanadi.

$C(I;\mathbb{C}^n)$ bilan barcha $w : I \to \mathbb{C}^n$ uzluksiz kompleks vektor-funksiyalar, $C(I;\mathbb{M}_{n\times n}(\mathbb{C}))$ bilan esa barcha $A : I \to \mathbb{M}_{n\times n}(\mathbb{C})$ uzluksiz matritsaviy funksiyalar sinfini belgilaymiz.

Endi

$$x' = A(t)x + g(t)$$

kompleks chiziqli sistemani qarash mumkin, bunda

$$A(t) \in C\left(I; \mathbb{M}_{n\times n}(\mathbb{C})\right),$$

$$g(t) = \left(g_1(t), g_2(t), \ldots, g_n(t)\right)^T \in C(I, \mathbb{C}^n),$$

$x = x(t)$ – noma'lum kompleks vektor-funksiya. Bu sistemaning nazariyasi haqiqiy sohadagiga juda ham o'xshash. Haqiqiy holdagi deyarli barcha teoremalar kompleks holda ham o'z kuchini saqlaydi. Endi faqat $\mathbb{R}^n$ chiziqli fazo o'rnida $\mathbb{C}^n$ chiziqli fazoni ishlatish kerak. Masalan, $x' = A(t)x$ bir jinsli sistemaning yechimlari to'plami n o'lchamli kompleks chiziqli fazoni tashkil etadi.

Haqiqiy sohada berilgan $x' = A(t)x$,

$A(t) \in C\left(I; \mathbb{M}_{n\times n}(\mathbb{R})\right)$ sistemani qaraylik. Bu sistemaning kompleks yechimlarini izlash sistemani komplekslashtirish deb ataladi. Agar bu sistemaning kompleks yechimi topilgan bo'lsa, uning haqiqiy va mavhum qismlari ham bu sistemaning yechimi bo'ladi. Haqiqatan ham, faraz qilaylik, $x = u(t) + iv(t)$ haqiqiy va mavhum qismlari ajratilgan kompleks yechim bo'lsin. Demak,

$$u'(t) + iv'(t) = A(t)(u(t) + iv(t)), t \in I,$$

yoki

$$u'(t) + iv'(t) = A(t)u(t) + iA(t)v(t), t \in I.$$

$A(t)$ – haqiqiy matritsa bo'lgani uchun oxirgi tenglikdan

$$u'(t) = A(t)u(t),\ v'(t) = A(t)v(t), t \in I,$$

ayniyatlarni hosil qilamiz. Ular $u(t)$ va $v(t)$ larning yechim ekanligini anglatadi.

Masalalar

1. Haqiqiy sohada $x' = A(t)x$ va $x' = A(t)x + g(t)$ sistemalar uchun ma'lum tasdiqlarni kompleks holga o'tkazing.

2. Agar $x = x(t)$ kompleks vektor-funksiya ushbu

$$x' = A(t)x + g(t),$$

bunda $A(t) \in C(I;\mathbb{M}_{n\times n}(\mathbb{R}))$ (haqiqiy) va $g(t) \in C(I;\mathbb{C}^n)$ (kompleks), sistemaning yechimi bo'lsa, u holda $x = \mathrm{Re}\,x(t)$ ($x = \mathrm{Im}\,x(t)$) haqiqiy vektor-funksiya $x' = A(t)x + \mathrm{Re}\,g(t)$ (mos ravishda $x' = A(t)x + \mathrm{Im}\,g(t)$) haqiqiy sistemaning yechimi ekanligini isbotlang.

IV.7. O'zgarmas koeffitsientli bir jinsli sistemani eksponensial matritsa yordamida yechish

Ushbu

$$x' = Ax \tag{IV.7.1}$$

o'zgarmas koeffitsientli bir jinsli sistemani yechish maqsadida **matritsaning ko'rsatkichli funksiyasi** tushunchasini kiritamiz; bu yerda $x = x(t)$ – noma'lum vektor-funksiya ($x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$),

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$ – o'zgarmas haqiqiy $n \times n$-matritsa,

ya'ni $A \subset \mathbb{M}_{n\times n}(\mathbb{R})$.

Ma'lumki,

$$\begin{cases} x' = ax \\ x(0) = c \end{cases} \quad (a, c \text{ – berilgan o'zgarmas sonlar})$$

skalyar Koshi masalasining $(-\infty,+\infty)$ da aniqlangan yagona yechimi

$$x = e^{at}c = \left(1 + at + \frac{a^2}{2!}t^2 + \dots + \frac{a^n}{n!}t^n + \dots\right)c,\ t \in \mathbb{R},$$

formula bilan beriladi.

Ushbu

$$\begin{cases} x' = Ax \\ x(0) = c \end{cases} \tag{IV.7.2}$$

($A \in \mathbb{M}_{n\times n}(\mathbb{R})$ – berilgan matritsa, $c \in \mathbb{R}^n$ – berilgan vektor)

Koshi masalasining yechimini shunga o'xshash formula bilan aniqlash uchun bu masalaga teng kuchli bo'lgan

$$x = c + \int_0^t Ax(s)ds$$

integral tenglamani ketma-ket yaqinlashishlar metodi yordamida yechamiz. Ketma-ket yaqinlashishlarni hisoblaymiz:

$x^0 = c$,

$$x^1 = c + \int_0^t Ax^0(s)ds = c + \int_0^t Acds = (E + At)c,$$

$$x^2 = c + \int_0^t Ax^1(s)ds = c + \int_0^t A(E + As)cds = (E + At + \frac{A^2}{2!}t^2)c,$$

....

$$x^k = c + \int_0^t Ax^{k-1}(s)ds = (E + At + \frac{A^2}{2!}t^2 + \dots + \frac{A^k}{k!}t^k)c,$$

....

Endi (IV.7.2) masalaning yechimi

$$x = e^{At}c, \tag{IV.7.3}$$

bunda

$$e^{At} \overset{def}{=} E + At + \frac{A^2}{2!}t^2 + \dots + \frac{A^k}{k!}t^k + \dots \quad \text{yoki}$$

$$\exp(At)=\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k \quad (A^0=E), \qquad \text{(IV.7.4)}$$

ko'rinishda bo'lishi kerakligini topamiz. $\exp(At)=e^{At}$ deb tushuniladi. (IV.7.4) qatorning ixtiyoriy segmentda tekis yaqinlashuvchi va (IV.7.3) vektor-funksiya (IV.7.2) Koshi masalasining yechimi ekanligi bizga ma'lum. Hozir biz bu tasdiqlarning mustaqil isbotini keltiramiz.

Teorema 1. *Ushbu* $\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k$ *matritsaviy darajali qator ixtiyoriy* $|t|\le\delta$ $(\delta>0)$ *oraliqda tekis va absolyut yaqinlashuvchi. Uni hadma-had differensiallash mumkin va* $\frac{de^{At}}{dt}=Ae^{At}$ *formula o'rinli.*

Ma'lumki, ushbu $\sum_{k=0}^{\infty}\frac{(\|A\|\cdot\delta)^k}{k!}$ sonli qator yaqinlashuvchi (uning yig'indisi $e^{\|A\|\delta}$ ga teng). Demak, u fundamental, ya'ni ixtiyoriy $\varepsilon>0$ songa ko'ra shunday ν natural sonni topish mumkinki, barcha $m>\nu$ nomerlar va ixtiyoriy $p\in\mathbb{N}$ uchun $\sum_{k=m}^{m+p}\frac{(\|A\|\cdot\delta)^k}{k!}<\varepsilon$ bo'ladi. $\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k$ va $\sum_{k=0}^{\infty}\left\|\frac{A^k}{k!}t^k\right\|$ funksional qatorlar $|t|\le\delta$ oraliqda tekis yaqinlashishning Koshi mezonini qanoatlantiradi. Haqiqatan ham, o'sha ε,ν,m,p sonlar uchun

$$\left\|\sum_{k=m}^{m+p}\frac{A^k}{k!}t^k\right\|\le\sum_{k=m}^{m+p}\left\|\frac{A^k}{k!}t^k\right\|=\sum_{k=m}^{m+p}\frac{\|A^k\|\cdot|t|^k}{k!}\le$$

$$\le\sum_{k=m}^{m+p}\frac{\|A\|^k\cdot|t|^k}{k!}\le\sum_{k=m}^{m+p}\frac{\|A\|^k\cdot\delta^k}{k!}<\varepsilon.$$

Demak, $\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k$ qator $|t|\le\delta$ oraliqda tekis va absolyut yaqinlashuvchi. Bundan tashqari qator hadlarining hosilalaridan tuzilgan $\sum_{k=1}^{\infty}\frac{A^k}{(k-1)!}t^{k-1}=\sum_{k=0}^{\infty}\frac{A^{k+1}}{k!}t^k=A\cdot\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k$ qator ham $|t|\le\delta$ oraliqda tekis yaqinlashuvchi bo'lgani uchun analizdan ma'lum teoremaga ko'ra $\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k$ qatorni hadma-had differensiallash mumkin:

$$\frac{d}{dt}\exp(At)=\frac{d}{dt}\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k=\sum_{k=1}^{\infty}\frac{A^k}{(k-1)!}t^{k-1}=$$

$$=A\cdot\sum_{m=0}^{\infty}\frac{A^m}{m!}t^m=A\cdot\exp(At). \qquad \text{(IV.7.5)}$$

Natija. $\Phi:\mathbb{R}\to\mathbb{M}_{n\times n}(\mathbb{R})$, $\Phi(t)=\exp(At)$, ***matritsaviy*** *funksiya ushbu*

$$\begin{cases}\Phi'(t)=A\Phi(t)\\ \Phi(0)=E\end{cases}$$

masalaning yagona yechimini, $x=\exp(At)c$ *vektor-funksiya esa*

$$\begin{cases}x'=Ax\\ x(0)=c\end{cases}$$

Koshi masalasining yagona yechimini ifodalaydi. Demak, $\Phi(t)=\exp(At)$ – (IV.7.1) *sistemaning fundamental matritsasi.*

Endi $\exp(At)$ matritsaning xossalarini o'rganamiz.

(IV.7.4) formulada $t=1$ deb quyidagi tenglikni hosil qilamiz:

$$\exp(A)=\sum_{k=0}^{\infty}\frac{A^k}{k!} \qquad \text{(IV.7.6)}$$

Shunday qilib, ixtiyoriy $A \in \mathbb{M}_{n\times n}(\mathbb{R})$ matritsa uchun uning eksponentasi $\exp(A) \in \mathbb{M}_{n\times n}(\mathbb{R})$ matritsa aniqlandi. Ravshanki, $\exp(O) = E$ $(O \in A \in \mathbb{M}_{n\times n}(\mathbb{R})$ – nol-matritsa).

Agar $A \in \mathbb{M}_{n\times n}(\mathbb{R})$ matritsa katakli-diagonal tuzilishga ega, ya'ni

$A = \begin{pmatrix} B & O \\ O & C \end{pmatrix}$, $A \in \mathbb{M}_{p\times p}(\mathbb{R})$, $B \in \mathbb{M}_{q\times q}(\mathbb{R})$, $p+q=n$, O – nol-matritsa, bo'lsa, u holda

$$A^k = \begin{pmatrix} B^k & O \\ O & C^k \end{pmatrix}, k \in \mathbb{N}, \text{ va, demak, } e^A = \begin{pmatrix} e^B & O \\ O & e^C \end{pmatrix}$$

bo'ladi. Bu yerda O bilan kerakli tartibli nol-matritsalar belgilangan. Xuddi shuningdek, $A_1, A_2, \dots, A_m$ – kvadrat matritsalar uchun

$$\exp(\operatorname{diag}(A_1, A_2, \dots, A_m)) = \operatorname{diag}(\exp(A_1), \exp(A_2), \dots, \exp(A_m))$$

formula o'rinli. Endi tushunarliki, $e^E = eE$ tenglik ham o'rinlidir.

Teorema 2. *Matritsa eksponentasi quyidagi xossalarga ega:*

1. *Agar* $\{A, B, S\} \subset \mathbb{M}_{n\times n}(\mathbb{R})$ *matritsalar uchun* $A = S^{-1}BS$ *bo'lsa,* $e^A = S^{-1}e^B S$ *tenglik o'rinli;*
2. *Agar* $AB = BA$ *bo'lsa,* $e^{A+B} = e^A e^B$ *bo'ladi;*
3. $(e^A)^{-1} = e^{-A}$.

⊷ $A = S^{-1}BS$ bo'lsin. U holda, ravshanki, $A^k = S^{-1}B^k S$, $k \in \mathbb{N}$, bo'ladi. Demak,

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{S^{-1}B^k S}{k!} = S^{-1}\left(\sum_{k=0}^{\infty} \frac{B^k}{k!}\right) S = S^{-1}e^B S.$$

Endi $AB = BA$ deylik. U holda $A^k B = BA^k$, $k \in \mathbb{N}$, bo'ladi. Bundan (IV.7.4) formulaga ko'ra ixtiyoriy $t \in \mathbb{R}$ uchun

$$e^{tA}B = Be^{tA} \qquad \text{(IV.7.7)}$$

ekanligi kelib chiqadi. (IV.7.5) va (IV.7.7) formulalarga ko'ra

$$\left(e^{tA}e^{tB}\right)' = \left(e^{tA}\right)' e^{tB} + e^{tA}\left(e^{tB}\right)' = Ae^{tA}e^{tB} + e^{tA}Be^{tB}$$
$$= Ae^{tA}e^{tB} + Be^{tA}e^{tB} = (A+B)e^{tA}e^{tB}.$$

Demak, $X = X(t) = e^{tA}e^{tB}$ funksiya

$$X' = (A+B)X$$

sistemaning yechimi va $X(0) = E$. Bu sistemaning shu boshlang'ich shartni qanoatlantiruvchi yechimi, ravshanki, $e^{t(A+B)}$ hamdir. Yechimning yagonalik xossasiga ko'ra har qanday $t \in \mathbb{R}$ uchun $e^{t(A+B)} = e^{tA}e^{tB}$ bo'lishi kerak. Bu tenglikda $t = 1$ qo'yib, ikkinchi xossani isbotlaymiz.

Uchinchi xossa ikkinchisidan osongina kelib chiqadi: $A(-A) = (-A)A$ bo'lgani uchun

$e^A e^{-A} = e^{-A}e^A = e^{A-A} = e^O = E$ $(O \in \mathbb{M}_{n\times n}(\mathbb{R})$ – nol-matritsa),

ya'ni e^A matritsa teskarilanuvchi va uning teskarisi e^{-A} matritsadan iborat. ⊶

Yuqoridagi fikrlar (e^A ning ta'rifi, teorema va h.k.) kompleks elementli $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ matritsa uchun ham o'rinli.

Masalalar

1. $e^{\operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)} = \operatorname{diag}(e^{\lambda_1}, e^{\lambda_2}, \dots, e^{\lambda_n})$ tenglikni isbotlang.
2. Agar $\{A, B\} \subset \mathbb{M}_{n\times n}(\mathbb{C})$ matritsalar uchun $e^{A+B} = e^A e^B$ bo'lsa, $AB = BA$ ekanligini ko'rsating.
3. Aytaylik, $f: (-r, r) \to \mathbb{R}$ funksiya darajali qatorga yoyilsin:

$$f(t) = a_0 + a_1 t + a_2 \frac{t^2}{2!} + \dots + a_n \frac{t^n}{n!} + \dots, \ |t| < r,$$

$A \in \mathbb{M}_{n\times n}(\mathbb{R})$ matritsaning barcha λ xos sonlari uchun esa $|\lambda| < r$ tengsizlik o'rinli bo'lsin. U holda

$$f(A) = a_0 + a_1 A + \frac{a_2}{2!}A^2 + \dots + \frac{a_n}{n!}A^n + \dots,$$

matritsaviy qatorning absolyut yaqinlashuvchi ekanligini ko'rsating.

4. Agar $A \subset \mathbb{M}_{n\times n}(\mathbb{R})$ matritsa simmetrik ($A^T = A$) bo'sa, ixtiyoriy

$t \in \mathbb{R}$ uchun e^{tA} matritsa ham simmetrik bo'lishini isbotlang.

5. Agar $A \subset \mathbb{M}_{n\times n}(\mathbb{R})$ matritsa ortogonal ($A^T = A^{-1}$) bo'lsa, ixtiyoriy $t \in \mathbb{R}$ uchun e^{tA} matritsa ham ortogonal ($e^{tA}(e^{tA})^T = E$) bo'lishini ko'rsating.

6. $A \in \mathbb{M}_{n\times n}(\mathbb{R})$ va $t \in \mathbb{R}$ bo'lsin.

1°. $\cos tA = \sum_{j=0}^{\infty}(-1)^j \frac{t^{2j}}{(2j)!}A^{2j}$ va $\sin tA = \sum_{j=0}^{\infty}(-1)^j \frac{t^{2j+1}}{(2j+1)!}A^{2j+1}$

funksiyalarning aniqlangan ekanligini asoslang;

2°. $\frac{d}{dt}(\cos tA)$ va $\frac{d}{dt}(\sin tA)$ hosilalarni hisoblang;

3°. ushbu

$$\begin{cases} x' = -Ay \\ y' = Ax \end{cases} \quad (x = x(t) \in \mathbb{R}^n, y = y(t) \in \mathbb{R}^n)$$

sistemani yeching.

IV.8. e^{tA} ni matritsaning Jordan kanonik ko'rinishidan foydalanib hisoblash.

Biz bu bandda A matritsaning Jordan kanonik ko'rinishidan foydalanib e^{tA} matritsani hisoblash usulini keltiramiz.

Chiziqli algebradan ma'lumki, ixtiyoriy $A \in \mathbb{M}_{n\times n}(\mathbb{C})$ matritsani Jordan kanonik ko'rinishiga keltirish mumkin, ya'ni shunday teskarilanuvchi S matritsa topiladiki, uning uchun

$J = S^{-1}AS$ $(A = SJS^{-1})$,

$$J = \mathrm{diag}(J_{\lambda_1,n_1}, \ldots, J_{\lambda_2,n_2}, \ldots, J_{\lambda_k,n_k}) = \begin{pmatrix} J_{\lambda_1,n_1} & & & \\ & \ddots & & \\ & & J_{\lambda_2,n_2} & \\ & & & \ddots J_{\lambda_k,n_k} \end{pmatrix}, \quad \text{(IV.8.1)}$$

bo'ladi, bunda J Jordan (katakli-diagonal) matritsasining diagonali bo'ylab Jordan kataklari, boshqa o'rinlarda esa nollar joylashgan bo'lib, u quyidagicha tuziladi. Faraz qilaylik, A matritsaning turli xos sonlari $\lambda_1, \lambda_2, \ldots, \lambda_s$ $(s \le n)$ mos ravishda $k_1, k_2, \ldots, k_s$ karrali $(k_1 + k_2 + \ldots + k_s = n)$ hamda λ_q xos songa mos kelgan chiziqli erkli vektorlar soni p_q, ya'ni $\dim\{x \mid (A-\lambda_q E)x = 0\} = p_q$ $(p_q = n - \mathrm{rank}(A - \lambda_q E))$ bo'lsin. U holda λ_q xos songa p_q dona

$$J_{\lambda_q, d_{qj}} = \begin{pmatrix} \lambda_q & 1 & & & \\ & \lambda_q & 1 & & \\ & & \lambda_q & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_q \end{pmatrix} \in \mathbb{M}_{d_{qj}\times d_{qj}}(\mathbb{C}),$$

$$j = 1, 2, \ldots, p_q, \ (d_{q1} + d_{q2} + \ldots + d_{qp_q} = k_q)$$

$d_{q1}, d_{q2}, \ldots, d_{qp_q}$ o'lchamli Jordan kataklari mos keladi; bu yerda bo'sh o'rinlarda nollar yozilgan deb tushunish kerak. Ravshanki, λ_q ga mos kelgan Jordan kataklarining eng katta o'lchami $\tilde{k}_q \overset{def}{=} \max\{d_{q1}, d_{q2}, \ldots, d_{qp_q}\} \le k_q$ bo'ladi. (IV.8.1) formulada Jordan kataklari boshqacha indekslangan.

$A = SJS^{-1}$ bo'lgani uchun $tA = S(tJ)S^{-1}$ va oldingi banddagi teorema 2 ga ko'ra

$$e^{tA} = Se^{tJ}S^{-1} = S\mathrm{diag}(e^{tJ_{\lambda_1,n_1}}, \ldots, e^{tJ_{\lambda_2,n_2}}, \ldots, e^{tJ_{\lambda_k,n_k}})S^{-1}. \quad \text{(IV.8.2)}$$

Demak, biz Jordan katagini t ga ko'paytmasining eksponentasini hisoblashimiz kerak. Ushbu

$$J_{\lambda,p}=\begin{pmatrix}\lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1\\ & & & & \lambda\end{pmatrix}\in \mathbf{M}_{p\times p}(\mathbb{C})$$

tipik Jordan katagini olaylik. Qulaylik uchun

$$E_p=\begin{pmatrix}1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix},\quad N_p=\begin{pmatrix}0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1\\ & & & & 0\end{pmatrix}$$

$p\times p$ o'lchamli matritsalarni kiritib, $J_{\lambda,p}=\lambda E_p+N_p$ formulani hosil qilamiz. $E_pN_p=N_pE_p$ bo'lgani uchun teorema 2ga ko'ra

$$e^{tJ_{\lambda,p}}=e^{t\lambda E_p}e^{tN_p}=e^{t\lambda}e^{tN_p}. \qquad \text{(IV.8.3)}$$

Endi e^{tN_p} ni hisoblaymiz. Osongina ishonch hosil qilish mumkinki, $N_p^p, N_p^{p+1}\ldots$ matritsalar nol-matritsaga aylanadi. Demak, matritsaning eksponentasi ta'rifidan

$$e^{tN_p}=E_p+\frac{N_p}{1!}t+\frac{N_p^2}{2!}t^2+\ldots+\frac{N_p^{p-1}}{(p-1)!}t^{p-1}.$$

Kerakli hisoblashlarni bajarib, (IV.8.3) formulaga ko'ra topamiz:

$$e^{tJ_{\lambda,p}}=e^{t\lambda}e^{tN_p}=\begin{pmatrix} e^{t\lambda} & \frac{te^{t\lambda}}{1!} & \frac{t^2e^{t\lambda}}{2!} & \cdots & \frac{t^{p-1}e^{t\lambda}}{(p-1)!}\\ 0 & e^{t\lambda} & \frac{te^{t\lambda}}{1!} & \cdots & \frac{t^{p-2}e^{t\lambda}}{(p-2)!}\\ \vdots & & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{te^{t\lambda}}{1!}\\ 0 & 0 & 0 & \cdots & e^{t\lambda}\end{pmatrix}. \qquad \text{(IV.8.4)}$$

Shunday qilib, agar A matritsani Jordan kanonik ko'rinishiga keltiruvchi S matritsa ma'lum bo'lsa, u holda (IV.8.4) va (IV.8.2) formulalar orqali e^{tA} ($t\in\mathbb{R}$) matritsani hisoblash mumkin.

Misol. Ushbu

$$A=\begin{pmatrix}4 & 1 & 1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}$$

matritsa uchun e^{tA} ni hisoblang.

⊶ Dastlab A matritsaning xos sonlarini topaylik:

$$\begin{vmatrix}4-\lambda & 1 & 1\\ -1 & 2-\lambda & -1\\ -1 & -1 & 2-\lambda\end{vmatrix}=0 \Leftrightarrow (\lambda-2)(\lambda-3)^2=0\,;\ \lambda_1=2,\lambda_{2,3}=3.$$

Oddiy xarakteristik son $\lambda=2$ ga mos keluvchi xarakteristik vektor:

$$(A-2E)\mathbf{v}=0,\ \mathbf{v}=\begin{pmatrix}-1\\ 1\\ 1\end{pmatrix}.$$

Ikki karrali $\lambda=3$ xarakteristik songa ikkita chiziqli erkli vektor mos keladi:

$$(A-3E)\mathbf{w}=0;\ \mathbf{w}_1=\begin{pmatrix}1\\-1\\0\end{pmatrix},\ \mathbf{w}_2=\begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

Demak,

$$S=[\mathbf{v},\mathbf{w}_1,\mathbf{w}_2]=\begin{pmatrix}-1&1&-1\\1&-1&0\\1&0&1\end{pmatrix},\ S^{-1}=\begin{pmatrix}1&1&1\\1&0&1\\-1&-1&0\end{pmatrix},$$

matritsa A matritsani Jordan kanonik ko'rinishiga (misolda diagonal ko'rinishga) keltiradi:

$$A=S\begin{pmatrix}2&0&0\\0&3&0\\0&0&3\end{pmatrix}S^{-1}.$$

Endi (IV.8.4) va (IV.8.2) formulalarga ko'ra e^{At} matritsani hisoblash oson:

$$e^{At}=S\begin{pmatrix}e^{2t}&0&0\\0&e^{3t}&0\\0&0&e^{3t}\end{pmatrix}S^{-1}=\begin{pmatrix}-1&1&-1\\1&-1&0\\1&0&1\end{pmatrix}\begin{pmatrix}e^{2t}&0&0\\0&e^{3t}&0\\0&0&e^{3t}\end{pmatrix}\begin{pmatrix}1&1&1\\1&0&1\\-1&-1&0\end{pmatrix}$$

yoki ko'paytirishlarni bajarib,

$$e^{At}=\begin{pmatrix}2e^{3t}-e^{2t}&e^{3t}-e^{2t}&e^{3t}-e^{2t}\\-e^{3t}+e^{2t}&e^{2t}&-e^{3t}+e^{2t}\\-e^{3t}+e^{2t}&-e^{3t}+e^{2t}&e^{2t}\end{pmatrix}.$$

ekanligini topamiz. ♁

Umumiy holda (IV.8.4) va (IV.8.2) formulalardagi mos matritsalarni ko'paytirish amallarini bajarib, quyidagi teoremani hosil qilamiz.

Teorema 1. *Har qanday $A\in\mathbb{M}_{n\times n}(\mathbb{C})$ kompleks matritsa uchun e^{tA} $(t\in\mathbb{R})$ eksponensial matritsaning barcha elementlari*

$\sum_{j=1}^{s}p_j(t)e^{\lambda_j t}$ *ko'rinishga ega, bunda $p_j(t)-$ kompleks koeffitsientli ko'phadlar, $\deg p_j(t)\le\tilde{k}_j-1$, $\tilde{k}_j$ bilan A matritsaning k_j karrali λ_j xos qiymatiga mos kelgan Jordan kataklarining eng katta tartibi belgilangan ($\tilde{k}_j\le k_j$); $s-$ (turli) xos qiymatlar soni.*

Agar A matritsa haqiqiy bo'lsa, $e^{t(\alpha+i\beta)}=e^{\alpha t}\left(\cos(\beta t)+i\sin(\beta t)\right)$ Eyler formulasini hisobga olib, (IV.8.4) va (IV.8.2) formulalarga ko'ra kerakli hisoblashlarni bajaramiz. S,S^{-1} matritsalar kompleks bo'lsada, (IV.8.2) formuladan haqiqiy e^{tA} matritsa topiladi. Natijada quyidagi teoremani hosil qilamiz.

Teorema 2. *Ixtiyoriy $A\in\mathbb{M}_{n\times n}(\mathbb{R})$ haqiqiy matritsa uchun e^{tA} $(t\in\mathbb{R})$ eksponensial matritsaning barcha elementlari ushbu*

$$\sum_{j=1}^{s}e^{\alpha_j t}\left(p_j(t)\cos(\beta_j t)+q_j(t)\sin(\beta_j t)\right)\tag{IV.8.5}$$

ko'rinishga ega. Bu yerda $\alpha_j+i\beta_j,\ j=1,2,\dots,s,$ – A matritsaning turli xos (qiymatlari) sonlari; $p_j(t)$ va $q_j(t)-$ haqiqiy koeffitsientli ko'phadlar; agar $\alpha_j+i\beta_j$ xos songa mos kelgan Jordan kataklarining eng katta tartibi $\tilde{k}_j$ bo'lsa, u holda bu $p_j(t)$ va $q_j(t)$ ko'phadlarning darajalari $\tilde{k}_j-1$ dan oshmaydi.

e^{tA} matritsaning ko'rinishini bilgan holda uni hisoblash mumkin. Buning uchun uning (IV.8.5) elementlarini noma'lum koeffitsientlar orqali yozib, ushbu $(e^{tA})'=Ae^{tA},\ e^{0\cdot A}=E$ ayniyatdan noma'lum koeffitsientlar uchun chiziqli tenglamalarni tuzib, ularni yechish kerak. Lekin bu usul ba'zan uzoq hisoblashlarni talab etishi mumkin.

Matritsaning Jordan kanonik ko'rinishidan foydalanib matritsaning logarifmini aniqlash mumkin. $A\in\mathbb{M}_{n\times n}(\mathbb{R})$ **matritsaning logarifmi** deb $e^{B}=A$ tenglikni qanoatlantiruvchi B matritsaga aytiladi va u $B=\ln A$ kabi belgilanadi.

Teorema. *Har qanday teskarilanuvchi* A ($\det A\neq 0$) *matritsa logarifmga ega.*

🗝 Dastlab $A\in\mathbb{M}_{n\times n}(\mathbb{R})$ matritsa Jordan katagidan iborat bo'lgan holni qaraylik. Bizga teskarilanuvchi $J_{\mu,p}$ ($\mu\neq 0$) Jordan katagi berilgan bo'lsin:

$$J_{\mu,p}=\mu E_p+N_p \text{ yoki } J_{\mu,p}=\mu\Big(E_p+\frac{1}{\mu}N_p\Big).$$

Analizdan ma'lum ushbu

$$\ln(1+t)=\sum_{j=1}^{\infty}(-1)^{j+1}\frac{t^j}{j},\ (|t|<1)$$

yoyilmadan kelib chiqib,

$$\ln J_{\mu,p}=E_p\ln\mu+\ln\Big(E_p+\frac{1}{\mu}N_p\Big)=$$

$$=E_p\ln\mu+\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\Big(\frac{N_p}{\mu}\Big)^j$$

($\ln\mu=\ln|\mu|+i\arg\mu$)

deb qabul qilamiz. Bu yerda $O=N_p^p=N_p^{p+1}=\ldots$ bo'lgani uchun

$$\ln J_{\mu,p}=E_p\ln\mu+\sum_{j=1}^{p-1}\frac{(-1)^{j+1}}{j}\Big(\frac{N_p}{\mu}\Big)^j$$

formula hosil bo'ladi. Endi

$$\exp(\ln J_{\mu,p})=J_{\mu,p} \qquad (*)$$

ekanligini ko'rsatamiz. Ravshanki,

$$\exp(\ln(1+t))=\exp\Big(\sum_{j=1}^{\infty}(-1)^{j+1}\frac{t^j}{j}\Big)=1+t\quad (|t|<1).$$

Demak,

$$\exp\Big(\sum_{j=1}^{p-1}\frac{(-1)^{j+1}}{j}\Big(\frac{N_p}{\mu}\Big)^j\Big)=\exp\Big(\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\Big(\frac{N_p}{\mu}\Big)^j\Big)=1+\frac{N_p}{\mu}.$$

Bundan foydalanib, eksponensial matritsaning xossalariga ko'ra (*) ni isbotlaymiz:

$$\exp(\ln J_{\mu,p})=\exp\Big(E_p\ln\mu+\sum_{j=1}^{p-1}\frac{(-1)^{j+1}}{j}\Big(\frac{N_p}{\mu}\Big)^j\Big)=$$

$$=\exp(E_p\ln\mu)\exp\Big(\sum_{j=1}^{p-1}\frac{(-1)^{j+1}}{j}\Big(\frac{N_p}{\mu}\Big)^j\Big)=$$

$$=E_p\mu\Big(E_p+\frac{N_p}{\mu}\Big)=\mu E_p+N_p=J_{\mu,p}.$$

Endi ixtiyoriy teskarilanuvchi $A\in\mathbb{M}_{n\times n}(\mathbb{R})$ matritsaning logarifmini aniqlaymiz. A matritsani Jordan kanonik ko'rinishiga keltiramiz:

$$A=SJS^{-1},\ J=\mathrm{diag}(J_{\lambda_1,n_1},\ldots,J_{\lambda_2,n_2},\ldots,J_{\lambda_k,n_k}).$$

Bu formulaga ko'ra tabiiy ravishda

$$\ln A=S^{-1}\ln JS,\ \ln J=\mathrm{diag}(\ln J_{\lambda_1,n_1},\ldots,\ln J_{\lambda_2,n_2},\ldots,\ln J_{\lambda_k,n_k}),$$

deb qabul qilamiz; bu yerdagi barcha Jordan kataklari teskarilanuvchi va, demak, ularning logarifmi aniqlangan.

Endi $\exp(\ln A)=A$ ekanligini quyidagicha ko'rsatamiz:

$$\exp(\ln A)=S^{-1}\exp\big(\mathrm{diag}(\ln J_{\lambda_1,n_1},\ldots,\ln J_{\lambda_2,n_2},\ldots,\ln J_{\lambda_k,n_k})\big)S=$$

$$=S^{-1}\mathrm{diag}\big(\exp(\ln J_{\lambda_1,n_1}),\ldots,\exp(\ln J_{\lambda_2,n_2}),\ldots,\exp(\ln J_{\lambda_k,n_k})\big)S=$$

$$=S^{-1}\mathrm{diag}(J_{\lambda_1,n_1},\ldots,J_{\lambda_2,n_2},\ldots,J_{\lambda_k,n_k})S=$$

$$=A.$$ ♘

Natija. *Ixtiyoriy teskarilanuvchi* $A\in\mathbb{M}_{n\times n}(\mathbb{R})$ ($\det A\neq 0$) *matritsa uchun* $B^m=A,\ m\in\mathbb{N}$, *tenglikni qanoatlantiruvchi* B *matritsa mavjud* ($B=\sqrt[m]{A}$).

⊶ B matritsa bo'lib $B=\exp(\frac{1}{m}\ln A)$ matritsa xizmat qiladi. Haqiqatan ham,

$$B^m=\left(\exp(\frac{1}{m}\ln A)\right)^m=\exp(\ln A)=A. \;\;✍$$

Bu yerda matritsaning xos (xarakteristik) soni bilan bog'liq bir tasdiqni e'tirof etaylik. Agar A matritsaning xos soni λ bo'lsa ($Ax=\lambda x$, $x\neq 0$), λ^m soni, ravshanki, A^m $(m\in\mathbb{N})$ matritsaning xos soni bo'ladi ($A^m x=\lambda^m x$, $x\neq 0$). Agar teskarilanuvchi A matritsaning xos soni λ ($\lambda\neq 0$) bo'lsa, keltirilgan natijaga ko'ra $\sqrt[m]{\lambda}$ qiymatlarning (ular m dona) birortasi $\sqrt[m]{A}$ matritsaning xos sonidan iborat bo'ladi.

Masalalar

1. Formulani isbotlang:

$$e^{A}=\lim_{k\to\infty}\left(E+\frac{1}{k}A\right)^k \qquad (k\in\mathbb{N}, A\in\mathbb{M}_{n\times n}(\mathbb{R}), E\in\mathbb{M}_{n\times n}(\mathbb{R})-\text{birlik}$$

matritsa).

2. Agar $\det A\neq 0$ bo'lsa, $\det A=e^{\mathrm{tr}(\ln A)}$ formulani isbotlang.

3. Agar $X(t)$ kvadrat matritsa $(-\infty,+\infty)$ oralig'ida differensiallanuvchi va $X(t+s)=X(t)X(s)$, $\{t,s\}\subset\mathbb{R}$, bo'lsa, $X(t)=e^{tX(0)}$ bo'lishini ko'rsating. Polia bu tasdiqning uzluksiz $X(t)$ matritsa uchun ham o'rinli ekanligini isbotlagan.

IV.9. $x'=Ax$ sistema umumiy yechimining tuzilishi

Endi yuqoridagi natijalarni $x'=Ax$ sistemaning bazis yechimlarini qurish uchun tadbiq etaylik.

e^{tA} matritsa bu sistemaning fundamental matritsasi. (IV.8.2) formulaga ko'ra

$$\tilde{\Phi}(t)=e^{tA}S=Se^{tJ}=S\mathrm{diag}(e^{tJ_{\lambda_1,n_1}},\ldots,e^{tJ_{\lambda_2,n_2}},\ldots,e^{tJ_{\lambda_k,n_k}}) \quad \text{matritsa}$$

ham shu sistemaning fundamental matritsasi bo'ladi (chunki $\det S\neq 0$). Fundamental matritsaning ustunlari bazis yechimlarni tashkil etadi. $S\mathrm{diag}(e^{tJ_{\lambda_1,n_1}},\ldots,e^{tJ_{\lambda_2,n_2}},\ldots,e^{tJ_{\lambda_k,n_k}})$ ko'paytirishni bajarib, har bir Jordan katagi o'z chiziqli erkli yechimlar guruhini aniqlashini ko'ramiz, ya'ni chiziqli erkli yechimlarni har bir Jordan katagiga ko'ra alohida-alohida guruh sifatida topib, va ularni birlashtirib, bazis yechimlarni qurish mumkin.

A matritsa n ta har xil $\lambda_1;\lambda_2,\ldots,\lambda_n$ xarakteristik sonlarga ega bo'lsin. U holda A matritsaning Jordan ko'rinishi $J=\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$ diagonal matritsadan iborat bo'ladi. Demak,

$$\tilde{\Phi}(t)=Se^{tJ}=S\mathrm{diag}(e^{t\lambda_1},e^{t\lambda_2},\ldots,e^{t\lambda_n})=[e^{t\lambda_1}s^1,e^{t\lambda_2}s^2,\ldots,e^{t\lambda_n}s^n],$$

bunda $[s^1,s^2,\ldots,s^n]=S$, $(e^{t\lambda_j}s^j)'=Ae^{t\lambda_j}s^j\Leftrightarrow\lambda_j s^j=As^j$, ya'ni S matritsa xos vektorlarni ustunlar bo'ylab yozilishidan hosil bo'lgan. Shuning uchun xarakteristik sonlar oddiy (bir karrali) bo'lganda har bir λ_j xos songa mos keluvchi s^j xos vektorni topib, $x=e^{\lambda_1 t}s^1, x=e^{\lambda_2 t}s^2,\ldots,x=e^{\lambda_n t}s^n$ bazis yechimlarni qurish mumkin. Umumiy yechim endi

$$x=c_1e^{\lambda_1 t}s^1+c_2e^{\lambda_2 t}s^2+\cdots+c_ne^{\lambda_n t}s^n$$

formula bilan beriladi, bu yerda $c_1,c_2,\ldots,c_n$ – ixtiyoriy o'zgarmaslar.

Endi karrali xarakteristik son mavjud bo'lgan holni qaraylik. Faraz qilaylk, λ soni k karrali ($k\geq 2$, $k\leq n$) xarakteristik son bo'lsin. Bu λ xos songa mos keluvchi chiziqli erkli xos vektorlar soni p bo'lsin, ya'ni $\dim\{x\,|\,(A-\lambda E)x=0\}=p$, $p\geq 1$. Chiziqli algebradan ma'lumki, u $(A-\lambda E)$ matritsaning rangi r orqali $p=n-r$ formula bilan hisoblanishi ham mumkin.

Agar $p=k$ bo'lsa, A matritsaning k karrali λ xos soniga k dona bir o'lchamli Jordan kataklari mos keladi; A matritsaning k dona $s^1,s^2,\ldots,s^k$ chiziqli erkli xos vektorlari, (1) sistemaning esa k dona $e^{\lambda t}s^1, e^{\lambda t}s^2, \ldots, e^{\lambda t}s^k$ chiziqli erkli yechimlari mavjud bo'ladi.

Agar $p < k$ bo'lsa, $l = k - p$ deb, $x' = Ax$ sistema ushbu

$$x = (a_l t^l + \cdots + a_1 t + a_0)e^{\lambda t} \qquad \text{(IV.9.1)}$$

ko'rinishdagi k dona chiziqli erkli yechimlarga ega ekanligini ko'ramiz, bunda $\{a_0, a_1, \ldots, a_l\} \subset \mathbb{R}^n$ hozircha noma'lum vektor-koeffitsientlar. Ularni topish uchun (IV.9.1) ifodani sistemaga qo'yamiz va $e^{\lambda t}$ ga qisqartiramiz:

$$a_l l t^{l-1} + \cdots + a_2 2t + a_1 + \lambda(a_l t^l + \cdots + a_1 t + a_0) = A(a_l t^l + \cdots + a_1 t + a_0).$$

Bu ayniyatda o'xshash hadlar koffitsientlarini tenglashtirib, $a_0, a_1, \ldots, a_l$ vektorlarni topish uchun quyidagi chiziqli algebraik tenglamalar sistemasini hosil qilamiz:

$$\begin{aligned} &Aa_l = \lambda a_l, \\ &Aa_{l-1} = \lambda a_{l-1} + l a_l, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &Aa_1 = \lambda a_1 + 2a_2, \\ &Aa_0 = \lambda a_0 + a_1 . \end{aligned}$$

Oxirgi sistemada $l+1$ ta vektor tenglama bor. Undan $a_l, \ldots, a_1, a_0$ noma'lum vektorlar komponentalariga nisbatan $(l+1)n$ ta skalyar tenglama hosil bo'ladi. Bu $(l+1)n$ tenglamali sistemada k ta noma'lum erkli bo'lib, qolganlari shular orqali chiziqli ifodalanadi, ya'ni noma'lumlar k ta erkli skalyar o'zgarmasga bog'liq holda topiladi.

Yuqoridagi ishlarni har bir xarakteristik son uchun bajarib, topilgan yechimlardan berilgan (1) differensial tenglamalar sistemasining bazis yechimlarini (fundamental matritsasini) tuzamiz va, demak, umumiy yechimini topamiz.

Berilgan sistemada A haqiqiy matritsa bo'lsa, odatda haqiqiy yechimlarni topish masalasi qo'yiladi. Bunday holda kompleks sohaga chiqib, kompleks bazis yechimlarni topgach, ulardan Eyler formulalariga ko'ra haqiqiy bazis yechimlarni

quramiz.

Izoh. (IV.8.1) differensial tenglamalar sistemasining bazis yechimlaridan uning $\Phi(t)$ fundamental matritsasi tuzilgach, $e^{tA} = \Phi(t)\Phi^{-1}(0)$ formula yordamida e^{tA} matritsani ham hisoblash mumkin.

Misol 1. Ushbu

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \; A = \begin{pmatrix} 0 & -4 & -2 \\ -1 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}, \qquad \text{(IV.9.2)}$$

sistemaning umumiy yechimini quring. e^{At} matritsani hisoblang.

⊶ Berilgan sistema $n = 3$- tartibli, uning xarakteristik ko'phadi

$$\det(A - \lambda E) = \begin{vmatrix} -\lambda & -4 & -2 \\ -1 & -\lambda & -1 \\ 1 & 2 & 3-\lambda \end{vmatrix} = -(\lambda+1)(\lambda-2)^2 .$$

Xarakteristik sonlari $\lambda_1 = -1, \lambda_2 = \lambda_3 = 2$.

Oddiy xarakteristik son $\lambda = -1$ ga berilgan sistemaning

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{-t}\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \qquad \text{(IV.9.3)}$$

ko'rinishdagi yechimi mos keladi, bunda $s_1, s_2, s_3 -$ hozircha noma'lum sonlar. Ularni topish uchun (IV.9.3) ni berilgan sistemaga qo'yib, uning qanoatlanishini talab qilamiz:

$$\begin{cases} s_1 - 4s_2 - 2s_3 = 0 \\ -s_1 + s_2 - s_3 = 0 \\ s_1 + 2s_2 + 4s_3 = 0 \end{cases} \Rightarrow s_1 = -2s_3, s_2 = -s_3 .$$

Demak, $s_3 = -1$ deb, quyidagi yechimni topamiz:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{-t}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad \text{yoki} \quad \begin{cases} x = 2e^{-t}, \\ y = e^{-t}, \\ z = -e^{-t}. \end{cases} \tag{IV.9.4}$$

Endi ikki karrali $\lambda = 2$ xos songa mos kelgan yechimlarni quramiz. $\operatorname{rank}(A-2E) = \operatorname{rank}\begin{pmatrix} -2 & -4 & -2 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{pmatrix} = 1$ bo'lgani uchun bu xos songa $3-1=2$ ta chiziqli erkli xos vektor mos keladi (bizda $n=3$, $k=2$, $r=1$). Ularga ko'ra berilgan sistemaning

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{2t}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

ko'rinishdagi yechimlarini topish mumkin. Buni sistemaga qo'yib, a, b, c noma'lum koeffitsientlarni aniqlaymiz:

$$\begin{cases} -2a - 4b - 2c = 0, \\ -a - 2b - c = 0, \\ a + 2b + c = 0. \end{cases} \Rightarrow \begin{cases} a = c_2, \\ b = c_3, \\ c = -c_2 - 2c_3. \end{cases}$$

(c_2, c_3 – ixtiyoriy o'zgarmaslar)

Demak, ikki karrali $\lambda = 2$ xos songa mos kelgan yechimlar

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{2t}\begin{pmatrix} c_2 \\ c_3 \\ -c_2 - 2c_3 \end{pmatrix} \tag{IV.9.5}$$

formula bilan beriladi. Endi oldin topilgan (IV.9.4) yechimni c_1 ixtiyoriy o'zgarmasga ko'paytirib, uni (IV.9.5) yechimga qo'shamiz va berilgan sistemaning umumiy yechimini topamiz:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{-t}\begin{pmatrix} 2c_1 \\ c_1 \\ -c_1 \end{pmatrix} + e^{2t}\begin{pmatrix} c_2 \\ c_3 \\ -c_2 - 2c_3 \end{pmatrix},$$

yoki skalyar ko'rinishda

$$\begin{cases} x = 2c_1e^{-t} + c_2e^{2t}, \\ y = c_1e^{-t} + c_3e^{2t}, \\ z = -c_1e^{-t} - (c_2 + 2c_3)e^{2t}. \end{cases}$$

Umumiy yechim topilgach, A matritsa uchun endi e^{At} matritsani hisoblash qiyin emas. Topilgan umumiy yechimdan

$$\Phi(t) = \begin{pmatrix} 2e^{-t} & e^{2t} & 0 \\ e^{-t} & 0 & e^{2t} \\ -e^{-t} & -e^{2t} & -2e^{2t} \end{pmatrix}$$

fundamental matritsani tuzamiz. Ravshanki, e^{At} eksponensial matritsa

$$e^{At} = \Phi(t)\Phi^{-1}(0) = \begin{pmatrix} 2e^{-t} & e^{2t} & 0 \\ e^{-t} & 0 & e^{2t} \\ -e^{-t} & -e^{2t} & -2e^{2t} \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & -1 & -2 \end{pmatrix}^{-1},$$

ya'ni

$$e^{At} = \begin{pmatrix} \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} & -\frac{4}{3}e^{2t} + \frac{4}{3}e^{-t} & -\frac{2}{3}e^{2t} + \frac{2}{3}e^{-t} \\ -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} & -\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t} \\ \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} & \frac{2}{3}e^{2t} - \frac{2}{3}e^{-t} & \frac{4}{3}e^{2t} - \frac{1}{3}e^{-t} \end{pmatrix} \tag{IV.9.6}$$

ko'rinishda bo'ladi. ♣

Misol 2. Ushbu

$$\begin{cases} x' = x + 3y - z \\ y' = -x + 4y \\ z' = y + z \end{cases} \qquad \textbf{(IV.9.7)}$$

sistemaning umumiy yechimini toping.

Berilgan sistemaning tartibi $n = 3$, matritsasi

$$A = \begin{pmatrix} 1 & 3 & -1 \\ -1 & 4 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

xarakteristik tenglamasi

$$\begin{vmatrix} 1-\lambda & 3 & -1 \\ -1 & 4-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (2-\lambda)^3 = 0,$$

xarakteristik sonlari $\lambda_1 = \lambda_2 = \lambda_3 = 2$, ya'ni $k = 3$ karrali bitta $\lambda = 2$ xarakteristik son bor.

$$r = \mathrm{rank}(A - \lambda E) = \mathrm{rank}(A - 2E) = \mathrm{rank}\begin{pmatrix} -1 & 3 & -1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix} = 2$$

bo'lgani uchun bu xarakteristik songa $p = n - r = 3 - 2 = 1$ ta xos vektor (o'zgarmas skalyar ko'paytuvchi aniqligida) mos keladi. Demak, $s = k - p = 2$ va yechim ushbu

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} t^2 + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} t + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \right) e^{2t} \qquad \text{(IV.9.8)}$$

ko'rinishda bo'lishi kerak. Umumiy nazariyaga ko'ra bu yerdagi to'qqiz koeffitsientning $k = 3$ tasi erkli bo'lib (hozircha qaysilariligini bilmaymiz), qolganlari esa ular orqali chiziqli ifodalanadi. Yechim uchun (IV.9.8) formulani berilgan sistemaga qo'yib va e^{2t} ga qisqartirib, quyidagi t bo'yicha ayniyatlarni hosil

qilamiz:

$$2\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} t + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + 2\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} t^2 + 2\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} t + 2\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} =$$

$$= A\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} t^2 + A\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} t + A\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Bu tenglikning har ikkala tomonidagi t^2, t^1 va t^0 darajalar oldidagi koeffitsientlarni mos ravishda tenglashtirib, ushbu

$$(A-2E)\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, (A-2E)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 2\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, (A-2E)\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

vektorli tenglamalarni hosil qilamiz. Birinchi vektorli tenglamada $w_3 = c_1$ ni erkli noma'lum deb qabul qilamiz va w_1, w_2 larni c_1 orqali topamiz: $w_1 = 2c_1, w_2 = c_1, w_3 = c_1$. Bularga ko'ra ikkinchi vektorli tenglamada $v_3 = c_2$ ni erkli noma'lum deb $v_1 = 2c_1 + 2c_2, v_2 = 2c_1 + c_2, v_3 = c_2$ larni aniqlaymiz. Uchinchi vektorli tenglamada $u_3 = c_3$ ni erkli noma'lum deb hisoblaymiz va $u_1 = -2c_1 + c_2 + 2c_3$, $u_2 = c_2 + c_3$, $u_3 = c_3$ larni topamiz. Topilgan qiymatlarni (IV.9.8) formulaga qo'yib, berilgan sistemaning

$$\begin{cases} x = \left(-2c_1 + c_2 + 2c_3 + 2(c_1 + c_2)t + 2c_1 t^2\right)e^{2t} \\ y = \left(c_2 + c_3 + (2c_1 + c_2)t + c_1 t^2\right)e^{2t} \\ z = \left(c_3 + c_2 t + c_1 t^2\right)e^{2t} \end{cases} \qquad \textbf{(IV.9.9)}$$

umumiy yechimini hosil qilamiz.

Misol 3. Ushbu

$$\begin{cases} x' = 3x + 2y - z \\ y' = -x + 3y + z \\ z' = y + 2z \end{cases} \qquad \text{(IV.9.10)}$$

sistemani yeching.

Berilgan sistemaning tartibi $n = 3$, matritsasi

$$A = \begin{pmatrix} 3 & 2 & -1 \\ -1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Xarakteristik sonlar:

$$\begin{vmatrix} 3-\lambda & 2 & -1 \\ -1 & 3-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = 2, \lambda_{2,3} = 3 \pm i$$

$\lambda = 2$ xarakteristik songa

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{2t} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

ko'rinishdagi yechim mos keladi. Buni berilgan sistema (IV.9.10)ga qo'yib, s_1, s_2, s_3 larni aniqlaymiz ($s_3 = 1$ tanlangan):

$$\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Demak, $\lambda = 2$ xarakteristik songa mos yechim

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \qquad \text{(IV.9.11)}$$

Endi $\lambda = 3 + i$ xos songa mos kelgan

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{(3+i)t} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

yechimlarni topamiz. Buni berilgan sistemaga qo'yamiz va a, b, c larni topamiz:

$$\begin{pmatrix} -i & 2 & -1 \\ -1 & -i & 1 \\ 0 & 1 & -1-i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a = 2 - i, \\ b = 1 + i, \\ c = 1 (\text{tanlangan}). \end{cases}$$

Demak, sistema yechimi

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{(3+i)t} \begin{pmatrix} 2-i \\ 1+i \\ 1 \end{pmatrix} = e^{3t} e^{it} \begin{pmatrix} 2-i \\ 1+i \\ 1 \end{pmatrix}$$

yoki, $e^{it} = (\cos t + i \sin t)$ Eyler formulasiga ko'ra haqiqiy va mavhum qismlarni ajratsak,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{3t} \begin{pmatrix} 2\cos t + \sin t \\ \cos t - \sin t \\ \cos t \end{pmatrix} + i e^{3t} \begin{pmatrix} 2\sin t - \cos t \\ \cos t + \sin t \\ \sin t \end{pmatrix}.$$

Yechimning haqiqiy va mavhum qismlari ham yechim bo'lgani uchun bundan $\lambda = 3 \pm i$ xarakteristik sonlarga mos kelgan ikkita haqiqiy yechimni aniqlaymiz

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{3t} \begin{pmatrix} 2\cos t + \sin t \\ \cos t - \sin t \\ \cos t \end{pmatrix}, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{3t} \begin{pmatrix} 2\sin t - \cos t \\ \cos t + \sin t \\ \sin t \end{pmatrix}. \qquad \text{(IV.9.12)}$$

Endi sistemaning umumiy yechimini (IV.9.11) va (IV.9.12) bazis yechimlarning ixtiyoriy chiziqli kombinatsiyasi sifatida yozamiz

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 2\cos t + \sin t \\ \cos t - \sin t \\ \cos t \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 2\sin t - \cos t \\ \cos t + \sin t \\ \sin t \end{pmatrix},$$

bunda c_1, c_2, c_3 – ixtiyoriy haqiqiy o'zgarmaslar.

$x' = Ax$ sistemaning yechimini qurishning boshqa bir usuli bizga ma'lum. Bu usulga ko'ra to'g'ridan-to'g'ri e^{At} eksponensial matritsani hisoblab, $x = e^{At}c$, $c \in \mathbb{R}^n$, formulaga asosan umumiy yechimni yozamiz.

Masalalar

1. $x' = Ax$, $A \in \mathbf{M}_{n\times n}(\mathbb{C})$, sistema va A matritsani Jordan ko'rinishiga keltiruvchi $S \in \mathbf{M}_{n\times n}(\mathbb{C})$ matritsa ($A = SJS^{-1}$) berilgan bo'lsin. Sistemada $x = Sy$ (y – yangi noma'lum vektor-funksiya) almashtirishni bajaring va hosil bo'lgan sistemani skalyar ko'rinishda yozing. Bu sistemani yeching va eski x noma'lumga qayting. Topilgan umumiy yechim tuzilishini tahlil qiling.

2. Ushbu

$$x' = \frac{1}{t}Ax,\ A \in A \subset \mathbf{M}_{n\times n}(\mathbb{C}),$$

Eyler sistemasida $\tau = \ln t$ $(t > 0)$ almashtirish bajaring va uni yeching.

3. Ushbu

$$\begin{cases} X' = AX + XB \\ X(0) = C \end{cases},\{A, B, C\} \subset \mathbf{M}_{n\times n}(\mathbb{R}),$$

matritsaviy Koshi masalasining yechimi $X = e^{tA}Ce^{tB}$ formula bilan berilishini isbotlang.

IV.10. e^{tA} ni hisoblashning yana bir usuli

Keli-Hamilton teoremasiga ko'ra A matritsa o'zining xarakteristik tenglamasini qanoatlantiradi, ya'ni

$$\chi(A) = 0, \qquad \text{(IV.10.1)}$$

bunda $\chi(\lambda) \overset{def}{=} \det(A - \lambda E)$ – xarakteristik ko'phad.

Bayonning uzluksiz va mustaqil bo'lishi maqsadida bu tasdiqning isboti paragrafning oxirida keltirilgan.

(IV.10.1) tenglikni e^{tA} ga ko'paytiramiz va $\frac{d^k e^{tA}}{dt^k} = A^k e^{tA}$ $(k = 0, 1, 2, \ldots)$ ekanligini hisobga olib,

$$\chi\left(\frac{d}{dt}\right)e^{tA} = 0$$

munosabatni topamiz. Demak, e^{tA} matritsaning har bir elementi ushbu

$$\chi\left(\frac{d}{dt}\right)x(t) = 0$$

n- tartibli skalyar differensial tenglamani qanoatlantiradi. Bu tenglamaning quyidagi n ta chiziqli erkli $\varphi_j(t)$ yechimlarini topaylik:

$$\chi\left(\frac{d}{dt}\right)\varphi_j(t) = 0,\ \frac{d^i\varphi_j(0)}{dt^i} = \delta_{ij}\ \ (i, j = 0, 1, \ldots, n-1). \quad \text{(IV.10.2)}$$

Demak, e^{tA} matritsaning elementlari $c_0\varphi_0(t) + c_1\varphi_1(t) + \ldots + c_{n-1}\varphi_{n-1}(t)$ ko'rinishga, o'zi esa

$$e^{tA} = \varphi_0(t)B_0 + \varphi_1(t)B_1 + \varphi_2(t)B_2 + \ldots + \varphi_{n-1}(t)B_{n-1}$$

ko'rinishga ega. Bu yerdagi B_j $(j = 0, 1, \ldots, n-1)$ noma'lum matritsalarni topish uchun bu tenglikni i $(i = 0, 1, \ldots, n-1)$ marta differensiallaymiz va $t = 0$ deymiz:

$$\left.\frac{d^i e^{tA}}{dt^i}\right|_{t=0} = A^i = \sum_{j=0}^{n-1} \frac{d^i\varphi_j(0)}{dt^i}B_j = B_i \quad (i = 0, 1, \ldots, n-1).$$

Demak,

$$e^{tA} = \varphi_0(t)E + \varphi_1(t)A + \varphi_2(t)A^2 + \ldots + \varphi_{n-1}(t)A^{n-1}; \quad \text{(IV.10.3)}$$

bu yerdagi $\varphi_j(t)$ $(j = 0, 1, \ldots, n-1)$ funksiyalar (IV.10.2) masalalarning yechimi.

Misol. Ushbu

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad (\alpha, \beta - \text{haqiqiy sonlar}, \beta \neq 0)$$

matritsa uchun e^{tA} matritsani hisoblang.

⊶ Berilgan matritsaning xarakteristik ko'phadi

$$\chi(\lambda)=\det(A-\lambda E)=\begin{vmatrix}\alpha-\lambda & \beta\\ -\beta & \alpha-\lambda\end{vmatrix}=\lambda^2-2\alpha\lambda+\alpha^2+\beta^2.$$

$\chi(\frac{d}{dt})x(t)=0$ differensial tenglamaning xarakteristik sonlari

$$\lambda=\alpha\pm i\beta\ \ (i^2=-1).$$

Uning umumiy yechimi $x(t)=c_1e^{\alpha t}\cos\beta t+c_2e^{\alpha t}\sin\beta t$. Bundan foydalanib, $x(0)=1,\ x'(0)=0$ va $x(0)=0,\ x'(0)=1$ boshlang'ich shartlarga ko'ra bizga kerakli $x(t)=\varphi_0(t)$ va $x(t)=\varphi_1(t)$ yechimlarni topamiz:

$$\varphi_0(t)=e^{\alpha t}\cos\beta t-\frac{\alpha e^{\alpha t}}{\beta}\sin\beta t,\ \varphi_1(t)=\frac{e^{\alpha t}}{\beta}\sin\beta t.$$

Endi (IV.10.3) formulaga ko'ra e^{tA} matritsani hisoblaymiz:

$$e^{tA}=\left(e^{\alpha t}\cos\beta t-\frac{\alpha e^{\alpha t}}{\beta}\sin\beta t\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{e^{\alpha t}}{\beta}\sin\beta t\begin{pmatrix}\alpha&\beta\\-\beta&\alpha\end{pmatrix}=$$

$$=e^{\alpha t}\begin{pmatrix}\cos\beta t&\sin\beta t\\-\sin\beta t&\cos\beta t\end{pmatrix}.$$ ♘

Misol. Berilgan

$$A=\begin{pmatrix}2&2&-1\\-1&-1&1\\-1&-2&2\end{pmatrix}$$

matritsa uchun e^{tA} eksponensial matritsani hisoblaymiz.

⊶ A matritsaning xarakteristik ko'phadi

$$\chi(\lambda)=\det(A-\lambda E)=\begin{vmatrix}2-\lambda&2&-1\\-1&-1-\lambda&1\\-1&-2&2-\lambda\end{vmatrix}=(1-\lambda)^3.$$

$\chi(\frac{d}{dt})x(t)=0$ differensial tenglamaning xarakteristik soni bitta $\lambda=1$: uch karrali. Bu differensial tenglamaning umumiy yechimi $x(t)=c_1e^t+c_2te^t+c_3t^2e^t$. Bundan foydalanib $x(0)=1,\ x'(0)=0,\ x''(0)=0$; $x(0)=0,\ x'(0)=1,\ x''(0)=0$ va $x(0)=0,\ x'(0)=0,\ x''(0)=1$ boshlang'ich shartlarni qanoatlantiruvchi $x(t)=\varphi_0(t)$; $x(t)=\varphi_1(t)$ va $x(t)=\varphi_2(t)$ yechimlarni topamiz:

$$\varphi_0(t)=e^t-te^t+\frac{1}{2}t^2e^t;\ \varphi_1(t)=te^t-t^2e^t\ \text{va}\ \varphi_2(t)=\frac{1}{2}t^2e^t.$$

Endi (IV.10.3) formulaga ko'ra kerakli hisoblashlarni bajarib, e^{tA} matritsani quramiz:

$$e^{tA}=(e^t-te^t+\frac{1}{2}t^2e^t)E+(te^t-t^2e^t)A+\frac{1}{2}t^2e^tA^2,$$

$$e^{tA}=\begin{pmatrix}e^t+te^t&2te^t&-te^t\\-te^t&e^t-2te^t&te^t\\-te^t&-2te^t&e^t+te^t\end{pmatrix}.$$ ♘

Keli-Hamilton teoremasining isboti. S matritsa A matritsani Jordan ko'rinishiga keltirsin:

$$A=SJS^{-1},\ \ J=\mathrm{diag}(J_{\lambda_1,n_1},\dots,J_{\lambda_2,n_2},\dots,J_{\lambda_k,n_k}).$$

Aniqlik uchun

$$\chi(\lambda)\overset{def}{=}\det(A-\lambda E)=\sum_{j=0}^{n}a_j\lambda^j\ \ (a_n=(-1)^n,\dots,a_0=\det A)$$

deylik. Ravshanki, $A^2=SJ^2S^{-1},\dots,A^n=SJ^nS^{-1}$. Demak,

$$\chi(A)=\sum_{j=0}^{n}a_jA^j=\sum_{j=0}^{n}a_jSJ^jS^{-1}=S\sum_{j=0}^{n}a_jJ^jS^{-1}=S\chi(J)S^{-1}.$$

Xuddi shunga o'xshash shakl almashtirishlarni bajaramiz:

$$\chi(J)=\sum_{j=0}^{n}a_j(\mathrm{diag}(J_{\lambda_1,n_1},\dots,J_{\lambda_2,n_2},\dots,J_{\lambda_k,n_k}))^j=$$

$$=\sum_{j=0}^{n}a_j\mathrm{diag}(J^j_{\lambda_1,n_1},\dots,J^j_{\lambda_2,n_2},\dots,J^j_{\lambda_k,n_k})=$$

$$=\mathrm{diag}\Big(\sum_{j=0}^{n}a_jJ^j_{\lambda_1,n_1},\dots,\sum_{j=0}^{n}a_jJ^j_{\lambda_2,n_2},\dots,\sum_{j=0}^{n}a_jJ^j_{\lambda_k,n_k}\Big)=$$

$$=\mathrm{diag}\big(\chi(J_{\lambda_1,n_1}),\dots,\chi(J_{\lambda_2,n_2}),\dots,\chi(J_{\lambda_k,n_k})\big).$$

Shunday qilib,

$$\chi(A)=S\,\mathrm{diag}\big(\chi(J_{\lambda_1,n_1}),\dots,\chi(J_{\lambda_2,n_2}),\dots,\chi(J_{\lambda_k,n_k})\big)S^{-1}.$$

Endi $\chi(J_{\lambda_1,n_1})=O,\dots,\chi(J_{\lambda_2,n_2})=O,\dots,\chi(J_{\lambda_k,n_k})=O$ ekanligini ko'rsatsak, isbot tugaydi; bu yerda $O-$ kerakli tartibli nol-matritsa. Jordan katagining ixtiyoriy birini olib, uni $J_{\mu,p}$ bilan belgilaylik; bunda $\mu-$ A matritsaning k karrali xarakteristik soni, $p-$ katakning tartibi ($p\times p-$ uning o'lchami), $p\le k\le n$ bo'ladi.

Ushbu

$$E_p=\begin{pmatrix}1&&&\\&1&&\\&&\ddots&\\&&&1\end{pmatrix},\quad N_p=\begin{pmatrix}0&1&&\\&0&\ddots&\\&&\ddots&1\\&&&0\end{pmatrix}$$

$p\times p$ o'lchamli matritsalar uchun $J_{\mu,p}=\mu E_p+N_p$ bo'ladi Nyuton binomi formulasiga ko'ra (μE_p va N_p matritsala kommutatsiyalanuvchi) quyidagi hisoblashlarni bajaramiz:

$$\chi(J_{\mu,p})=\sum_{j=0}^{n}a_j(\mu E_p+N_p)^j=$$

$$=\sum_{j=0}^{n}a_j\sum_{r=0}^{j}C_j^r(\mu E_p)^{j-r}(N_p)^r=$$

$$=\sum_{j=0}^{n}a_j\sum_{r=0}^{j}\frac{j!}{r!(j-r)!}\mu^{j-r}N_p^r.$$

Oxirgi yig'indida qo'shish tartibini almashtirib, topamiz

$$\chi(J_{\mu,p})=\sum_{r=0}^{n}\frac{N_p^r}{r!}\sum_{j=r}^{n}a_j\frac{j!}{(j-r)!}\mu^{j-r}.$$

$\chi(\lambda)=\sum_{j=0}^{n}a_j\lambda^j$ ko'phadning r - tartibli hosilasi

$$\chi^{(r)}(\lambda)=\sum_{j=0}^{n}a_jj(j-1)\dots(j-r+1)\lambda^{j-r}=\sum_{j=r}^{n}a_j\frac{j!}{(j-r)!}\lambda^{j-r}.$$

Bundan

$$\sum_{j=r}^{n}a_j\frac{j!}{(j-r)!}\mu^{j-r}=\chi^{(r)}(\mu).$$

μ soni k karrali xarakteristik son bo'lgani uchun

$$\chi(\mu)=0,\ \chi'(\mu)=0,\dots,\chi^{(k-1)}(\mu)=0,\ \chi^{(k)}(\mu)\ne 0.$$

Tushunarliki, N_p matritsaning p - va undan yuqori darajalari nol-matritsaga aylanadi. $p-1\le k-1$ ekanligini ham hisobga olib,

$$\chi(J_{\mu,p})=\sum_{r=0}^{n}\frac{N_p^r}{r!}\chi^{(r)}(\mu)=\sum_{r=0}^{p-1}\frac{N_p^r}{r!}\chi^{(r)}(\mu)+\sum_{r=p}^{n}\frac{N_p^r}{r!}\chi^{(r)}(\mu)=O$$

tenglikni topamiz. ✍

IV.11. Chiziqli o'zgarmas koeffitsientli bir jinsli bo'lmagan sistemalar

Ushbu

$$x' = Ax + g(t),\ A \in \mathbb{M}_{n\times n}(\mathbb{C}),\ g(t) \in C(I;\mathbb{C}^n), \quad \text{(IV.11.1)}$$

chiziqli o'zgarmas koeffitsientli bir jinsli bo'lmagan sistemani ixtiyoriy o'zgarmaslarni variatsiyalash metodi yordamida yechish mumkin. Bu holda mos bir jinsli sistemaning fundamental matritsasi $\Phi(t) = e^{tA}$ va $\Phi^{-1}(s) = e^{-sA}$ bo'lgani uchun (IV.5.5) Koshi formulasiga ko'ra umumiy yechim

$$x(t) = \int_{t_0}^{t} e^{(t-s)A} g(s)ds + e^{tA}c,\ t_0 \in I,$$

ko'rinishda ifodalanadi, bunda $c \in \mathbb{C}^n$ – ixtiyoriy o'zgarmas vektor. Bu formuladagi integrallash amali qatnashgan had (IV.11.1) sistemaning xususiy yechimini ifodalaydi.

Agar sistemaning ozod hadi

$g(t) = p(t)e^{\gamma t},\ p(t) = b^m t^m + \cdots + b^1 t + b^0, \{b^m, \dots, b^1, b^0\} \subset \mathbb{C}^n,$

$\gamma \in \mathbb{C}$, m - darajali vektorli kvaziko'phaddan iborat, ya'ni berilgan sistema

$$x' = Ax + (b^m t^m + \cdots + b^1 t + b^0)e^{\gamma t} \quad \text{(IV.11.2)}$$

ko'rinishda bo'lsa, xususiy yechimni noma'lum koeffitsientlar metodi yordamida integrallash amalini ishlatmasdan turib topish mumkin.

Buni ko'rsatish uchun (IV.11.2) sistemada $x = Hy$ (y – yangi noma'lum vektor-funksiya) almashtirishni bajaramiz, unda H – bilan A matritsani Jordan ko'rinishiga keltiruvchi matritsa, $J = H^{-1}AH$, belgilangan. Natijada

$$Hy' = AHy + p(t)e^{\gamma t}, \text{ ya'ni } \quad y' = H^{-1}AHy + H^{-1}p(t)e^{\gamma t}$$

yoki

$$y' = Jy + \tilde{p}(t)e^{\gamma t}, \tilde{p}(t) = H^{-1}p(t), \quad \text{(IV.11.3)}$$

sistemani hosil qilamiz. Bu yerda $J = \mathrm{diag}(J_{\lambda_1,n_1}, \dots, J_{\lambda_2,n_2}, \dots, J_{\lambda_k,n_k})$ katakli-diagonal matritsa

bo'lganligi uchun (IV.11.3) sistema bir-biriga bog'liq bo'lmagan tenglamalar guruhlariga ajraladi. Aniqlik uchun J_{λ_1,n_1} Jordan katagiga mos kelgan tenglamalar guruhini yozaylik:

$$\begin{aligned}
y_1' &= \lambda_1 y_1 + y_2 + \tilde{p}_1(t)e^{\gamma t},\\
y_2' &= \lambda_1 y_2 + y_3 + \tilde{p}_2(t)e^{\gamma t},\\
&\cdots\cdots\cdots\cdots\\
y_{n_1-1}' &= \lambda_1 y_{n_1-1} + y_{n_1} + \tilde{p}_{n_1-1}(t)e^{\gamma t},\\
y_{n_1}' &= \lambda_1 y_{n_1} + \tilde{p}_{n_1}(t)e^{\gamma t}.
\end{aligned} \quad \text{(IV.11.4)}$$

Bu yerdagi $\tilde{p}_j(t)$ $(j = 1,2,\dots,n_1)$ ko'phadning darajasi $\deg \tilde{p}_j(t) \le m$. Qolgan Jordan kataklariga mos tenglamalar guruhi ham (IV.11.4) ga o'xshash bo'ladi. (IV.11.4) sistemada $y_j = e^{\lambda_1 t} z_j$, $j = 1,2,\dots,n_1$, almashtirishni bajaramiz va

$$\begin{aligned}
z_1' &= z_2 + \tilde{p}_1(t)e^{(\gamma-\lambda_1)t},\\
z_2' &= z_3 + \tilde{p}_2(t)e^{(\gamma-\lambda_1)t},\\
&\cdots\cdots\cdots\cdots\\
z_{n_1-1}' &= z_{n_1} + \tilde{p}_{n_1-1}(t)e^{(\gamma-\lambda_1)t},\\
z_{n_1}' &= \tilde{p}_{n_1}(t)e^{(\gamma-\lambda_1)t}
\end{aligned} \quad \text{(IV.11.5)}$$

sistemani hosil qiamiz. Bu tenglamalarni oxirgisidan boshlab ketma-ket yechamiz. Bunda ikki hol bo'lishi mumkin: $\gamma - \lambda_1 \ne 0$ va $\gamma - \lambda_1 = 0$.

Faraz qilaylik, $\gamma - \lambda_1 \ne 0$ bo'lsin. U holda bo'laklab integrallashlarni bajarib,

$$z_{n_1} = \int \tilde{p}_{n_1}(t)e^{(\gamma-\lambda_1)t}dt = \tilde{q}_{n_1}(t)e^{(\gamma-\lambda_1)t}$$

xususiy yechimni topamiz, bunda $\tilde{q}_{n_1}(t)$ ko'phadning darajasi $\tilde{p}_{n_1}(t)$ nikiga teng. Topilgan z_{n_1} ni (IV.11.5) sistemaning oxiridan ikkinchisiga qo'yib, z_{n_1-1} ni topamiz va h.k. Natijada

$$z_j = \tilde{q}_j(t)e^{(\gamma-\lambda_1)t},\ j=1,2,\dots,n_1,\ \deg\tilde{q}_j(t)\le m,$$

yechimni hosil qilamiz. $y_j = e^{\lambda_1 t}z_j$, $j=1,2,\dots,n_1$, almashtirish formulalariga ko‘ra mos

$$y_j = \tilde{q}_j(t)e^{\gamma t},\ j=1,2,\dots,n_1,\ \deg\tilde{q}_j(t)\le m,$$

larni topamiz.

Endi faraz qilaylik, $\gamma-\lambda_1=0$ bo‘lsin. Bu holda $e^{(\gamma-\lambda_1)t}\equiv 1$ va (IV.11.5) sistemadagi ozod hadlar ko‘phadlardan iborat bo‘ladi. Har bir integrallashda ko‘phadning darajasi bittaga ortadi. n_1 marta integrallashni bajarib, (IV.11.5) sistemaning

$$z_j = \tilde{q}_j(t),\ j=1,2,\dots,n_1,\ \deg\tilde{q}_j(t)\le m+n_1,$$

yechimini topamiz. Bularni almashtirish formulalariga qo‘yib, mos y_j larni aniqlaymiz:

$$y_j = \tilde{q}_j(t)^{\gamma t},\ j=1,2,\dots,n_1,\ \deg\tilde{q}_j(t)\le m+n_1.$$

Yuqoridagicha ish tutib, (IV.11.3) sistemani yechamiz va $x=Hy$ almashtirish formulasiga ko‘ra izlangan noma’lum funksiyaga qaytamiz. Bunda quyidagi xulosaga kelamiz:

agar γ son A matritsaning xos qiymati bo‘lmasa, u holda (IV.11.2) sistema $x=q(t)e^{\gamma t}$ ko‘rinishdagi yechimga ega, bunda $q(t)$ – vektor-koeffitsientli ko‘phad va $\deg q(t)=\deg p(t)$;

agar γ son A matritsaning k karrali xos qiymatidan iborat bo‘lsa, u holda (IV.11.2) sistema $x=q(t)e^{\gamma t}$ ko‘rinishdagi yechimga ega, bunda $q(t)$ – vektor-koeffitsientli ko‘phad va $\deg q(t)=\deg p(t)+k$.

Shunday qilib, (IV.11.2) sistemaning xususiy yechimini topish uchun quyidagicha ish tutish mumkin:

Dastlab k sonni aniqlaymiz: agar γ son A matritsaning xos qiymati bo‘lmasa, $k=0$ deymiz; aks holda esa k bilan γ ning necha karrali xos son ekanligini belgilaymiz. Endi yechimni

hozircha noma’lum bo‘lgan vektor-koeffitsientli $(m+k)$ - darajali kvaziko‘phad

$$x(t)=e^{\gamma t}(d^{m+k}t^{m+k}+\dots+d^1t+d^0)$$

ko‘rinishida yozamiz. Nihoyat, buni (IV.11.2) sistemaga qo‘yib, uning qanoatlanishidan izlangan noma’lum vektor-koeffitsientlarni topamiz va xususiy yechimni quramiz.

Endi haqiqiy sohada

$$x' = Ax + g(t),\ A\in\mathbb{M}_{n\times n}(\mathbb{R}),\ g(t)\in C(I;\mathbb{R}^n), \quad \text{(IV.11.6)}$$

sistemani qaraylik. Maqsad – bu sistemaning haqiqiy xususiy yechimini topish.

Faraz qilaylik, ozod had

$$g(t)=e^{\alpha t}\left(p(t)\cos\beta t+q(t)\sin\beta t\right) \quad \text{(IV.11.7)}$$

haqiqiy kavaziko‘phaddan iborat bo‘lsin, ya’ni

$$x' = Ax + e^{\alpha t}\left(p(t)\cos\beta t+q(t)\sin\beta t\right) \quad \text{(IV.11.8)}$$

ko‘rinishdagi sistemani qaraylik, bunda α,β – haqiqiy sonlar va $p(t), q(t)$ – haqiqiy vektor-koeffitsientli ko‘phadlar. Haqiqiy xususiy yechimni kompleks sohaga chiqib topish mumkin. Eyler formulasiga ko‘ra

$$\cos\beta t=\operatorname{Re}e^{i\beta t},\ \sin\beta t=-\operatorname{Re}(ie^{i\beta t}).$$

Demak,

$$g(t)=e^{\alpha t}\left(p(t)\cos\beta t+q(t)\sin\beta t\right)=\operatorname{Re}\left(p(t)-iq(t)\right)e^{(\alpha+i\beta)t}.$$

Endi ushbu

$$x' = Ax + \left(p(t)-iq(t)\right)e^{(\alpha+i\beta)t} \quad \text{(IV.11.9)}$$

sistemaning biror kompleks yechimini topib, uning haqiqiy qismini ajratsak, (IV.11.8) sistemaning haqiqiy xususiy yechimini topgan bo‘lamiz.

(IV.11.8) sistemaning xususiy yechimini kompleks sohaga chiqmasdan ham topish mumkin. Buning uchun yechimni to‘g‘ridan-to‘g‘ri ushbu

$$x(t)=e^{\alpha t}\left(r(t)\cos\beta t+s(t)\sin\beta t\right)$$

ko‘rinishda izlash lozim, bunda $r(t)$ va $s(t)$ lar $(m+k)$ - darajali haqiqiy vektor-koeffitsientli ko‘phadlar,

$m=\max\{\deg p(t),\deg q(t)\}$, k bilan $\alpha+i\beta$ xarakteristik sonning karralilik darajasi belgilangan; $\alpha+i\beta$ xarakteristik son bo‘lmaganda esa $k=0$ deb hisoblanadi.

Misol. Ushbu

$$\begin{cases} x'=x+y+4e^{2t} \\ y'=-x+3y+25\cos t \end{cases} \qquad \text{(IV.11.10)}$$

sistemaning umumiy yechimini toping.

Izlanayotgan umumiy yechim mos bir jinsli sistemaning umumiy yechimiga berilgan sistemaning xususiy yechimini qo‘shishdan hosil bo‘ladi.

Bir jinsli sistema

$$\begin{cases} x'=x+y \\ y'=-x+3y \end{cases}, \qquad \text{(IV.11.11)}$$

uning matritsasi

$$A=\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix},$$

xarakteristik tenglama

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix}=(\lambda-2)^2=0,$$

demak, bitta $k=2$ karrali xarakteristik son $\lambda=2$ mavjud. Shuning uchun (IV.11.11) sistemaning umumiy yechimi

$$\begin{pmatrix} x \\ y \end{pmatrix}=\left(\begin{pmatrix} c \\ d \end{pmatrix}t+\begin{pmatrix} a \\ b \end{pmatrix}\right)e^{2t} \qquad \text{(IV.11.12)}$$

ko‘rinishda bo‘lishi kerak. (IV.11.12) ni (IV.11.11) ga qo‘yib, uning qanoatlanishi shartidan

$$\begin{pmatrix} c \\ d \end{pmatrix}+2\left(\begin{pmatrix} c \\ d \end{pmatrix}t+\begin{pmatrix} a \\ b \end{pmatrix}\right)=A\left(\begin{pmatrix} c \\ d \end{pmatrix}t+\begin{pmatrix} a \\ b \end{pmatrix}\right)$$

ayniyatni hosil qilamiz. Oxirgi ayniyatdan

$$A\begin{pmatrix} c \\ d \end{pmatrix}=2\begin{pmatrix} c \\ d \end{pmatrix},\quad A\begin{pmatrix} a \\ b \end{pmatrix}=2\begin{pmatrix} a \\ b \end{pmatrix}+\begin{pmatrix} c \\ d \end{pmatrix}$$

shartlarni hosil qilamiz. Bu yerdagi birinchi sistema

$$\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}=2\begin{pmatrix} c \\ d \end{pmatrix}.$$

Bundan $c=d=c_1$ – ixtiyoriy o‘zgarmas ekanligini topamiz. Ikkinchi sistema endi

$$\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=2\begin{pmatrix} a \\ b \end{pmatrix}+\begin{pmatrix} c_1 \\ c_1 \end{pmatrix}$$

ko‘rinishni oladi. Bu sistemadan $a=c_2, b=c_1+c_2$ (c_2 – ixtiyoriy o‘zgarmas), yechimlarni aniqlaymiz. Topilgan qiymatlarni (y) ga qo‘yib (b) sistemaning

$$\begin{pmatrix} x \\ y \end{pmatrix}=\left(\begin{pmatrix} c_1 \\ c_1 \end{pmatrix}t+\begin{pmatrix} c_2 \\ c_1+c_2 \end{pmatrix}\right)e^{2t} \qquad \text{(IV.11.13)}$$

umumiy yechimini hosil qilamiz.

Endi (IV.11.10) (IV.11.10) sistemaning xususiy yechimini topishimiz kerak. Bu yechimni superpozitsiya prinsipiga ko‘ra

$$\begin{cases} x'=x+y+4e^{2t} \\ y'=-x+3y \end{cases} \qquad \text{(IV.11.14)}$$

va

$$\begin{cases} x'=x+y \\ y'=-x+3y+25\cos t \end{cases} \qquad \text{(IV.11.15)}$$

sistemalarning xususiy yechimlari yig‘indisi sifatida topamiz.

(IV.11.14) sistema uchun $\deg p(t)=m=0$ va $\gamma=2$ ikki karrali xarakteristik son. Demak, $k=2$ va (IV.11.14) sistemaning xususiy yechimini

$$\begin{pmatrix} x \\ y \end{pmatrix}=\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}t^2+\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}t+\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right)e^{2t} \qquad \text{(IV.11.16)}$$

ko‘rinishda izlash mumkin. (IV.11.16) ni (IV.11.14) ga qo‘yamiz va

e^{2t} ga qisqartiramiz

$$2\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}t+\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}+2\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}t^2+2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}t+2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=$$

$$=A\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}t^2+A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}t+A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}+\begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

Oxirgi ayniyatdan

$$A\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}=2\begin{pmatrix} w_1 \\ w_2 \end{pmatrix},\ A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}+2\begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

$$A\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=2\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}+\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}-\begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

tenglamalarni hosil qilamiz. Bu tenglamalar chiziqli bog'langan. Shuning uchun yechimlar cheksiz ko'p. Biz ushbu

$$w_1=w_2=-2,\ v_1=4,\ v_2=u_2=u_1=0$$

yechimni tanlaymiz va ularni (IV.11.16) ga qo'yib, (IV.11.14) ning

$$\begin{pmatrix} x \\ y \end{pmatrix}=\left(\begin{pmatrix} -2 \\ -2 \end{pmatrix}t^2+\begin{pmatrix} 4 \\ 0 \end{pmatrix}t\right)e^{2t} \qquad \text{(IV.11.17)}$$

xususiy yechimini hosil qilamiz.

(IV.11.15) sistemaning xususiy yechimi ushbu

$$\begin{cases} x'=x+y \\ y'=-x+3y+25e^{it} \end{cases} \qquad \text{(IV.11.18)}$$

kompleks sistema yechimining haqiqiy qismidan iborat, chunki $\mathrm{Re}(25e^{it})=25\cos t$. (IV.11.18) sistema uchun $\deg p(t)=m=0$, $\gamma=i$ va bu λ xarakteristik son bo'lmaganligi sababli $k=0$. Shuning uchun (IV.11.18) ning xususiy yechimini

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} a \\ b \end{pmatrix}e^{it} \qquad \text{(IV.11.19)}$$

ko'rinishda izlaymiz. (IV.11.19) ni (IV.11.18) ga qo'yib, hosil bo'lgan

$$ia=a+b,\ ib=-a+3b+25$$

tenglamalardan $a=3+4i,\ b=-7-i$ qiymatlarni topamiz. Ularni (IV.11.19) ga qo'yib, (IV.11.18) ning yechimini topamiz

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 3+4i \\ -7-i \end{pmatrix}e^{it}=\left(\begin{pmatrix} 3 \\ -7 \end{pmatrix}+i\begin{pmatrix} 4 \\ -1 \end{pmatrix}\right)(\cos t+i\sin t)=$$

$$=\begin{pmatrix} 3 \\ -7 \end{pmatrix}\cos t-\begin{pmatrix} 4 \\ -1 \end{pmatrix}\sin t+i\left(\begin{pmatrix} 3 \\ -7 \end{pmatrix}\sin t+\begin{pmatrix} 4 \\ -1 \end{pmatrix}\cos t\right)$$

Bu yechimning haqiqiy qismi bo'lmish

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 3 \\ -7 \end{pmatrix}\cos t-\begin{pmatrix} 4 \\ -1 \end{pmatrix}\sin t \qquad \text{(IV.11.20)}$$

funksiya (IV.11.15) sistemaning xususiy yechimini beradi. (IV.11.17) va (IV.11.20) yechimlarni qo'shib, (IV.11.10) sistemaning

$$\begin{pmatrix} x \\ y \end{pmatrix}=\left(\begin{pmatrix} -2 \\ -2 \end{pmatrix}t^2+\begin{pmatrix} 4 \\ 0 \end{pmatrix}t\right)e^{2t}+\begin{pmatrix} 3 \\ -7 \end{pmatrix}\cos t-\begin{pmatrix} 4 \\ -1 \end{pmatrix}\sin t \qquad \text{(IV.11.21)}$$

xususiy yechimini topamiz. Nihoyat, (IV.11.13) va (IV.11.21) larni qo'shib, berilgan sistemaning

$$\begin{cases} x=(-2t^2+(c_1+4)t+c_2)e^{2t}+3\cos t-4\sin t \\ y=(-2t^2+c_1t+c_1+c_2)e^{2t}+\sin t-7\cos t \end{cases}$$

umumiy yechimini hosil qilamiz. ♘

IV.12. Chiziqli davriy sistemalar

1°. Ushbu

$$x'=Ax+f(t) \qquad \text{(IV.12.1)}$$

vektor-matritsa ko'rinishida ifodalangan o'zgarmas koeffitsiyentli differentsial tenglamalar sistemasini qaraylik; bu yerda $A=\|a_{kj}\|_{n\times n}\in\mathbb{M}_{n\times n}(\mathbb{R})$ - o'zgarmas matritsa, $f\in C(\mathbb{R},\mathbb{R}^n)$ davriy funksiya, $f(t+\mathrm{T})\equiv f(t)$, $t\in\mathbb{R}$, $\mathrm{T}>0$. Tabiiy ravishda

"(IV.12.1) sistemaning T davrli yechimi bormi?" degan savol tug'iladi. Bu savolga quyidagi teorema javob beradi.

Teorema 1. *Agar A matritsaning spektri (xos sonlar to'plami)ga mavhum o'qning* $\frac{2\pi k}{T}i,\ k\in\mathbb{Z}$, *nuqtalari tegishli bo'lmasa, u holda* (IV.12.1) *tenglama o'ng tomonidagi ixtiyoriy uzluksiz* T *davrli* $f(t)$ *funksiya uchun* T *davrli yagona yechimga ega.*

⊸ Ma'lumki, (IV.12.1) ning umumiy yechimi

$$x(t)=e^{At}x^0+\int_0^t e^{A(t-s)}f(s)ds \qquad \text{(IV.12.2)}$$

ko'rinishda ifodalanadi; bunda $x^0=x(0)$.

Yechimning yagonalik xossasiga ko'ra (IV.12.1) sistemaning T davrli yechimga ega bo'lishi uchun uning $x(0)=x(T)$ shartni qanoatlantiruvchi yechimga ega bo'lishi yetarli va zarurdir. Demak, davriy yechimning mavjudlik sharti (IV.12.2) formulaga ko'ra

$$x^0=e^{AT}x^0+\int_0^T e^{A(T-s)}f(s)ds \qquad \text{(IV.12.3)}$$

kabi yoziladi. Bu tenglikdagi integralda $\tau=s-T$ o'zgaruvchini kiritib, va $f(\tau+T)=f(\tau)$ ekanligini hisobga olib, (IV.12.3) shartni

$$-(e^{AT}-E)x^0=\int_{-T}^0 e^{-A\tau}f(\tau)d\tau \qquad \text{(IV.12.4)}$$

ko'rinishga keltiramiz.

Agar A matritsaning $\lambda_1,\lambda_2,\ldots,\lambda_m$ xos sonlari mos ravishda $k_1, k_2, \ldots, k_m$ karrali bo'lsa, u holda e^{AT} matritsaning xos sonlari $e^{\lambda_1 T}, e^{\lambda_2 T},\ldots,e^{\lambda_m T}$ mos ravishda $k_1, k_2, \ldots, k_m$ karrali bo'ladi. Shuning uchun

$$\det(e^{AT}-E)=(e^{\lambda_1 T}-1)^{k_1}(e^{\lambda_2 T}-1)^{k_2}\ldots(e^{\lambda_m T}-1)^{k_m}.$$

Demak, agar

$$\lambda_l T\neq 2\pi k i,\ k\in\mathbb{Z},$$

ya'ni

$$\lambda_l\neq\frac{2\pi k}{T}i,\ l=1,\ldots,m,\ k\in\mathbb{Z},$$

bo'lsa, u holda $e^{AT}-E$ matritsaning determinanti noldan farqli va (IV.12.4) ga ko'ra $x^0=-(e^{AT}-E)^{-1}\int_{-T}^0 e^{-A\tau}f(\tau)d\tau$ boshlang'ich shartli yechim T davrga ega bo'lgan yagona yechimni aniqlaydi. ⊹

Bu yerda shuni ta'kidlaylikki, bu holda bir jinsli tenglama notrivial T davrli yechimga ega bo'la olmaydi, chunki aks holda $e^{AT}x^0=x^0$ tenglama $x^0\neq 0$ yechimga ega bo'lar edi; bu esa $(e^{AT}-E)^{-1}$ ning mavjudligiga zid. Shunday qilib, quyidagi tasdiq isbotlandi:

Agar A matritsa $\frac{2\pi k}{T}i,\ k\in\mathbb{Z}$, ko'rinishdagi xos qiymatga ega bo'lsa, u holda (IV.12.1) sistema ba'zi T davrli f larda T davrli yechimga ega bo'ladi, ba'zi T davrli f larda esa u birorta ham davriy yechimga ega bo'lmaydi.

Eslatma. Agar A matritsa xos qiymatlarining haqiqiy qismlari 0 dan kichik bo'lsa, u holda barcha uzluksiz T davriy f funksiyalar uchun (IV.12.1) ning davriy yechimi mavjud bo'lib, $t\to+\infty$ bo'lganda (IV.12.1) ning har qanday yechimi shu davriy yechimga yaqinlashadi. Nega?

2°. Koeffitsientlari T davrli funksiyalardan iborat bo'lgan ushbu

$$x'=A(t)x \qquad \text{(IV.12.5)}$$

chiziqli bir jinsli tenglamalar sistemasini qaraylik; bu yerda

$$A(t)\in C(\mathbb{R},\mathbb{M}_{n\times n}(\mathbb{R}));\ A(t+T)\equiv A(t),\ t\in\mathbb{R},\ T>0. \quad \text{(IV.12.6)}$$

Tushunarliki, agar $x=\varphi(t)$ funksiya (IV.12.5) ning yechimi bo'lsa, u holda $x=\varphi(t+T)$ ham (IV.12.5) ning yechimi. Lekin bu $x=\varphi(t)$ yechim davriy bo'lishi shart emas, chunki uning argumenti T ga ortganda u o'zgarmasga ko'payishi yoki unga chiziqli bog'liq bo'lmagan yechim qo'shilishi mumkin.

$\mathbb{R}^n$ fazoning $e^1, e^2, \dots, e^n$ standart bazisiga ko'ra tuzilgan $x(0) = e^k$ $(k = 1, \dots, n)$ boshlang'ich shartni qanoatlantiruvchi (IV.12.5) sistemaning yechimini $x = \varphi^k(t)$ bilan belgilab, $\varphi^1(t), \dots, \varphi^n(t)$ bazis yechimlardan $\Phi(t) = [\varphi^1(t), \dots, \varphi^n(t)]$ fundamental matritsani tuzaylik. Bu fundamental matritsa $t = 0$ nuqtada normalangan, ya'ni $\Phi(0) = E$. (IV.12.5) sistemaning har qanday yechimi

$$x(t) = \Phi(t)x(0)$$

formula bilan ifodalanadi. $\Phi(t)$ fundamental matritsa quyidagi Koshi masalasining yechimidir:

$$\begin{cases} \Phi'(t) = A(t)\Phi(t) \\ \Phi(0) = E \end{cases}$$

Tushunarliki, ixtiyoriy $t \in \mathbb{R}$ uchun $\det \Phi(t) \neq 0$ ham bo'ladi.

Ravshanki,

$$\Phi'(t + \mathrm{T}) = A(t + \mathrm{T})\Phi(t + \mathrm{T}) = A(t)\Phi(t + \mathrm{T}).$$

Shuning uchun, $\Phi(t)$ bilan birgalikda $\Phi(t + \mathrm{T})$ ham fundamental matritsa va u $\Phi(t)$ni o'ngdan biror o'zgarmas teskarilanuvchi C ($\det C \neq 0$) matritsaga ko'paytirishdan hosil bo'lishi kerak:

$$\Phi(t + \mathrm{T}) = \Phi(t)C. \qquad \text{(IV.12.7)}$$

Bu tenglikda $t = 0$ desak, $\Phi(\mathrm{T}) = C$ hosil bo'ladi .Demak, (IV.12.7) ga ko'ra

$$\Phi(t + \mathrm{T}) = \Phi(t)\Phi(\mathrm{T}). \qquad \text{(IV.12.8)}$$

Bu yerdagi $\Phi(\mathrm{T})$ matritsa (IV.12.5) sistemaning monodromiya matritsasi deyiladi. $\Phi(\mathrm{T})$ matritsaning xos qiymatlari (IV.12.5) sistemaning multiplikatorlari deyiladi.

Monodromiya matritsasi haqiqiy, lekin multiplikatorlar kompleks bo'lishi mumkin.

Teorema 2. λ *kompleks son* (IV.12.5) *sistemaning multiplikatori bo'lishi uchun shu sistemaning*

$$x(t + \mathrm{T}) = \lambda x(t) \qquad \text{(IV.12.9)}$$

shartni qanoatlantiruvchi $x(t)$ *notrivial yechimga ega bo'lishi zarur va yetarlidir.*

Zarurligi. Faraz qilaylik, λ son (IV.12.5) sistemaning multiplikatori bo'lsin. U holda

$$\exists x^0 \neq 0 \;\; \Phi(\mathrm{T})x^0 = \lambda x^0.$$

$x(t) = \Phi(t)x^0$ vektor-funksiya (IV.12.5) sistemaning notrivial yechimidir. (IV.12.8) ga ko'ra

$$x(t + \mathrm{T}) = \Phi(t + \mathrm{T})x^0 = \Phi(t)\Phi(\mathrm{T})x^0 =$$

$$= \Phi(t)\lambda x^0 = \lambda\Phi(t)x^0 = \lambda x(t).$$

Demak, (IV.12.9) shart bajariladi.

Yetarliligi. $x(t)$ vektor-funksiya (IV.12.5) ning (IV.12.9) shartni qanoatlantiruvchi notrivial yechimi bo'lsin. (IV.12.9) da $t = 0$ deb, topamiz:

$$x(\mathrm{T}) = \lambda x(0) \qquad \text{(IV.12.10)}$$

$x(t)$ yechim $x(t) = \Phi(t)x(0)$ ko'rinishga ega. Bundan $x(\mathrm{T}) = \Phi(\mathrm{T})x(0)$ va buni (IV.12.10) bilan taqqoslab,

$$\Phi(\mathrm{T})x(0) = \lambda x(0) \qquad \text{(IV.12.11)}$$

tenglikni topamiz. Berilgan notrivial $x(t)$ yechim uchun $x(0) \neq 0$ bo'lishi kerak, chunki aks holda $x(t) \equiv 0$ trivial yechim bo'lardi. Endi (IV.12.11) tenglikdan $x(0)$ vektor $\Phi(\mathrm{T})$ monodromiya matritsaning xos vektori, λ esa uning xos soni, ya'ni (IV.12.5) sistemaning multiplikatori ekanligini ko'ramiz.

Natija. (IV.12.5) *sistema* T *davrli notrivial yechimga ega bo'lishi uchun uning biror multiplikatori* $\lambda = 1$ *bo'lishi yetarli va zarurdir.*

Teorema. (IV.12.5) *sistemaning* $\Phi(t)$ *fundamental matritsasi* T *davrli biror* $\Psi(t)$ *matritsaviy funksiya va o'zgarmas* M *matritsa orqali*

$$\Phi(t) = \Psi(t)e^{tM}$$

formula bilan ifodalanadi.

🗝 $\det\Phi(\mathrm{T}) = \det C \neq 0$ bo'lgani uchun monodromiya matritsasining logarifmi mavjud. $M = \frac{1}{\mathrm{T}}\ln C$, ya'ni $\Phi(\mathrm{T}) = C = e^{\mathrm{T}M}$ deylik. Demak,

$$\Phi(t+\mathrm{T}) = \Phi(t)e^{\mathrm{T}M}.$$

Ushbu

$$\Psi(t) = \Phi(t)e^{-tM}$$

matritsaviy funksiya T davrli va $\det\Psi(t) \neq 0$ dir. Haqiqatan ham

$$\Psi(t+\mathrm{T}) = \Phi(t+\mathrm{T})e^{-(t+\mathrm{T})M} = \Phi(t)e^{\mathrm{T}M}e^{-\mathrm{T}M}e^{-tM} = \Phi(t)e^{-tM} = \Psi(t)$$

va $\det\Psi(t) = \det\Phi(t)\cdot\det e^{-tM} \neq 0$. ☝.

Natija. (IV.12.5) *sistema* $m\mathrm{T}$ $(m\in\mathbb{R})$ *davrli notrivial yechimga ega bo'lishi uchun* $\sqrt[m]{1}$ *ning biror qiymati multiplikatordan iborat bo'lishi yetarli va zarurdir.*

Isboti. Fundamental matritsaning $\Phi(t) = \Psi(t)e^{tM}$ ko'rinishida ifodalanishidan (IV.12.5) sistemaning yechimi uchun quyidagi formulani yozamiz:

$$x(t) = \Psi(t)e^{tM}x^0.$$

Bundan

$$x(t+m\mathrm{T}) = \Psi(t+m\mathrm{T})e^{(t+m\mathrm{T})M}x^0 = \\ = \Psi(t)e^{tM}e^{m\mathrm{T}M}x^0.$$

Endi, ravshanki,

$$x(t+m\mathrm{T}) = x(t) \Leftrightarrow \Psi(t)e^{tM}e^{m\mathrm{T}M}x^0 = \Psi(t)e^{tM}x^0 \Leftrightarrow e^{m\mathrm{T}M}x^0 = x^0,$$

ya'ni

$$x(t+m\mathrm{T}) = x(t) \Leftrightarrow \left(e^{\mathrm{T}M}\right)^m x^0 = x^0. \text{ ☝}$$

3⁰. Endi bir jinsli bo'lmagan tenglamalar sistemasini qaraylik:

$$x' = A(t)x + g(t), \qquad \text{(IV.12.12)}$$

bu yerda $A(t)\in C(\mathbb{R},\mathbb{M}_{n\times n}(\mathbb{R}))$, $g(t)\in C(\mathbb{R},\mathbb{R}^n)$ va

$$A(t+\mathrm{T}) = A(t),\quad g(t+\mathrm{T}) = g(t),\ t\in\mathbb{R},\ \mathrm{T}>0.$$

Teorema 3. *Agar* (IV.12.5) *bir jinsli sistema* T *davrli notrivial yechimga ega bo'lmasa (barcha multiplikatorlari 1 dan farqli), u holda* (IV.12.12) *bir jinsli bo'lmagan sistema ixtiyoriy* T *davrli* $g(t)\in C(\mathbb{R},\mathbb{R}^n)$ *o'ng tomon uchun yagona* T *davrli yechimga ega*

🗝 Isbotni teorema 1 ning isboti kabi bajaramiz. (IV.12.12) ning ixtiyoriy yechimi, ma'lumki, quyidagicha ifodalanadi:

$$x(t) = \Phi(t)x^0 + \int_0^t \Phi(t)\Phi^{-1}(s)g(s)ds, \qquad \text{(IV.12.13)}$$

bu yerda $\Phi(t)$ matritsa (IV.12.5) sistemaning yuqorida kiritilgan fundamental matritsasi. $A(t)$ va $g(t)$ larning T davrliligidan $x(t)$ yechim T davrli bo'lishi uchun

$$x(\mathrm{T}) = x(0) \qquad \text{(IV.12.14)}$$

shartning bajarilishi yetarli va zarur ekanligi kelib chiqadi. (IV.12.14) davriylik shartini (IV.12.13) dan foydalanib

$$\Phi(\mathrm{T})x^0 + \Phi(\mathrm{T})\int_0^{\mathrm{T}} \Phi^{-1}(s)g(s)ds = x^0$$

yoki

$$(\Phi(\mathrm{T}) - E)x^0 = -\Phi(\mathrm{T})\int_0^{\mathrm{T}} \Phi^{-1}(s)g(s)ds \qquad \text{(IV.12.15)}$$

ko'rinishga keltiramiz.

Endi faraz qilaylik, (IV.12.5) sistemaning multiplikatorlari 1 dan farqli bo'lsin. U holda $\det(\Phi(\mathrm{T}) - E) \neq 0$, chunki aks holda $(\Phi(\mathrm{T}) - E)a = 0$ bir jinsli algebraik sistema $a \neq 0$ yechimga ega, ya'ni $\lambda = 1$ (IV.12.5) sistema uchun multiplikator bo'lardi. Demak, $(\Phi(\mathrm{T}) - E)^{-1}$ teskari matritsa mavjud. Shuning uchun (IV.12.15) tenglamadan x^0 bir qiymatli aniqlanadi. (IV.12.12) sistemaning ana shu x^0 boshlang'ich qiymatli yechimi T davrlidir. ♤

Masalalar

1. $y' = f(x,y)$ tenglamada $f(x,y) \in C(\mathbb{R}^2)$ bo'lsin. Agar bu tenglama T davrli yechimga ega bo'lsa, $f(x,y)$ funksiya x bo'yicha T davrli bo'lishini isbotlang.

2. Faraz qilaylik, $f(x,y)$ funksiya $C^1(\mathbb{R}^2)$ sinfga tegishli, x bo'yicha davriy va $\dfrac{\partial f(x,y)}{\partial y} > 0$ bo'lsin. U holda $y' = f(x,y)$ tenglama bittadan ko'p davriy yechimga ega bo'lolmasligini ko'rsating.

V BOB. AVTONOM SISTEMALAR

V.1. Avtonom sistema yechimlarining umumiy xossalari

Ushbu

$$\begin{cases} x_1' = f_1(x_1, x_2, \dots, x_n) \\ x_2' = f_2(x_1, x_2, \dots, x_n) \\ \dots\dots\dots\dots\dots\dots \\ x_n' = f_n(x_1, x_2, \dots, x_n) \end{cases} \qquad (V.1.1)$$

differensial tenglamalar sistemasini qaraylik. Bu yerda sistemaning o'ng tomoni erkli haqiqiy o'zgaruvchi t ga bog'liq emas. Shu sababli bu sistema **avtonom (muxtor) sistema** deyiladi. Harakati avtonom sistemalar bilan tavsiflanadigan mexanik sistemalar harakatini boshqaruvchi qonunlar, va kuchlar vaqt o'tishi bilan o'zgarmaydi, ya'ni ular t vaqtga bog'liq bo'lmaydi.

(V.1.1) sistemada erkli o'zgaruvchi t vaqt deb tushunilganda, bu sistema **dinamik sistema** deb yuritiladi.

(V.1.1) avtonom sistemani

$$x' = f(x) \qquad (V.1.2)$$

vektorli ko'rinishda ifodalaylik; bu yerda

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x' = \begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix}, \quad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}, \quad x \in G \subset \mathbb{R}^n,$$

$x = x(t)$ noma'lum funksiya $n \times 1$ o'lchamli vektor-funksiyadan iborat. Sistemaning o'ng tomoni aniqlangan va $x = x(t)$ yechimlarning qiymatlari joylashgan G soha bu sistema uchun **fazalar fazosi** deb ataladi.

Biz bu paragrafda avtonom (dinamik) sistemalarni o'rganamiz. G sohada $f(x)$ vektor-funksiya o'zining barcha birinchi tartibli xususiy hosilalari bilan birgalikda uzluksiz, ya'ni $f \in C^1(G;\mathbb{R}^n)$ deb hisoblaymiz. U holda ixtiyoriy $x^0 \in G$ uchun $x|_{t_0} = x^0$ shartni qanoatlantiruvchi yagona yechim $x = x(t)$ mavjud bo'ladi. Bu $x = x(t)$ yechim fazalar fazosida parametrik ko'rinishda berilgan chiziqni (yoki nuqtani) ifodalaydi. U (V.1.2) ning ($t = t_0$ da $x = x^0 \in G$ nuqtadan o'tuvchi) **fazaviy traektoriyasi** deb ataladi. Vaqt o'tishi bilan $x(t) \in G$ nuqta traektoriya bo'ylab harakat qiladi; bu harakat yo'nalishini rasmda odatda strelka bilan ko'rsatiladi. (V.1.2) avtonom sistemaning barcha fazaviy traektoriyalari uning fazaviy tasviri (portreti) deyiladi.

(V.1.2) ning o'ng tomonidagi $f(x)$ funksiya vektor maydonni (tezliklar maydonini) aniqlaydi. (V.1.2) tenglama traektoriyasining ixtiyoriy $x = x(t)$ nuqtasidagi $x'(t)$ tezlik vektor maydonning shu nuqtadagi $f(x(t))$ qiymatiga teng: $x'(t) = f(x(t))$. Traektoriya o'zining har bir x nuqtasida shu nuqtadagi $f(x)$ vektorga urinadi.

Agar $a \in G$ nuqta uchun $f(a) = 0$ bo'lsa, a nuqta $f(x)$ vektor maydonning maxsus (kritik) nuqtasi deyiladi. Bunda hosil bo'luvchi $x(t) = a$ o'zgarmas yechimning fazaviy traektoriyasi a

nuqtadan iborat bo'ladi va u (V.1.2) sistemaning muvozanat nuqtasi yoki muvozanat (statsionar) holati deb ataladi.

Shuni ta'kidlaylikki, integral chiziq yechimning $1+n$ o'lchamli $\mathbb{R}^{1+n}=\left\{(t,x)\mid t\in\mathbb{R},x\in\mathbb{R}^n\right\}$ fazodagi grafigidan iborat bo'lib, uning $\mathbb{R}^n$ (o'lchami bittaga kam) fazodagi ortogonal proyeksiyasi fazaviy traektoriyani aniqlaydi. Shuning uchun avtonom sistemani uning traektoriyalari orqali o'rganish qulay.

Eslatma. Avtonom bo'lmagan

$$x'=f(t,x)$$

sistemaga qo'shimcha $y=t$ funksiyani kiritib, uni ushbu

$$\begin{cases} y'=1 \\ x'=f(y,x)\end{cases}$$

avtonom ko'rinishga keltirish mumkin. Lekin bu yerda $y=t$ va $x_1,\ldots,x_n$ o'zgaruvchilarning mohiyatlari har xil va fazalar fazosi (y,x) nuqtalar to'plami bo'lmish $1+n$ o'lchamli $\mathbb{R}^{1+n}$ fazo qismidan iborat bo'ladi.

Avtonom sistema (V.1.2) ning yechimlari va traektoriyalarining ba'zi xossalarini keltiramiz.

1^0. *Agar $x=x(t)$, $t\in(a;b)$, yechim bo'lsa, ixtiyoriy $c\in\mathbb{R}$ uchun $x=x(t+c)$, $t\in(a-c;b-c)$, funksiya ham yechim hamda ularning traektoriyalari ustma-ust tushadi.*

⊶ Shartga ko'ra

$$x'(t)=f(x(t))\ \ (t\in(a;b))\Rightarrow x'(t+c)=f(x(t+c))\ \ (t\in(a-c;b-c))$$

va, demak,

$$\frac{d}{dt}x(t+c)=x'(t+c)\cdot 1=f(x(t+c))\ \ (t\in(a-c;b-c)).$$

Bu $x=x(t)$ va $x=x(t+c)$ yechimlarning traektoriyalari bir xil bo'ladi, chunki $x=x(t)$ traektoriyaning $x(\tilde{t})$ nuqtasidan $x=x(t+c)$ yechim $t=\tilde{t}-c$ bo'lganda, $x=x(t+c)$ traektoriyaning $x(\tilde{t}+c)$ nuqtasidan esa $x=x(t)$

yechim $t=\tilde{t}+c$ bo'lganda o'tadi. ♘

Demak, $x=\varphi(t;t_0,x^0)$ fazaviy traektoriyani $x=\varphi(t-t_0,x^0)$ ko'rinishda yozish mumkin, ya'ni avtonom sistema uchun vaqt boshi t_0 ahamiyatsiz.

2^0. *Agar $x=\varphi(t)$ va $x=\psi(t)$ yechimlar uchun $\varphi(t_1)=\psi(t_2)$ bo'lsa, $\varphi(t)=\psi(t+t_2-t_1)$ bo'ladi. Demak, umumiy nuqtaga ega bo'lgan traektoriyalar ustma – ust tushadi, ya'ni fazalar fazosining har bir nuqtasidan yagona fazaviy traektoriya o'tadi.*

⊶ Yuqoridagi 1^0 xossaga ko'ra $x=\psi(t)$ yechim bo'lganligi uchun $x=\psi(t+t_2-t_1)$ ham yechim va ularning traektoriyalari bir xil. $x=\varphi(t)$ va $x=\psi(t+t_2-t_1)$ yechimlarning qiymatlari $t=t_1$ da teng bo'lganligi uchun yechimning yagonalik xossasiga ko'ra bu yechimlar ustma-ust tushadi, ya'ni $\varphi(t)=\psi(t+t_2-t_1)$. ♘

Natija. *Ixtiyoriy ikki traektoriya yo umumiy nuqtaga ega emas, yoki ustma-ust tushadi.*

3^0. *Agar $x=x(t)$ traektoriya uchun $\lim\limits_{t\to+\infty}x(t)=a$ ($a=(a_1,a_2,\ldots,a_n)^T\in G\subset\mathbb{R}^n$) bo'lsa, a nuqta muvozanat nuqtadir.*

⊶ Teskarisini faraz qilaylik, ya'ni a nuqta muvozanat (maxsus, kritik) nuqta bo'lmasin. U holda $f(a)\neq 0$ bo'ladi. Aniqlik uchun $f(a)$ vektorning birinchi koordinatasi noldan farqli, ya'ni $f_1(a)\neq 0$ deylik. Fikrlashni yanada aniqlashtirish uchun $f_1(a)=\alpha>0$ deb ham hisoblaymiz. f uzluksiz bo'lgani uchun $\lim\limits_{t\to+\infty}f_1(x(t))=f_1(a)=\alpha>0$. Limitning ta'rifiga ko'ra shunday t_* topiladiki, har qanday $t\geq t_*$ uchun $f_1(x(t))>\alpha/2$ bo'ladi. Demak, shu $t\geq t_*$ lar uchun $x_1'(t)=f_1(x(t))>\alpha/2$. Buni integrallab, $x_1(t)-x_1(t_*)>(t-t_*)\alpha/2$, $t\ge t_*$, tengsizlikni topamiz. Oxirgi tengsizlikdan $\lim_{t\to+\infty}x_1(t)=+\infty$. Lekin berilganga ko'ra $\lim_{t\to+\infty}x_1(t)=a_1\in\mathbb{R}$ bo'lishi kerak edi. ♁

4^0. *Agar o'zgarmasdan (muvozanat nuqtadan) farqli biror $x=\varphi(t)$ yechim uchun $\varphi(t_1)=\varphi(t_2)$, $t_1\ne t_2$, bo'lsa, bu yechim eng kichik musbat davrga ega bo'lgan davriy funksiyadan, uning traektoriyasi esa sodda (o'z-o'zini kesmaydigan) yopiq chiziqdan iborat bo'ladi.*

⊶ Aniqlik uchun $\tau=t_2-t_1>0$ deylik. Yuqoridagi 2^0. xossaga ko'ra $\varphi(t)=\varphi(t+\tau)$. Bu tenglikdan foydalanib yechimni o'ngga va chapga cheksiz davom ettiramiz va $x=\varphi(t)$ yechimning $(-\infty;+\infty)$ oraliqda aniqlangan $\tau>0$ davrli funksiyadan iborat ekanligini topamiz. Endi bu yechimning eng kichik musbat davrga ega ekanligini isbotlaymiz. T bilan $x=\varphi(t)$ funksiyaning barcha musbat davrlari to'plamini belgilaylik:

$$T=\{\theta>0\mid \forall t\in\mathbb{R}\ \varphi(t+\theta)=\varphi(t)\}.$$

$T\ne\varnothing$, chunki $\tau\in T$. T quyidan nol bilan chegaralangan, demak, T aniq quyi chegaraga ega. $\tau_0=\inf T$ deylik. Ravshanki, $0\le\tau_0\le\tau$. Aniq quyi chegara ta'rifiga ko'ra τ_0 ga intiluvchi musbat davrlar ketma-ketligi θ_j, $j\in\mathbb{N}$, mavjud: $\theta_j\to\tau_0$, $\theta_j\in T$. $x=\varphi(t)$ funksiya uzluksiz bo'lgani uchun $\varphi(t+\theta_j)=\varphi(t)$ $(\theta_j\in T)$ munosabatda limitga o'tib, $\varphi(t+\tau_0)=\varphi(t)$ ayniyatni topamiz. Agar $\tau_0\ne 0$ bo'lishini ko'rsatsak, τ_0 eng kichik musbat davr ekanligini isbotlagan bo'lamiz. Ko'rsatilishi kerak bo'lgan tasdiqning teskarisini faraz qilaylik, ya'ni $\tau_0=0$ deylik. Demak, infimumning ta'rifiga ko'ra, xohlagancha kichik musbat davrlar mavjud. $x=\varphi(t)$ yechim o'zgarmasdan farqli bo'lgani uchun shunday $t_3\in\mathbb{R}$ topiladiki,

uning uchun $\varepsilon\overset{def}{=}\|\varphi(t_3)-\varphi(t_2)\|>0$ bo'ladi. $x=\varphi(t)$ funksiya uzluksiz bo'lgani uchun esa $t_3\in\mathbb{R}$ nuqtaning shunday $B(t_3)$ atrofini topamizki, bu atrofdagi barcha t lar uchun $\|\varphi(t)-\varphi(t_3)\|<\varepsilon$ bo'ladi. Yetarlicha kichik musbat θ davrni va $k\in\mathbb{Z}$ sonni tanlash evaziga $t=t_2+k\theta$ nuqtani $B(t_3)$ atrofga tushirish mumkin. Ana shunday $t=t_2+k\theta\in B(t_3)$ nuqta uchun $\varepsilon=\|\varphi(t_2)-\varphi(t_3)\|=\|\varphi(t_2+k\theta)-\varphi(t_3)\|<\varepsilon$ ziddiyat hosil bo'ladi. Demak, farazimiz noto'g'ri va τ_0 eng kichik musbat davr. Bundan $x=\varphi(t)$ $(0\le t\le\tau_0)$ traektoriyaning sodda yopiq chiziq ekanligi kelib chiqadi, chunki $\varphi(0)=\varphi(\tau_0)$. Agar bu yopiq chiziq o'z-o'zini kesganda, ya'ni $[0;\tau_0]$ oraliqdagi biror t_1 va t_2 $(t_2>t_1)$ lar $(\{t_1,t_2\}\subset[0;\tau_0],\ t_2-t_1<\tau_0)$ uchun $\varphi(t_1)=\varphi(t_2)$ bo'lganda edi, u holda $x=\varphi(t)$ yechim $\theta=t_2-t_1<\tau_0$ musbat davrga ega bo'lardi. Bu esa τ_0 ning eng kichik musbat davr ekanligiga zid. Demak, $x=\varphi(t)$ (yopiq) traektoriya o'z-o'zini kesmaydi.

Osongina ko'rsatish mumkinki, $x=\varphi(t)$ yechimning ixtiyoriy T davri τ_0 ga karrali bo'ladi. Faraz qilatlik, $T>0$ davr τ_0 ga karrali bo'lmasin, ya'ni ixtiyoriy $k\in\mathbb{N}$ uchun $T\ne k\tau_0$ bo'lsin. Tushunarliki, $T>\tau_0$ bo'lishi kerak. $k\tau_0<T$ tengsizlikni qanoatlantiruvchi eng katta $k\in\mathbb{N}$ ni m deylik, ya'ni $m\tau_0<T$, $(m+1)\tau_0>T$ $(m\in\mathbb{N})$. Demak, $T-m\tau_0<\tau_0$ davr mavjud. Bu esa τ_0 ning eng kichik musbat davr ekanligiga zid. Shunday qilib, farazimiz noto'g'ri va har qanday davr τ_0 ga karrali. ♁

Teorema. *Yuqorida aytilgan $f\in C^1(G;\mathbb{R}^n)$ shart bajarilganda $x'=f(x)$ avtonom sistemaning har qanday traektoriyasi (yechimi) quyidagi uch turning bittasiga mansub bo'ladi:*

— *nuqta, ya'ni muvozanat nuqtasi (yechimning davri ixtiyoriy son);*

— *o'z-o'zini kesmaydigan yopiq chiziq (eng kichik musbat davrli yechim);*

— *o'z-o'zini kesmaydigan yopiqmas chiziq (davrsiz yechim).*

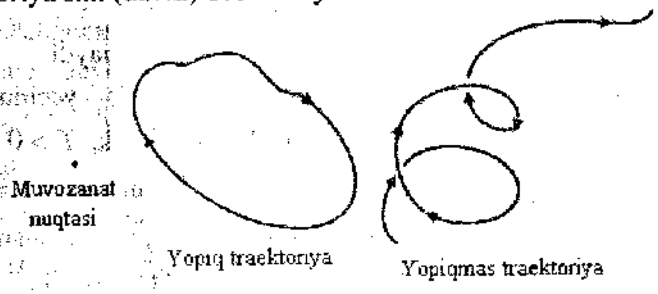
⊶ Ixtiyoriy $x = \varphi(t)$ **yechimni** qaraylik. Mantiqan quyidagi uch hol bo'lishi **mumkin** xolos.

1). $\varphi(t) = \text{const}$; bu holda traektoriya nuqtadan iborat.

2). $\varphi(t) \neq \text{const}$, lekin biror t_1 va $t_2 \neq t_1$ lar uchun $\varphi(t_1) = \varphi(t_2)$; bu holda 3° xossaga ko'ra traektoriya o'z-o'zini kesmaydigan yopiq chiziqan iborat.

3). Barcha t_1 va $t_2 \neq t_1$ lar uchun $\varphi(t_1) \neq \varphi(t_2)$; bu holda traektoriya o'z-o'zini kesmaydigan va yopiq bo'lmagan chiziqan iborat (V.1- rasm). ☝

O'z-o'zini kesmaydigan yopiq chiziqdan iborat bo'lgan traektoriya **sikl** (davra) deb ham yuritiladi.



V.1-rasm.

5°. Avtonom sistema yechimlarining gruppaviy xossasi. Bu bandda (V.1.2) avtonom sistemaning barcha yechimlarini $(-\infty;+\infty)$ oraliqda aniqlangan deb hisoblaymiz.

$x = \varphi(t,\xi)$ bilan $x(0) = \xi$ boshlang'ich shartni qanoatlantiruvchi yechimni belgilab, ixtiyoriy $t \in \mathbb{R}$ uchun $g^t : G \to G$ akslantirishni $g^t(\xi) = \varphi(t,\xi)$ ($\xi \in G$, t – parametr) formula

bilan kiritaylik. U holda bir parametrli $g^t : G \to G$ akslantirishlar oilasi hosil bo'ladi. $f \in C^1(G;\mathbb{R}^n)$ bo'lganligi uchun $g^t(\xi) = \varphi(t,\xi) \in C^1(\mathbb{R} \times G; G)$ ham bo'ladi.

Jumla. *$g^t : G \to G$ almashtirishlar oilasi kompozitsiya amaliga nisbatan Abel gruppasini tashkil etadi, ya'ni*

1). $(g^r \circ g^s) \circ g^t = g^r \circ (g^s \circ g^t)$, $\{r,s,t\} \subset \mathbb{R}$, *(assotsiativlik);*

2). $g^s \circ g^t = g^t \circ g^s = g^{t+s}$ *(kommutativlik);*

3). $g^0 \circ g^t = g^t \circ g^0$ *(birlik elementning mavjudligi);*

4). $g^{-t} \circ g^t = g^t \circ g^{-t} = g^0$ *(teskari elementning mavjudligi).*

⊶ Assotsiativlik xossasi **har** qanday almashtirishlar uchun o'rinli. Agar $g^t \circ g^s = g^{t+s}$ munosabatni isbotlasak, undan isbotlanishi kerak bo'lgan boshqa xossalar bevosita kelib chiqadi. $g^t \circ g^s = g^{t+s}$ tenglik $\varphi(t,\varphi(s,\xi)) = \varphi(t+s,\xi)$ $(\xi \in G)$ ekanligini anglatadi. Oxirgi tenglik $x = \varphi(t,\varphi(s,\xi))$ va $x = \varphi(t+s,\xi)$ yechimlarning $t = 0$ nuqtada tengligi va yechimning yagonalik xossasidan ravshan. ☝

Agar $g^t : G \to G,\ t \in \mathbb{R}$, bir parmetrli almashtirishlar majmuasi uchun ushbu

– $g^0 = 1$ – G ni ayniy almashtirish, ya'ni $g^0(\xi) = \xi,\ \xi \in G$;

– $g^{t+s} = g^t \circ g^s$, $\{t,s\} \subset \mathbb{R}$;

– $g^t(\xi) \in C(\mathbb{R} \times G; G)$

shartlar o'rinli bo'lsa, u holda G da $g^t : G \to G,\ t \in \mathbb{R}$, **dinamik sistema** berilgan deb ataladi. Agar bundan tashqari $g^t(\xi) \in C^1(\mathbb{R} \times G; G)$ ham bo'lsa, qaralayotgan $g^t : G \to G,\ t \in \mathbb{R}$, dinamik sistema **fazaviy oqim deyiladi**.

Shunday qilib, yuqorida kiritilgan $g^t : G \to G,\ g^t(\xi) = \varphi(t,\xi),\ t \in \mathbb{R}$, almashtirishlar fazaviy oqimni tashkil etadi.

(V.1.2) avtonom sistema G da aniqlangan g^t fazaviy oqim orqali bir qiymatli tiklanadi.

Jumla. *Ushbu*

$$\left.\frac{dg^t}{dt}\right|_{t=0}=f$$

formula o'rinli.

🗝 Ixtiyoriy $\xi\in G$ uchun quyidagi hisoblashlarni bajaramiz:

$$\left.\frac{dg^t}{dt}\right|_{t=0}(\xi)=\left.\frac{dg^t(\xi)}{dt}\right|_{t=0}=\left.\frac{d\varphi(t,\xi)}{dt}\right|_{t=0}=f(\xi).$$ ♤

6⁰. Traektoriyalarning limit to'plamlari. (V.1.2) avtonom sistemanining $x=\varphi(t)$ yechimi berilgan bo'lsin. $\Gamma=\{x\in\mathbb{R}^n \mid x=\varphi(t),\ t\in(-\infty,+\infty)\}$ traektoriyaning yoki uning $\Gamma^+=\{x\in\mathbb{R}^n \mid x=\varphi(t),\ t\in[t_0,+\infty)\}$ musbat yarimtraektoriyasi uchun ω-limit nuqta deb shunday $p\in\mathbb{R}^n$ nuqtaga aytiladiki, uning uchun biror $t_1,t_2,\dots,t_k,\dots\to+\infty$ ketma-ketlik topilib, $p=\lim\limits_{k\to+\infty}\varphi(t_k)$ bo'ladi. Γ traektoriyaning barcha ω-limit nuqtalari to'plami Γ ning ω-limit to'plami deb ataladi va $\Omega(\Gamma)$ bilan belgilanadi. $t_1,t_2,\dots,t_k,\dots\to-\infty$ bo'lganda yuqoridagiga o'xshash α-limit nuqtalar va α-limit to'plamlar tushunchalari kiritiladi.

Agar $x=\varphi(t)$ yechim uchun $\lim\limits_{t\to+\infty}\varphi(t)=a$ (masalan, $x=\varphi(t)\equiv a$ – muvozanat holat) bo'lsa, bu yechim traektoriyasining ω-limit to'plami, tushunarliki, bitta a nuqtadan iborat bo'ladi. Agar $x=\varphi(t)$ – eng kichik musbat τ davrli yechim bo'lsa, unga mos yopiq traektoriya Γ yopiq chiziqdan iborat, ixtiyoriy t uchun

$$\lim_{k\to+\infty}\varphi(t+k\tau)=\lim_{k\to+\infty}\varphi(t)=\varphi(t)$$

va $p\notin\Gamma$ nuqtaga Γ ning nuqtalari bo'ylab yaqinlashib bo'lmaganligi uchun mos yopiq traektoriyaning ω-limit to'plami shu Γ dan iborat, $\Omega(\Gamma)=\Gamma$. Agar $x=\varphi(t)$ yechimning $\tilde{\Gamma}$ traektoriyasi $t\to+\infty$ da Γ yopiq traektoriyaga yaqinlashsa (spiralsimon buralsa), u holda $\tilde{\Gamma}$ traektoriyaning ω-limit to'plami Γ yopiq traektoriyadan iborat bo'ladi, $\Omega(\tilde{\Gamma})=\Gamma$.

Yopiq bo'lmagan traektoriyaning ω-limit to'plamini tekshirish katta ahamiyatga ega, chunki bu to'plam traektoriyaning $t\to+\infty$ dagi tabiatini aniqlaydi.

Jumla. $\Gamma^+=\{x\in\mathbb{R}^n \mid x=\varphi(t),\ t\in[t_0,+\infty)\}$ *(yarim) traektoriyaning ω-limit to'plami bo'sh to'plamdan iborat bo'lishi uchun ushbu* $\|\varphi(t)\|\xrightarrow[t\to+\infty]{}+\infty$ *shartning bajarilishi yetarli va zarurdir.*

🗝 **Yetarliligi.** Agar $\|\varphi(t)\|\xrightarrow[t\to+\infty]{}+\infty$ bo'lsa, ravshanki, $\Gamma^+=\{x\in\mathbb{R}^n \mid x=\varphi(t),\ t\in[t_0,+\infty)\}=\varnothing$.

Zarurligi. Agar $\|\varphi(t)\|\xrightarrow[t\to+\infty]{}+\infty$ shart bajarilmasa, u holda shunday B_ρ shar va $t_1,t_2,\dots,t_k,\dots\to+\infty$ ketma-ketlik topiladiki, ular uchun $\varphi(t_k)\in B_\rho$ $(k\in\mathbb{N})$ bo'ladi. Bu $\varphi(t_k)$ $(k\in\mathbb{N})$ chegaralangan ketma-ketlikdan yaqinlashuvchi qismiy ketma-ketlik ajratib, Γ^+ ning ω-limit nuqtasini topamiz, y'ani $\Gamma^+\neq\varnothing$. ♤

Teorema. *Aytaylik,* $\Gamma^+=\{x\in\mathbb{R}^n \mid x=\varphi(t),\ t\in[t_0,+\infty)\}$ *yarim traektoriya chegaralangan va o'zining biror atrofi bilan birgalikda G sohada joylashgan bo'lsin. U holda $\Omega(\Gamma^+)$ to'plam bo'shmas, chegaralangan, yopiq va butun traektoriyalardan tashkil topgan bo'ladi.*

🗝 $\Omega(\Gamma^+)$ to'plamning bo'shmasligi hozirgina isbotlangan jumladan kelib chiqadi.

$\Omega(\Gamma^+)$ chegaralangan, chunki Γ^+ chegaralangan.

$\Omega(\Gamma^+)$ to'plamning yopiqligini ko'rsataylik. p nuqta $\Omega(\Gamma^+)$ uchun limit nuqta bo'lsin. Demak, p ga intiluvchi $p_k \in \Omega(\Gamma^+)$ $(k=1,2,\dots)$ nuqtalar ketma-ketligi mavjud, $p_k \xrightarrow[k\to+\infty]{} p$. Ixtiyoriy $\varepsilon = 2^{-k}$ sonni qaraylik. $\varepsilon/2$ ga ko'ra shunday p_k topamizki, uning uchun $\| p_k - p \| < \varepsilon/2$ bo'ladi. $p_k \in \Omega(\Gamma^+)$ nuqtaga ko'ra shunday $t_{k,j}$ $(j=1,2,\dots, t_{k,j} \xrightarrow[j\to+\infty]{} +\infty)$ ketma-ketlikni topamizki, uning uchun $\varphi(t_{k,j}) \xrightarrow[j\to+\infty]{} p_k$ bo'ladi. Limitning ta'rifiga ko'ra esa shunday $j = j(k)$ nomer mavjudki, uning uchun $t_{k,j(k)} > k$ va $\| \varphi(t_{k,j(k)}) - p_k \| < \varepsilon/2$ bo'ladi. Endi ravshanki,

$$\| \varphi(t_{k,j(k)}) - p \| \le \| \varphi(t_{k,j(k)}) - p_k \| + \| p_k - p \| < \\ < \varepsilon/2 + \varepsilon/2 = \\ = \varepsilon = 2^{-k}$$

va $t_{k,j(k)} \xrightarrow[k\to+\infty]{} +\infty$, demak, ya'ni $p \in \Omega(\Gamma^+)$.

$\Omega(\Gamma^+)$ butun traektoriyalardan tuzilgan, ya'ni ixtiyoriy $b \in \Omega(\Gamma^+)$ nuqtadan o'tgan Γ_b traektoriya to'laligicha $\Omega(\Gamma^+)$ to'plamda yotadi. Shu tasdiqni isbotlaymiz. $t=0$ da $b \in \Omega(\Gamma^+)$ nuqta orqali o'tuvchi yechimni $x = \psi(t)$ $(\psi(0) = b)$ bilan belgilaylik. b nuqta Γ^+ yarimtraektoriyaning biror F $(F \subset G)$ yopiq ε-atrofida yotadi $(\varepsilon > 0)$. $D = \{(t,x) \mid t \in \mathbb{R}, x \in F\}$ deylik. Tushunarliki, $(0,b) \in D$. $x = \psi(t)$ yechim yo barcha $t \in (-\infty,+\infty)$ larda aniqlangan, yoki u biror $t = \tilde{t}$ da F ning chegarasiga chiqadi, $\tilde{p} = \psi(\tilde{t}) \in \partial F$. Demak,

$$\operatorname{dist}(\tilde{p}, \Gamma^+) = \varepsilon. \qquad (V.1.3)$$

$b \in \Omega(\Gamma^+)$ nuqtaga ko'ra shunday $t_1, t_2, \dots, t_k, \dots \to +\infty$ ketma-ketlik topamizki, uning uchun $\varphi(t_k) \xrightarrow[k\to+\infty]{} b$ bo'ladi.

$\chi_k(t) \equiv \varphi(t + t_k)$ $(k = 1,2,\dots)$ funksiyalarni qaraylik. $x = \varphi(t)$ bilan birgalikda ular ham yechim va $\chi_k(0) \xrightarrow[k\to+\infty]{} b = \psi(0)$. Yechimning boshlang'ich qiymatlarga uzluksiz bog'liqligi xossasiga ko'ra (ssilka)

$$\chi_k(\tilde{t}) \xrightarrow[k\to+\infty]{} \psi(\tilde{t}) = \tilde{p}. \qquad (V.1.4)$$

Lekin yetarlicha katta t_k lar uchun $\chi_k(\tilde{t}) \equiv \varphi(\tilde{t} + t_k) \in \Gamma^+$ va (V.1.4) munosabat (V.1.3) ga zid. Demak, $x = \psi(t)$ yechim F ning chegarasiga chiqa olmaydi va u $(-\infty,+\infty)$ intervalgacha davom etadi. Yana yuqoridagiga o'xshash ixtiyoriy $t \in (-\infty,+\infty)$ uchun $\varphi(t + t_k) = \chi_k(t) \xrightarrow[k\to+\infty]{} \psi(t)$. Bundan barcha t lar uchun $\psi(t) \in \Omega(\Gamma^+)$ ekanligi ravshan. ♘

Eslatma. Teoremaning shartlarida Γ^+ yarim traektoriyaning ω-limit to'plami bog'lanishli to'plamdan iborat bo'ladi. Buning isboti, masalan, [17] da keltirilgan.

$\mathbb{R}^n, n \ge 3$, fazoda ω-limit to'plamlar tuzilishi kam o'rganilgan. Tekislikda traektoriyalar xossalari bilan V.2- V.4 bandlarda tanishamiz. ♘

7⁰. Tekislikda avtonom sistemalar. Bu bandda ikki o'lchamli ushbu

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \quad (x,y) \in G \subset \mathbb{R}^2 \qquad (V.1.5)$$

avtonom sistemani qaraymiz; bu yerda $\{f,g\} \subset C^1(G;\mathbb{R})$. Sistemaning ($\big(f(x,y), g(x,y)\big)$ vektor maydonning) maxsus nuqtalari to'plamini G_0,

$G_0 = \{(x,y) \in G \mid f^2(x,y) + g^2(x,y) = 0\}$, bilan belgilaylik. U holda $\tilde{G} = G \setminus G_0$ ochiq to'plamda (aniqrog'i, uning bog'lanishli komponentalarida) ushbu

$$g(x,y)dx = f(x,y)dy, \ (x,y) \in \tilde{G}, \qquad (V.1.6)$$

differensial tenglamani hosil qilamiz.

Jumla. *Faraz qilaylik,* $\{f,g\}\subset C^1(G;\mathbb{R})$ *bo'lsin. U holda* (V.1.5) *sistemaning muvozanat holatidan farqli har qanday traektoriyasi* (V.1.6) *tenglamaning integral chizig'idan iborat va aksincha, ya'ni* (V.1.6) *tenglamaning ixtiyoriy integral chizig'i* (V.1.5) *sistemaning muvozanat holatidan farqli traektoriyasidan iborat bo'ladi.*

(V.1.5) sistemaning muvozanat holatidan farqli

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad \text{(V.1.7)}$$

traektoriyasini qaraylik. Bu traektoriya muvozanat nuqtadan farqli bo'lganligi uchun u G_0 bilan umumiy nuqtaga ega emas. Traektoriya $t=t_0$ paytda $(x_0,y_0)\in\tilde{G}$ nuqtadan o'tgan bo'lsin. Aniqlik uchun $f(x_0,y_0)\neq 0$ deylik. U holda $x'(t_0)=f(x(t_0),y(t_0))=f(x_0,y_0)\neq 0$ va, demak, (V.1.7) sistemadan $(x_0,y_0)\in\tilde{G}$ nuqtaning yetarlicha kichik atrofida y ni x ning $y=y(t(x))$ funksiyasi sifatida ifodalash mumkin hamda

$$\frac{dy}{dx}=\frac{dy(t(x))}{dx}=\frac{y'(t)}{x'(t)}=\frac{g(x,y(t))}{f(x,y(t))}\quad (t=t(x)).$$

Bu tenglik $y=y(t(x))$ funksiyaning (V.1.6) tenglama yechimi ekanligini anglatadi. Shunday qilib, (V.1.7) fazaviy traektoriya o'zining ixtiyory $(x_0,y_0)\in\tilde{G}$ nuqtasi atrofida (V.1.6) tenglamaning integral chizig'idan iborat.

Endi (V.1.6) tenglamaning ixtiyoriy $\gamma\subset\tilde{G}$ integral chizig'ini qaraylik. Biz uning (V.1.5) avtonom sistemaning muvozanat nuqtadan farqli bo'lgan traektoriyasi ekanligini ko'rsatishimiz kerak. γ ning nuqtadan farqli ekanligi ravshan, chunki u o'zining ixtiyoriy $(x_0,y_0)\in\gamma\subset\tilde{G}$ nuqtasi atrofida $f(x_0,y_0)\neq 0$ bo'lganda $\dfrac{dy}{dx}=f(x,y)$ tenglama, ya'ni $y=y(x)$ $(y(x_0)=y_0)$

yoki $g(x_0,y_0)\neq 0$ bo'lganda $\dfrac{dx}{dy}=g(x,y)$ tenglama, ya'ni $x=x(y)$ $(x(y_0)=x_0)$ ko'rinishda ifodalanadi. Aniqlik uchun $f(x_0,y_0)\neq 0$ deylik. U holda $t_0\in\mathbb{R}$ nuqtaning kichik atrofida $x=x(t)$ funksiyani ushbu $\dfrac{dx}{dt}=f(x,y(x)),\ x(t_0)=x_0$, masalaning yechimi sifatida aniqlab, $y=y(x(t))$ funksiya uchun

$$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}=\frac{g(x,y)}{f(x,y)}f(x,y)=g(x,y),\ y(x(t_0))=y_0,$$

munosabatlarni hosil qilamiz. Demak, qurilgan $x=x(t)$ va $y=y(x(t))$ funksiyalar (V.1.5) sistemaning yechimi, ya'ni (V.1.6) tenglamaning ixtiyoriy $\gamma\subset\tilde{G}$ integral chizig'i o'zining ixtiyoriy (x_0,y_0) nuqtasining yetarlicha kichik atrofida (V.1.5) sistemaning shu nuqta orqali o'tuvchi traektoriyasi bilan ustma-ust tushadi. γ integral chiziqni shunaqa atroflar bilan qoplab va yechimning yagonalik xossasidan foydalanib, γ integral chiziqning to'laligicha (V.1.5) sistema traektoriyasidan iborat ekanligiga ishonch hosil qilamiz.

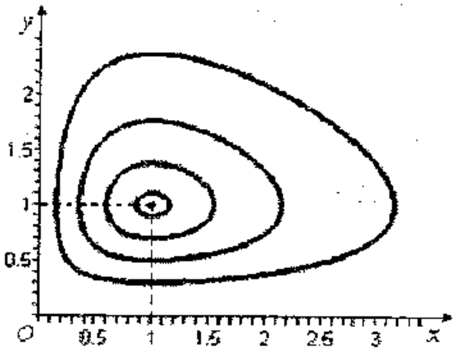
Misol 1. Ushbu

$$\begin{cases} x'=(1-y)x \\ y'=\alpha(x-1)y \end{cases}\quad (\alpha>0-\text{o'zgarmas},\ x>0,y>0)$$

Volterra-Lotka sistemasini qaraylik. Bu sistemadan $\dfrac{dy}{dx}=\dfrac{\alpha(x-1)y}{(1-y)x}$ tenglamani yozib, va uni yechib, traektoriyalarning

$$\frac{yx^{\alpha}}{e^{y}e^{\alpha x}}=c,\ c-\text{o'zgarmas son},$$

tenglama bilan oshkormas ko'rinishda berilishini topamiz. Ravshanki, $x=1,y=1$ o'zgarmas yechim; u (1;1) muvozanat nuqtani aniqlaydi. Traektoriyalar ($\alpha=1/2$ holida) V.2-rasmda keltirilgan.



V.2- rasm.

Misol 2. Tekislikda ushbu

$$\begin{cases} x' = -y + x(1-x^2-y^2) \\ y' = x + y(1-x^2-y^2) \end{cases} \qquad \text{(V.1.8)}$$

avtonom sistemani qaraylik. Bu sistemani tekshirish uchun fazaviy tekislikda $(r,\varphi), r \geq 0$, qutb koordinatalarini kiritamiz: $x = r\cos\varphi, y = r\sin\varphi$. (V.1.8) sistemada bu almashtirishlarni bajaramiz:

$$\begin{cases} r'\cos\varphi - r\varphi'\sin\varphi = -r\sin\varphi + r(1-r^2)\cos\varphi, \\ r'\sin\varphi + r\varphi'\cos\varphi = r\cos\varphi + r(1-r^2)\sin\varphi. \end{cases}$$

Bundan

$$\begin{cases} r' = r(1-r^2), \\ \varphi' = 1. \end{cases}$$

Bu sistemaning tenglamalari ajralgan. Ularni alohoda-alohida yechib, topamiz:

$$r = 0, \quad \begin{cases} r = \dfrac{1}{\sqrt{1+c_1e^{-2t}}}, \\ \varphi = t + c_2. \end{cases}$$

Demak, (V.1.8) sistemaning traektoriyalari:

$x = y = 0$ – muvozanat nuqtasi hamda

$$\left\{ x = \frac{\cos(t+c_2)}{\sqrt{1+c_1e^{-2t}}},\ y = \frac{\sin(t+c_2)}{\sqrt{1+c_1e^{-2t}}};\ t \in \mathbb{R} \right\}$$ – spirallar ($c_1 \neq 0$) va

birlik aylana ($c_1 = 0$).

Bundan ravshanki, bitta yopiq traektoriya $x^2 + y^2 = 1$ ($c_1 = 0$ da hosil bo'luvchi) mavjud bo'lib, muvozanat nuqtasidan boshqa barcha traektoriyalar vaqt o'tishi bilan shu davraga intiladi.

V.2. Tekislikda chiziqli avtonom sistemalar fazaviy portreti

Ushbu

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \quad \text{yoki} \quad \begin{pmatrix} x \\ y \end{pmatrix}' = A\begin{pmatrix} x \\ y \end{pmatrix},\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{(V.2.1)}$$

sistemaning yechimlari tabiatini (x,y) fazalar tekisligida o'rganamiz. Bunda a,b,c,d koeffitsientlar haqiqiy sonlar va bu sistema yagona maxsus nuqtaga ega, ya'ni $\det A \neq 0$ deb faraz qilinadi. Demak, (V.2.1) sistema $x = x(t) = 0, y = y(t) = 0$ bir dona muvozanat nuqtasiga ega.

A matritsaning xos (xarakteristik) sonlari

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \text{yoki} \quad \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

tenglamadan topiladi. Xarakteristik sonlarni λ_1 va λ_2 bilan belgilaylik. $\det A = ad - bc \neq 0$ bo'lgani uchun $\lambda_1 \neq 0$ va $\lambda_2 \neq 0$.

Dastlab xarakteristik sonlar kompleks sonlardan iborat bo'lgan holni qaraylik. A matritsa haqiqiy bo'lgani uchun uning λ_1 va λ_2 xos sonlari o'zaro qo'shma bo'ladi: $\lambda_{1,2} = \alpha \pm i\beta, \{\alpha,\beta\} \subset \mathrm{R}, \beta \neq 0$; aniqlik uchun $\beta > 0$ deb hisoblaymiz. Mos xos vektorlar $\boldsymbol{a} \mp i\boldsymbol{b}$ ham o'zaro qo'shma ($\boldsymbol{a}, \boldsymbol{b}$ - haqiqiy vektorlar). U holda

$$A(\boldsymbol{a}-i\boldsymbol{b})=(\alpha+i\beta)(\boldsymbol{a}-i\boldsymbol{b}) \text{ yoki } \begin{cases} A\boldsymbol{a}=\alpha\boldsymbol{a}+\beta\boldsymbol{b} \\ A\boldsymbol{b}=-\beta\boldsymbol{a}+\alpha\boldsymbol{b} \end{cases}.$$

Demak, agar $\boldsymbol{a}, \boldsymbol{b}$ vektorlar koordinatalarini ustunlar bo'ylab yozib, $T=[\boldsymbol{a}, \boldsymbol{b}]$ matritsani tuzsak, u holda

$$AT=A[\boldsymbol{a},\boldsymbol{b}]=[A\boldsymbol{a},A\boldsymbol{b}]=[\alpha\boldsymbol{a}+\beta\boldsymbol{b},-\beta\boldsymbol{a}+\alpha\boldsymbol{b}]=$$

$$=[\boldsymbol{a},\boldsymbol{b}]\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}=T\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

bo'ladi. Bundan T teskarilanuvchi bo'lgani uchun

$$T^{-1}AT=\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \quad (V.2.2)$$

tenglik kelib chiqadi. Endi (V.2.1) sistemada

$$\begin{pmatrix} u \\ v \end{pmatrix}=T^{-1}\begin{pmatrix} x \\ y \end{pmatrix} \quad \left(\text{ya'ni } \begin{pmatrix} x \\ y \end{pmatrix}=T\begin{pmatrix} u \\ v \end{pmatrix}\right) \quad (V.2.3)$$

chiziqli almashtirishni bajarib,

$$\begin{pmatrix} u \\ v \end{pmatrix}'=\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \text{ yoki } \begin{cases} u'=\alpha u-\beta v \\ v'=\beta u+\alpha v \end{cases} \quad (V.2.4)$$

sistemani hosil qilamiz. (u,v) fazalar tekisligida (r,φ) qutb koordinatalarini ma'lum

$$\begin{cases} u=r\cos\varphi \\ v=r\sin\varphi \end{cases}$$

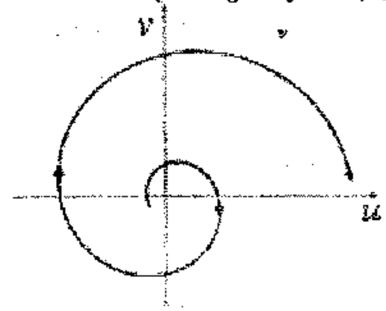
formulalar bilan kiritib, (V.2.4) sistemani ushbu

$$\begin{cases} r'=\alpha r \\ \varphi'=\beta \end{cases}$$

sodda ko'rinishga keltiramiz. Bu sistema osongina yechiladi:

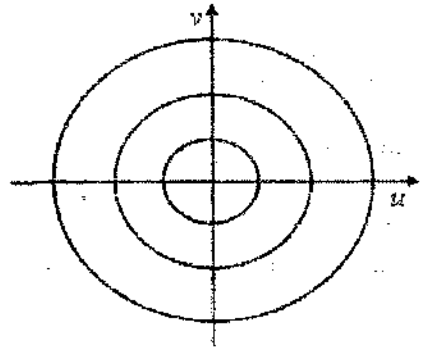
$$\begin{cases} r=r_0e^{\alpha t} \\ \varphi=\beta t+\varphi_0 \end{cases} \quad (r_0 \ge 0, \varphi_0 - \text{ixtiyoriy o'zgarmaslar}). \quad (V.2.5)$$

Vaqt o'tishi bilan harakatlanuvchi nuqtaning φ qutb koordinatasi ortadi ($\beta>0$). Agar $\alpha=\mathrm{Re}\lambda_1=\mathrm{Re}\lambda_2 \ne 0$ bo'lsa, (V.2.5) yechim ***spiral*** deb ataluvchi traektoriyalarni aniqlaydi. Bu holda (0,0) maxsus nuqta ***fokus*** deb ataladi. $\alpha<0$ bo'lganda bu spiralning radiusi vaqt o'tishi bilan kamayadi (***turg'un fokus***), $\alpha>0$ bo'lganda esa – ortadi (***noturg'un fokus***) (V.3- rasm).



V.3-rasm. Turg'un va noturg'un fokuslar.

Agar $\alpha=0$, ya'ni xos sonlar sof mavhum bo'lsa, traektoriyalar markazlari O(0,0) nuqtada joylashgan aylanalar oilasidan ($r=r_0, r_0$ – const) iborat bo'ladi (V.4- rasm).



V.4-rasm. Markaz

Oxy tekisligida esa traektoriyalar $O(0,0)$ markazli ellipslar kabi tasvirlanadi. Bu holda $(0,0)$ maxsus nuqta ***markaz*** deb yuritiladi.

Endi xarakteristik sonlar haqiqiy bo'lgan holni qaraymiz. Bunda $\lambda_1 \neq \lambda_2$ yoki $\lambda_1 = \lambda_2$ bo'ladi. Xos sonlar turli bo'lsin. Ma'lumki, bu turli xos sonlarga mos keluvchi $\boldsymbol{a}_1$ va $\boldsymbol{a}_2$ xos vektorlar ($A\boldsymbol{a}_1 = \lambda_1\boldsymbol{a}_1, A\boldsymbol{a}_2 = \lambda_2\boldsymbol{a}_2$) chiziqli erkli. Bu vektorlar koordinatalarini ustunlar bo'ylab yozib, $S=[\boldsymbol{a}_1,\boldsymbol{a}_2]$ matritsani tuzaylik. $\boldsymbol{a}_1$ va $\boldsymbol{a}_2$ vektorlar chiziqli erkli bo'lgani uchun $\det S \neq 0$, ya'ni S matritsa teskarilanuvchi. Ravshanki,

$$AS = A[\boldsymbol{a}_1,\boldsymbol{a}_2]=[A\boldsymbol{a}_1,A\boldsymbol{a}_2]=[\lambda_1\boldsymbol{a}_1,\lambda_2\boldsymbol{a}_2]=[\boldsymbol{a}_1,\boldsymbol{a}_2]\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}=S\Lambda,\ \Lambda=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}.$$

Demak,

$$A = S^{-1}\Lambda S. \qquad (V.2.6)$$

Yangi u,v noma'lumlarni

$$\begin{pmatrix}u\\ v\end{pmatrix}=S^{-1}\begin{pmatrix}x\\ y\end{pmatrix}\quad\left(\begin{pmatrix}x\\ y\end{pmatrix}=S\begin{pmatrix}u\\ v\end{pmatrix}\right) \qquad (V.2.7)$$

formula yordamida kiritaylik. Bu chiziqli almashtirish natijasida (V.2.1) sistema

$$\begin{pmatrix}u\\ v\end{pmatrix}'=\Lambda\begin{pmatrix}u\\ v\end{pmatrix}\ \text{yoki}\ \begin{cases}u'=\lambda_1 u\\ v'=\lambda_2 v\end{cases} \qquad (V.2.8)$$

ko'rinishga o'tadi. Oxirgi (V.2.8) sistemaning yechimlari quyidagicha:

$$u=c_1e^{\lambda_1 t}, v=c_2e^{\lambda_2 t}\ (c_1,c_2 - \text{ixtiyoriy o'zgarmaslar}). \qquad (V.2.9)$$

Bu formulalar fazaviy traektoriyalarning parametrik tenglamasini ifodalaydi. t ni yo'qotib, traektoriyalarni ushbu

$v=c_2\left(\frac{u}{c_1}\right)^{\frac{\lambda_2}{\lambda_1}}$ $(c_1 \neq 0)$ va $u=0$ $(c_1=0)$, bunda $v>0$ yoki $v<0$,

o'shkor ko'rinishda ham yozish mumkin.

(V.2.1) sistemaning umumiy yechimi

$$\begin{pmatrix}x\\ y\end{pmatrix}=S\begin{pmatrix}u\\ v\end{pmatrix}=[\boldsymbol{a}_1,\boldsymbol{a}_2]\begin{pmatrix}c_1e^{\lambda_1 t}\\ c_2e^{\lambda_2 t}\end{pmatrix}=c_1e^{\lambda_1 t}\boldsymbol{a}_1+c_2e^{\lambda_2 t}\boldsymbol{a}_2$$

formula bilan beriladi.

Dastlab λ_1,λ_2 xos sonlar bir xil ishorali, ya'ni $\lambda_1\cdot\lambda_2>0$ holni qaraylik. Bu holda $(0,0)$ maxsus nuqta ***tugun*** deb ataladi. $\lambda_2/\lambda_1>1$ $(|\lambda_2|>|\lambda_1|)$ bo'lganda $v=c_2\left(\frac{u}{c_1}\right)^{\frac{\lambda_2}{\lambda_1}}$ traektoriyalar Ou o'qiga urinadi, $\lambda_2/\lambda_1<1$ $(|\lambda_2|<|\lambda_1|)$ bo'lganda esa ular Ov o'qiga urinadi. (V.2.7) chiziqli almashtirishda Ou o'qi $\boldsymbol{a}_1$ xos vektor, Ov o'qi esa $\boldsymbol{a}_2$ xos vektor orqali o'tgan o'qqa almashinadi:

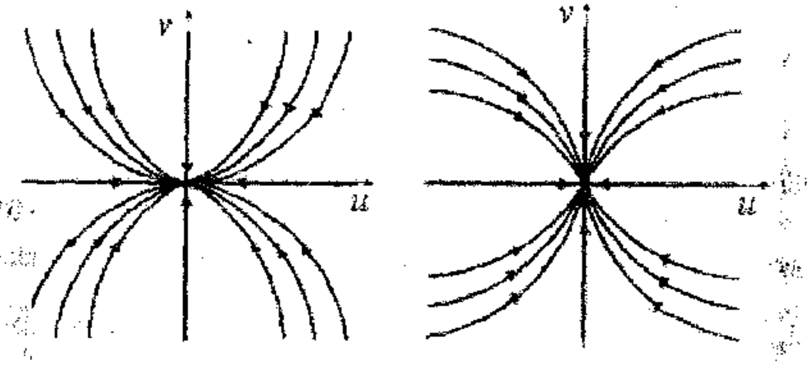
$$\begin{pmatrix}x\\ y\end{pmatrix}=S\begin{pmatrix}u\\ 0\end{pmatrix}=[\boldsymbol{a}_1,\boldsymbol{a}_2]\begin{pmatrix}u\\ 0\end{pmatrix}=u\boldsymbol{a}_1,\quad \begin{pmatrix}x\\ y\end{pmatrix}=[\boldsymbol{a}_1,\boldsymbol{a}_2]\begin{pmatrix}0\\ v\end{pmatrix}=v\boldsymbol{a}_2.$$

Demak, Oxy tekisligida traektoriyalar moduli bo'yicha kichik xos songa mos kelgan xos vektorga urinadi.

Agar xos sonlarning ikkalasi ham manfiy bo'lsa ($\lambda_1<0,\lambda_2<0,\lambda_1\neq\lambda_2$), (V.2.9) yechimlarning (u,v) fazalar tekisligidagi tasviri V.5- rasmda ko'rsatilgan tabiatli bo'ladi. Vaqt o'tishi bilan fazaviy nuqta koordinatalar boshiga (muvozanat nuqtaga) intiladi: $\lim\limits_{t\to+\infty}u=\lim\limits_{t\to+\infty}c_1e^{\lambda_1 t}=0$, $\lim\limits_{t\to+\infty}v=\lim\limits_{t\to+\infty}c_2e^{\lambda_2 t}=0$.

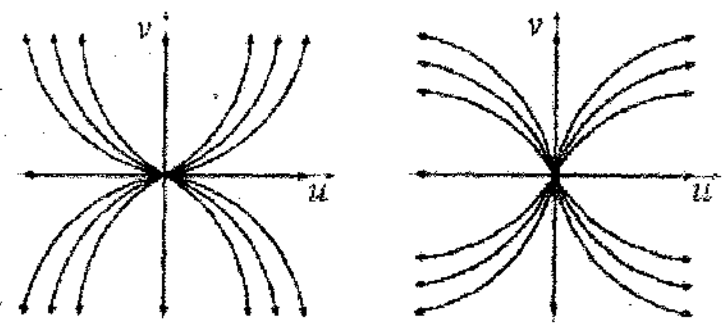
Bu $(0,0)$ maxsus nuqta ***turg'un tugun*** deb ataladi.

Turg'un tugunga kiruvchi to'rt dona koordinata yarim o'qlaridan iborat bo'lgan traektoriyalar ham mavjud.



V.5-rasm. Turg'un tugun.

Agar ikkala xos son ham musbat bo'lsa ($\lambda_1 > 0, \lambda_2 > 0, \lambda_1 \neq \lambda_2$), (V.2.9) yechimlarning fazaviy tasviri V.6- rasmda keltirilgan. Bu holda $(0,0)$ maxsus nuqta ***noturg'un tugun*** deb yuritiladi. Noturg'un tugundan chiquvchi to'rt dona koordinata yarim o'qlaridan iborat bo'lgan traektoriyalar mavjud.

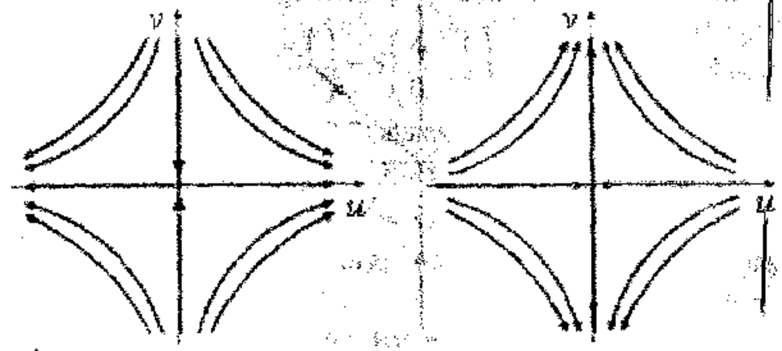


Noturg'un tugun, $\lambda_2 > \lambda_1 > 0$ Noturg'un tugun, $\lambda_1 > \lambda_2 > 0$

V.6- rasm. Noturg'un tugun.

Agar xos sonlar turli ishorali bo'lsa ($\lambda_1\lambda_2 < 0, \lambda_1 \neq \lambda_2$), faza tasviri, masalan, V.7-rasmdagidek bo'ladi. Bu holda

$(0,0)$ maxsus nuqta ***egar*** deb ataladi.

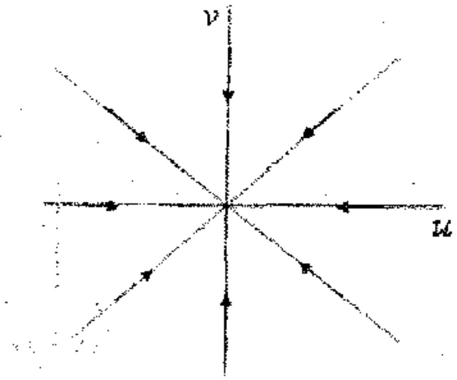


Egar, $\lambda_2 < 0 < \lambda_1$ Egar, $\lambda_1 < 0 < \lambda_2$

V.7-rasm. Egar.

Egardan chiquvchi yoki unga kiruvchi hamda traektoriyalar oilasini to'rt qismga ajratuvchi to'rt dona traektoriya (koordinata yarim o'qlari) ***separatrisalar*** deb yuritiladi.

Endi A matritsa karrali xos sonlarga ega bo'lgan $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ holni qaraymiz. Agar A matritsaning bu xos soniga ikki dona chiziqli erkli xos vektorlar mos kelsa, ya'ni A matritsa diagonallashtiriluvchi bo'lsa, bir xil ishorali turli xos sonlar ($\lambda_1\lambda_2 > 0$) holidagi fikr yuritishlar bu holda ham o'z kuchini saqlaydi. Bu holda maxsus nuqta ***dikritik tugun*** deyiladi (V.8-rasm). Traektoriyalar maxsus nuqtaga kiruvchi (***turg'un dikritik tugun***) yoki undan chiquvchi (***noturg'un dikritik tugun***) nurlardan iborat bo'ladi.



V.8-rasm. Dikritik tugun $\lambda_1 = \lambda_2 < 0$

($\lambda_1 = \lambda_2 > 0$ holda yo'nalishlar teskari)

Endi A matritsa diagonallashtiriluvchi bo'lmasin deyli... Bu holda shunday $b \neq 0$ vektor topiladiki, uning uchun $(A-\lambda E)b \neq 0$ va $(A-\lambda E)^2 b = 0$ bo'ladi; bunda $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ – birlik matritsa. Haqiqatan ham, agar bunday vektor topilmaganda edi, u holda har qanday $b \neq 0$ vektor uchun $(A-\lambda E)b = 0$ yoki $(A-\lambda E)^2 b \neq 0$ bo'lardi. Bundan esa b sifatida A matritsaning xos vektorini tanlab, ziddiyat hosil qilardik. Shunday qilib, $(A-\lambda E)b \neq 0$ va $(A-\lambda E)^2 b = 0$ shartlarni qanoatlantiruvchi $b \neq 0$ vektor bor. Shu vektorga ko'ra $(A-\lambda E)b = a$ vektorni tuzaylik. U holda $(A-\lambda E)a = 0$, ya'ni $Aa = \lambda a$, $a \neq 0$, va $Ab = a + \lambda b$ bo'ladi. Topilgan $a \neq 0$ va $b \neq 0$ vektorlar chiziqli erkli, chunki aks holda $a = \mu b$ ($\mu \neq 0$) bo'lib, $0 = \frac{1}{\mu}(A-\lambda E)a = \frac{1}{\mu}(A-\lambda E)\mu b = (A-\lambda E)b$ ziddiyat hosil bo'lardi. Demak, ushbu $S = [a, b]$ matritsa teskarilanuvchi. Quyidagilarga egamiz:

$$AS = A[a, b] = [Aa, Ab] = [\lambda a, a + \lambda b] =$$

$$= [a, b]\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = S\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad A = S^{-1}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}S.$$

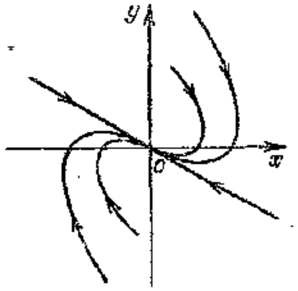
Endi (V.2.1) sistemada (V.2.7) almashtirishni bajarib, uni

$$\begin{cases} u' = \lambda u + v \\ v' = \lambda v \end{cases}$$

ko'rinishga keltiramiz. Oxirgi sistemaning yechimi osongina topiladi:

$$\begin{cases} u = (c_1 t + c_2)e^{\lambda t} \\ v = c_1 e^{\lambda t} \end{cases}$$

Qaralayotgan ($\lambda_1 = \lambda_2$) holdagi maxsus nuqta ***aynigan tugun*** deb ataladi. Ouv tekisligida traektoriyalar Ou o'qiga, Oxy tekisligida esa ular a xos vektorga urinadi. Fazaviy traektoriyalar V.9-rasmda ko'rsatilgan.



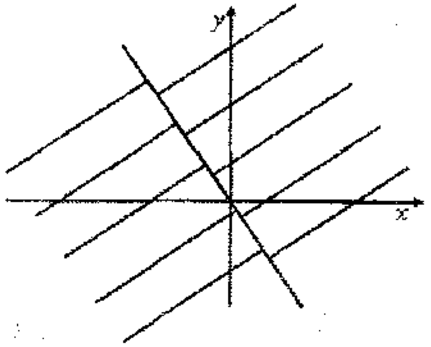
V.9-rasm. Aynigan tugun $\lambda_1 = \lambda_2 < 0$

($\lambda_1 = \lambda_2 > 0$ holda yo'nalishlar teskari)

$\lambda < 0$ bo'lganda (0,0) maxsus nuqta turg'un aynigan tugundan (rasm), $\lambda > 0$ bo'lganda esa u noturg'un aynigan tugundan iborat.

Oxy faza(lar) tekisligidagi traektoriyalar mazmunan Ouv faza tekisligidagi traektoriyalarga o'xshash bo'ladi, chunki (x, y) va (u, v) o'zgaruvchilar chiziqli almashtirish bilan o'zaro bog'langan. Bunda Ouv tekisligida yarim to'g'ri chiziqdan iborat bo'lgan traektoriyalar Oxy tekisligada ham yarim to'g'ri chiziqdan iborat bo'lgan traektoriyalar sifatida tasvirlanadi. Traektoriyalar bo'ylab harakat yo'nalishi berilgan sistemaga bog'liq. Shunday qilib, Oxy tekisligidagi traektoriyalar joylashishi, ya'ni maxsus nuqtaning tipi A matritsaning λ_1, λ_2 xos sonlari bilan aniqlanadi.

Biz yuqorida (V.2.1) sistemada $\det A \neq 0$ deb uni tekshirdik. Agar $\det A = 0$ bo'lsa, u holda (V.2.1) sistemaning muvozanat nuqtalari cheksiz ko'p bo'ladi. Masalan, $ad - bc = 0$, lekin $|a| + |b| \neq 0$ bo'lsa, muvozanat nuqtalari butun bir to'g'ri chiziq $ax + by = 0$ni tashkil etadi, traektoriyalar esa ushbu $\frac{dy}{dx} = \text{const}$ ko'rinishdagi osongina yechiladigan tenglamaning yechimlaridan iborat bo'ladi. Traektoriyalar uchi $ax + by = 0$ to'g'ri chiziqda joylashgan o'zaro parallel nurlardan iborat bo'ladi (V.10- rasm).



V.10-rasm.

Bu yerda shuni e'tirof etaylikki, ushbu $(x, y) \to (\mu x, \mu y)$ yoki $(u, v) \to (\mu u, \mu v)$ gomotetik almashtirishda traektoriya yana traektoriyaga akslanadi. Demak, traektoriyalar o'zaro gomotetik chiziqlardan iborat bo'ladi. Bu izoh ba'zan fazaviy portretni chizishda qo'l keladi. Bundan, masalan, (V.2.1) chiziqli avtonom

sistema yakkalangan (alohida ajralgan) yopiq traektoriyaga ega emasligi kelib chiqadi.

Misol 1. Ushbu

$$\begin{cases} x' = 2x - y \\ y' = x + 2y \end{cases}$$

sistemaning maxsus nuqtasini tekshiring va maxsus nuqta atrofida traektoriyalarini chizing.

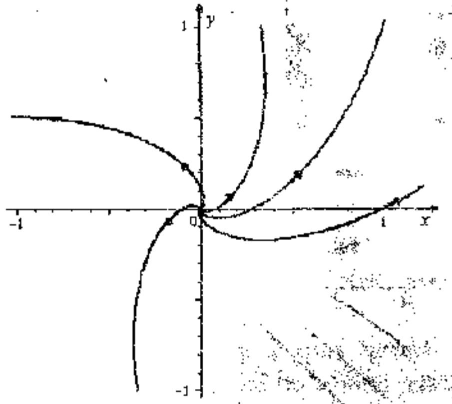
Xarakteristik sonlarni topamiz

$$\begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = 0, \quad \lambda_{1,2} = 2 \pm i .$$

$\operatorname{Re}\lambda_1 > 0$ bo'lganligi uchun $(0;0)$ maxsus nuqta noturg'un fokusdan iborat. Traektoriyalarning (spirallarning) buralish yo'nalishini aniqlash uchun, masalan, $(1;0)$ nuqtada tezlik vektorini quramiz:

$$x' = 2, \; y' = 1 .$$

Demak, traektoriyalar bo'ylab harakatlanuvchi nuqta soat mili aylanishiga teskari yo'nalishda harakatlanadi va u $(0;0)$ nuqtadan uzoqlashadi (V.11-rasm).



V.11-rasm.

Misol 2. Ushbu

$$\begin{cases} x' = 2x + 3y \\ y' = x + 4y \end{cases}$$

sistemaning maxsus nuqtasini tekshiring va maxsus nuqta atrofida traektoriyalarini chizing.

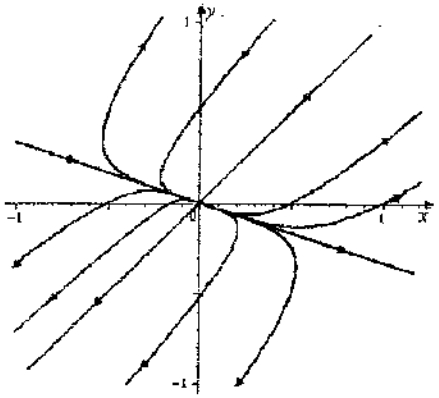
Berilgan sistemaning xarakteristik sonlari

$$\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = 0; \quad \lambda_1 = 1,\ \lambda_2 = 5.$$

$(0;0)$ muvozanat nuqta – tugun. Endi yarim to'g'ri chiziqlardan iborat bo'lgan $x = t,\ y = kt$ $(t > 0$ yoki $t < 0)$ traektoriyalarni aniqlaymiz. Bularni berilgan sistemaga qo'yib

$$\begin{cases} 1 = 2t + 3kt \\ k = t + 4kt \end{cases}; \quad 3k^2 - 2k - 1 = 0\ ; \quad k_1 = 1,\ k_2 = -1/3$$

ekanligini topamiz. Demak, $y = x$, $y = -x/3$, $x > 0$ yoki $x < 0$, yarim to'g'ri chiziqlar izlangan traektoriyalarni ifodalaydi. Traektoriyalar bo'ylab harakat yo'nalishini topish uchun $(x_1; y_1) = (1; 0)$ va $(x_2; y_2) = (-0,5; -0,5)$ nuqtalarda tezlik vektorlarini hisoblaymiz: $(x'; y') = (2;1)$ va $(x'; y') = (-2,5; -2,5)$. Traektoriyalar portreti V.12- rasmda tasvirlangan.



V.12- rasm.

V.3. Tekislikda nochiziqli avtonom sistemalar fazaviy portreti

Endi

$$\begin{cases} x' = f(x,y), \\ y' = g(x,y) \end{cases} \qquad \text{(V.3.1)}$$

nochiziqli avtonom sistemani qaraylik. Bu yerda, soddalik uchun, f va g funksiyalarni ikki marta uzluksiz differensiallanuvchi $(\{f, g\} \subset C^2)$ deb hisoblaymiz. Sistemaning maxsus nuqtalari

$$f(x,y) = 0\ ,\ g(x,y) = 0$$

tenglamalardan topiladi. (x_0, y_0) maxsus nuqtani tekshirish uchun bu nuqta atrofida $f(x,y)$ va $g(x,y)$ funksiyalarni Teylor formulasiga ko'ra $r = \sqrt{(x-x_0)^2 + (y-y_0)^2} \to 0$ bo'lganda

$$f(x,y) = a(x-x_0) + b(y-y_0) + O(r^2),$$

$$a = \frac{\partial f(x_0,y_0)}{\partial x},\ b = \frac{\partial f(x_0,y_0)}{\partial y},$$

$$g(x,y) = c(x-x_0) + d(y-y_0) + O(r^2),$$

$$c = \frac{\partial g(x_0,y_0)}{\partial x},\ d = \frac{\partial g(x_0,y_0)}{\partial y},$$

ko'rinishda tasvirlab, (V.3.1) sistemada $x = x_0 + u$, $y = y_0 + v$ almashtirishni bajaramiz, ya'ni koordinatalar boshini (x_0, y_0) maxsus nuqtaga ko'chiramiz. Natijada

$$\begin{cases} u' = au + bv + O(\rho^2) \\ v' = cu + dv + O(\rho^2) \end{cases}, \quad \rho = \sqrt{u^2 + v^2} \to 0$$

tenglamalarni topamiz. Bu yerdagi yuqori tartibli cheksiz kichik miqdorlarni tashlab yuborishdan hosil bo'luvchi ushbu

$$\begin{cases} u' = au + bv \\ v' = cu + dv \end{cases} \text{ yoki } \begin{pmatrix} u \\ v \end{pmatrix}' = A\begin{pmatrix} u \\ v \end{pmatrix},\ \left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \qquad \text{(V.3.2)}$$

chiziqli avtonom sistema (V.3.1) nochiziqli avtonom sistemaning $(0,0)$ maxsus nuqta atrofida chiziqlilashtirilishi (yoki birinchi yaqinlashishi) deyiladi. (V.3.2) sistemaning $(0,0)$ maxsus nuqtasi, ya'ni (V.3.1) sistemaning (x_0, y_0) maxsus nuqtasi tabiatini quyidagi teorema ochadi.

Teorema. *Agar A matritsaning xos sonlari uchun* $\operatorname{Re}\lambda_1 \neq 0$ *va* $\operatorname{Re}\lambda_2 \neq 0$ *bo'lsa,* (V.3.2) *nochiziqli sistema* $(0,0)$ *maxsus nuqtasining tipi (turi) chiziqlilashtirilgan* (V.3.2) *sistemaning maxsus nuqtasi tipi (turi) bilan bir xil. Bunda traektoriyalarning buralish va maxsus nuqtaga yaqinlashish yoki undan uzoqlashish yo'nalishlari hamda turg'unlik tabiatlari saqlanadi.*

Bu teoremaga qaraganda (kuchliroq) umumiyroq teoremaning isboti [9] da keltirilgan.

Barcha maxsus nuqtalarning tabiatini tekshirib, ba'zi sohalarda tezliklar maydoni yo'nalishlarini aniqlab, sistemaning traektoriyalari manzarasini chizamiz.

Misol 1. Ushbu

$$\begin{cases} x' = 2x + y^2 - 1 \\ y' = 6x - y^2 \end{cases}$$

sistemaning traektoriyalar portretini chizing.

Sistemaning muvozanat (kritik) nuqtalarini topamiz:

$$\begin{cases} 2x + y^2 - 1 = 0 \\ 6x - y^2 = 0 \end{cases}$$

Ular ikkita: (0;-1) va (0;1). Har bir kritik nuqta atrofida berilgan sistemani chiziqlilashtiramiz.

(0;1) nuqta atrofida chiziqlashtirilgan sistema

$$\begin{pmatrix} u \\ v \end{pmatrix}' = A_1 \begin{pmatrix} u \\ v \end{pmatrix}, A_1 = \begin{pmatrix} (2x+y^2-1)'_x & (2x+y^2-1)'_y \\ (6x-y^2)'_x & (6x-y^2)'_y \end{pmatrix}\Bigg|_{(0;-1)} = \begin{pmatrix} 2 & -2 \\ 6 & 2 \end{pmatrix},$$

(0;-1) nuqta atrofida chiziqlashtirilgan sistema

$$\begin{pmatrix} u \\ v \end{pmatrix}' = A_2 \begin{pmatrix} u \\ v \end{pmatrix}, A_2 = \begin{pmatrix} (2x+y^2-1)'_x & (2x+y^2-1)'_y \\ (6x-y^2)'_x & (6x-y^2)'_y \end{pmatrix}\Bigg|_{(0;1)} = \begin{pmatrix} 2 & 2 \\ 6 & -2 \end{pmatrix}. A_1$$

matritsaning xarakteristik sonlari kompleks $\lambda_1 = 2 + i2\sqrt{5}$, $\lambda_2 = 2 - i2\sqrt{5}$ va $\operatorname{Re}(\lambda_1) = \operatorname{Re}(\lambda_2) = 2 > 0$. Demak, (0;-1) muvozanat nuqta – noturg'un fokus.

A_2 matritsaning xarakteristik sonlari turli ishorali: $\lambda_1 = -4, \lambda_2 = 4$. Demak, (0;1) muvozanat nuqta – egar. Egar orqali o'tgan to'rtta traektoriya yo'nalishini aniqlaymiz. Buning uchun

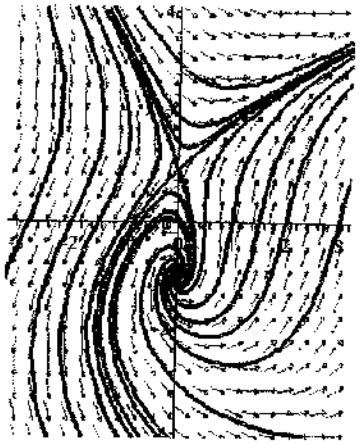
$$\begin{pmatrix} u \\ v \end{pmatrix}' = A_2 \begin{pmatrix} u \\ v \end{pmatrix}, \text{ ya'ni } \begin{cases} u' = 2u + 2v \\ v' = 6u - 2v \end{cases}$$

sistemada $v = ku$ deymiz va noma'lum k sonni topamiz:

$$\begin{cases} u' = 2u + 2ku \\ kv' = 6u - 2ku \end{cases}, \ k^2 + 2k - 3 = 0, \ \{k_1 = 1, k_2 = -3\}.$$

Demak, (0;1) muvozanat nuqtadan traektoriyalar $\alpha_1 = \operatorname{arctg}(1) = 45^0$ va $\alpha_2 = \operatorname{arctg}(-3) \approx -72^0$ burchak ostida o'tadi.

$2x + y^2 - 1 = 0$ va $6x - y^2 = 0$ parabolalar tekislikni beshta bo'lakka ajratadi. Har bir bo'lakda tezlik vektorlarini tasvirlab, ularga urintirib bir nechta traektoriyalarni quramiz (V.13-rasm).



V.13-rasm.

Misol 2. Ushbu

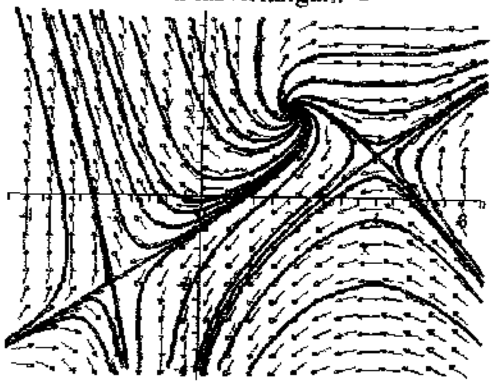
$$\begin{cases} x' = xy - 4 \\ y' = (x-4)(y-x) \end{cases}$$

sistemaning traektoriyalar portreti (manzarasi) ni tasvirlang.

⊶ *Muvozanat nuqtalari:*

$(4;1)$ – egar, $(2;2)$ – noturg'un fokus, $(-2;-2)$ – egar.

Traektoriyalar V.14- rasmda tasvirlangan. ♁



V.14-rasm.

V.4. Tekislikda avtonom sistemalarning sikllari (davralari)

Tekislikda ushbu

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}, \ \{f,g\} \subset C^1(G;\mathbb{R}), \qquad (V.4.1)$$

avtonom sistemani qaraylik.

Dastlab ba'zi tushuncha va tasdiqlarni eslaylik.

O'z-o'zini kesmaydigan yopiq traektoriyani sikl (davra) deb atagan edik. Sodda, ya'ni o'z-o'zini kesmaydigan va yopiq uzluksiz chiziq Jordan chizig'i deb ataladi. Jordan chizig'i aylananing uzluksiz biyektiv aksidan iborat bo'ladi.

Jordan teoremasiga ko'ra har qanday Jordan chizig'i tekislikni chegaralari shu chiziqdan iborat bo'lgan ikkita chegaralangan va chegaralanmagan sohalarga ajratadi; chegaralangan soha berilgan Jordan chizig'ining ichki tomoni (qismi), chegaralanmagan soha esa uning tashqi tomoni (qismi) deb ataladi. Agar $G \subset \mathbb{R}^2$ sohada joylashgan har qanday Jordan chizig'ining ichki tomoni (qismi) ham to'laligicha G da joylashsa, G soha bir bog'lamli soha deb ataladi. Noaniqroq aytadigan bo'lsak, bir bog'lamli soha – "teshiklarga" ega bo'lmagan soha.

Sikllarning mavjudligini isbotlashda quyidagi teoremadan foydalanish mumkin.

Teorema (Puankare). *Agar G – bir bog'lamli soha, Γ – (V.4.1) sistemaning G da joylashgan yopiq traektoriyasi bo'lsa, u holda Γ ning ichki qismida kamida bitta kritik (statsionar) nuqta mavjud.*

⊶ Qat'iy va to'la isbot topologiya elementlarini bilishni talab qiladi [1]. Biz isbotning asosiy g'oyalarini keltiramiz. G – bir bog'lamli bo'lgani uchun $\Gamma \subset G$ yopiq traektoriyani uning ichidagi ixtiyoriy O^* nuqtaga uzluksiz deformatsiyalash mumkin, ya'ni shunday $s \in [0;1]$ parametrli va uzluksiz o'zgaruvchi Γ_s sodda yopiq chiziqlar oilasi mavjudki, uning uchun $\Gamma_1 = \Gamma$ va $\Gamma_0 = O^*$ bo'ladi. Faraz qilaylik, (f,g) vektor-maydon Γ ning ichki qismida nolga aylanmasin. U holda Γ da va uning ichidagi har qanday nuqtada $(f,g)\neq(0,0)$ bo'lgani uchun shu vektor bilan Ox o'qi orasidagi θ burchakni uzluksiz o'zgaruvchi funksiya sifatida aniqlash mumkin. Γ_s ni bir marta o'tilganda (aylanib chiqilganda) θ burchak biror $2\pi k,\ k\in\mathbb{Z}\ \ (k\neq 0)$, orttirma oladi. Bu yerdagi k butun sonni Γ_s ning (f,g) vektor-maydonga nisbatan tartibi deymiz va uni T_s bilan belgilaymiz. T_s butun qiymatlar qabul qiladi va u $s\in[0;1]$ ning uzluksiz funksiyasidir. Demak, T_s – o'zgarmas, ya'ni $T_s=\text{const}=k_0\in\mathbb{Z}$, $s\in[0;1]$, bo'lishi kerak. Lekin $\Gamma=\Gamma_1$ traektoriyaning ixtiyoriy nuqtasidagi (f,g) tezlik vektori shu nuqtada Γ ga urinadi, demak, shu traektoriya bo'ylab soat mili yo'nalishida (yoki teskari yo'nalishda) bir marta to'la aylanib chiqilganda θ burchak $+2\pi$ (mos ravishda -2π) orttirma oladi, ya'ni $T_s=1$ (mos ravishda $T_s=-1$). T_0 esa 0 ga teng, chunki $\Gamma_0=O^*$ nuqtaning yetarlicha kichik atrofida (f,g) uzluksiz vektor-maydon deyarli o'zgarmaydi. Hosil bo'lgan ziddiyat farazimizning noto'g'riligini va (f,g) vektor-maydonning Γ ning ichki qismida kamida bitta kritik (statsionar) nuqtaga ega ekanligini isbotlaydi. ♁

Natija. *Agar bir bog'lamli G sohada* (V.4.1) *sistemaning kritik nuqtasi bo'lmasa, uning G da yopiq traektoriyasi ham mavjud emas.*

Sikllarning mavjud emasligini isbotlashda quyidagi teoremadan ham foydalanish mumkin.

Teorema (Bendikson-Dyulak). *Agar G bir bog'lamli soha bo'lib, biror* $h\in C^1(G)$ *funksiya uchun G sohada*

$$\frac{\partial(hf)}{\partial x}+\frac{\partial(hg)}{\partial y}>0$$

tengsizlik bajarilsa, u holda (V.4.1) *sistema G sohada siklga ega emas.*

⊸ Teskarisini faraz qilaylik, ya'ni (V.4.1) sistemaning Γ sikli mavjud bo'lsin. Γ ning ichki qismini G^* bilan belgilaylik. U holda analizdan ma'lum bo'lgan Grin formulasiga ko'ra

$$\iint_{G^*}\left(\frac{\partial(hf)}{\partial x}+\frac{\partial(hg)}{\partial y}\right)dx\,dy=\oint_{\Gamma}h(g\,dx-f\,dy).$$

Bu tenglikning chap tomoni musbat son, o'ng tomoni esa nolga teng, chunki Γ – (V.4.1) sistemaning traektoriyasi, ya'ni Γ da $g\,dx-f\,dy=0$. Hosil bo'lgan ziddiyat siklning mavjud emasligini isbotlaydi. ♁

Limit davralar

(V.4.1) sistemaning limit davrasi deb uning (yakkalangan) ajratilgan sikliga aytiladi. Aniqrog'i, agar Γ siklning yetarlicha kichik atrofida Γ dan boshqa sikl mavjud bo'lmasa, u holda Γ limit sikl (yoki limit davra) deb ataladi. Turli traektoriyalar umumiy nuqtaga ega bo'lmaganligi sababli Jordan teoremasiga ko'ra Γ sikldan farqli har qanday traektoriya to'laligicha yo Γ ning ichki qismida, yoki uning tashqi qismida joylashadi.

Misol. Ushbu

$$\begin{cases}x'=-y\\ y'=x\end{cases}$$

sistema cheksiz ko'p yakkalanmagan sikllarga ega:

$$\begin{cases}x=c_1\cos(t+c_2)\\ y=c_1\sin(t+c_2)\end{cases},\quad x^2+y^2=c_1^2.$$

Qutb koordinatalariga o'tib, ushbu

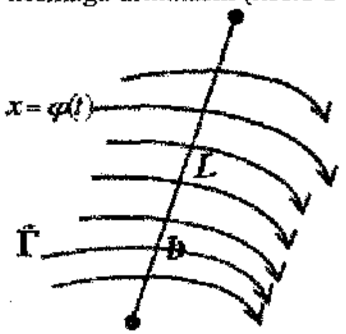
$$\begin{cases}x'=-y+x\sin(x^2+y^2)\\ y'=x+y\sin(x^2+y^2)\end{cases}$$

sistema cheksiz ko'p

$$\begin{cases}x=\sqrt{k\pi}\cos t\\ y=\sqrt{k\pi}\sin t\end{cases},\ k=1,2,\ \dots,\ (x^2+y^2=k\pi)$$

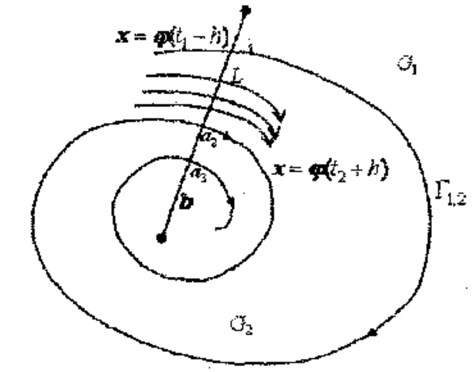
yakkalangan sikllarga (limit davralarga) ega ekanligini va sikldan iborat va bu holda teorema isbot bo'ldi.

Endi faraz qilaylik, $x=\varphi(t)$ yechim davriy bo'lmasin. Tushunarliki, na faqat $\Omega(\Gamma^{+})$ ω-limit to'plamda, balki uning yetarlicha kichik atrofida ham muvozanat nuqtalari mavjud emas. Biz ana shu atrof bilan chegaralanamiz. Ixtiyoriy $b\in\Omega(\Gamma^{+})$ nuqtani tayinlaylik va u orqali o'tgan traektoriyani $\tilde{\Gamma}$ bilan belgilaylik. b nuqta orqali $(f(b),g(b))\neq 0$ ($\Omega(\Gamma^{+})$ da muvozanat nuqtalari yo'q) vektorga kollinear bo'lmagan shunday kichik L kesma o'tkazaylikki, uning nuqtalari orqali o'tkazilgan traektoriyalar L kesmaga urinmasin (kesib o'tsin) (V.16- rasm).



V. 16-rasm

Γ^{+} yopiq traektoriya emasligi va b nuqta Γ^{+} ning ω-limit nuqtasi bo'lgani sababli Γ^{+} traektoriya L kesmani cheksiz ko'p turli nuqtalarda kesib o'tadi (V.16- rasm). Bu kesishish nuqtalarining ixtiyoriy ikkita bevosita ketma-ket kelganini $a_1=\varphi(t_1)$ va $a_2=\varphi(t_2)$ $(t_0\le t_1<t_2)$ bilan Γ^{+} ning $\{x\in\mathbb{R}^n \mid x=\varphi(t),\ t\in[t_1,t_2]\}$ qismini esa $\Gamma_{1,2}$ bilan belgilaylik. $[a_1;a_2]$ kesma va $\Gamma_{1,2}$ egri chiziqlar birlashmasi Λ yopiq chiziqni tashkil etadi va u tekislikni tashqi G_1 va ichki G_2 sohalarga ajratadi (V.17- rasm).



V.17-rasm.

$h>0$ sonni shunday kichik tanlaylikki, $\varphi(t_1-h)\in G_1$ va $\varphi(t_2+h)\in G_2$ bo'lsin, ya'ni $\varphi(t_1-h)$ va $\varphi(t_2+h)$ nuqtalar Λ ning turli tomonlarida joylashgan. Hech qanday traektoriya $\Gamma_{1,2}$ orqali G_1 sohadan G_2 sohaga yoki aksincha o'tolmaydi, chunki traektoriyalar kesisholmaydi va $\Gamma_{1,2}$ – traektoriya qismi. $[a_1;a_2]$ kesma orqali esa traektoriyalar G_1 soha orqali G_2 sohaga kiradi, lekin aksincha emas. $\Gamma_{1,2}$ traektoriya qismi L kesmani o'zining chetlari bo'lmish a_1 va a_2 nuqtalardagina kesadi xolos. Shuning uchun L kesmaning uchlari Λ yopiq chiziqning turli tomonlarida joylashgan. L kesmaning G_2 sohadagi uchini a deylik. Γ^{+} traektoriyaning $t>t_2+h$ qismi to'laligicha G_2 sohada yotadi va u $[a_1;a_2]$ kesma bilan umumiy nuqtaga ega bo'lolmaydi. Demak, b nuqta $[a_1;a_2]$ kesmaga tegishli emas, u $[a;a_2]$ kesmada yotishi kerak. Γ^{+} traektoriyaning L kesma bilan a_2 nuqtadan bevosita keyingi uchrashish nuqtasini $a_3=\varphi(t_3)$ $(t_2<t_3)$ deylik. Yuqoridagi fikrlashlardan ravshanki, $a_3\in[b;a_2]$ bo'ladi. Γ^{+} traektoriyaning L kesma bilan navbatdagi ketma-ket uchrashish nuqtalarini $a_4=\varphi(t_4),\ \ldots,\ a_k=\varphi(t_k),\ \ldots$ $(t_4<\ldots<t_k<\ldots)$

bilan belgilaylik. Yuqoridagilardan tushunarliki, $a_1, a_2, \ldots, a_k, \ldots$ nuqtalar L kesmaning $[a_1;b]$ qismida joylashgan monoton ketma-ketlikni tashkil etadi. Demak, ular yaqinlashuvchi, ya'ni $\lim\limits_{k\to+\infty} a_k = \tilde{b}$. $\tilde{b} = b$ ekanligini ko'rsatamiz. Dastlab $t_k \to +\infty$ bo'lishini isbotlaylik. Teskarisini faraz qilaylik, ya'ni $\lim\limits_{k\to+\infty} t_k = \tau < +\infty$ bo'lsin. U holda

$$\varphi(\tau) = \varphi(\lim_{k\to+\infty} t_k) = \lim_{k\to+\infty} a_k = \tilde{b} \quad \text{va} \quad \varphi'(\tau) = \lim_{k\to+\infty} \frac{\varphi(\tau) - \varphi(t_k)}{\tau - t_k}$$

tezlik vektori birinchidan, $\varphi(\tau) = \tilde{b}$ nutadagi $(f(\tilde{b}), g(\tilde{b}))$ vektorga teng, ikkinchidan u shu vektorga kollinear bo'lmagan L kesma bo'ylab yo'nalgan. Bu ziddiyat $\lim\limits_{k\to+\infty} t_k = +\infty$ ekanligini isbotlaydi. Demak, $x = \varphi(t)$ traektoriya $t \geq t_1$ bo'lganda L kesmani $a_1, a_2, \ldots, a_k, \ldots$ nuqtalardagina kesadi xolos, ya'ni bu traektoriya L kesmada bir dona $\tilde{b}$ ω-limit nuqtaga ega. Shuning uchun $\tilde{b} = b$ bo'lishi kerak. Bu yerda shuni e'tirof etaylikki, hozirgacha biz bor yo'g'i $b \in \Omega(\Gamma^+)$ nuqtaning muvozanat nuqta emasligidan foydalandik xolos.

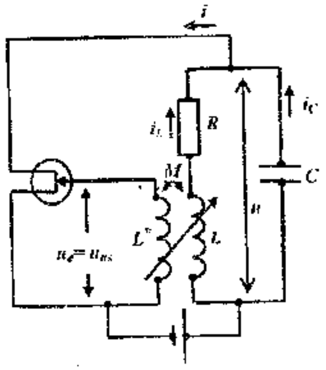
Endi $x = \varphi(t)$ traektoriya hech qanday boshqa $x = \psi(t)$ traektoriyanaing ω-limit to'plamida yotmasligini ko'rsatamiz. Teskarisini faraz qilaylik. U holda Γ^+ traektoriyaning har bir nuqtasi, xususan $a_1 = \varphi(t_1)$ nuqta ham, $x = \psi(t)$ uchun ω-limit nuqta bo'ladi. a_1 nuqta muvozanat nuqta bo'lmaganligi sababli yuqoridagi fikrlashlarda $x = \varphi(t)$ traektoriyani $x = \psi(t)$ bilan, b nuqtani esa a_1 bilan almashtirib, L kesmada $x = \psi(t)$ traektoriyanaing bor yo'g'i bir dona ω-limit nuqtasi borligini (u ham bo'lsa a_1) topamiz. Bu esa $a_k = \varphi(t_k) \in \Gamma^+$ nuqtalarning barchasi $x = \psi(t)$ traektoriya uchun ω-limit nuqta ekanligiga zid. Shunday qilib, quyidagi jumla isbotlandi.

Jumla. *Agar biror traektoriya yopiqmas va uning ω-limit nuqtalari orasida muvozanat nuqtalar bo'lmasa, u holda bu traektoriya hech qanday traektoriya uchun ω-limit to'plam bo'lolmaydi.*

Qurilgan $\tilde{\Gamma}$ traektoriya to'laligicha $\Omega(\Gamma^+)$ ω-limit to'plamda joylashadi, ya'ni $\tilde{\Gamma}$ traektoriyaning ixtiyoriy nuqtasi Γ^+ uchun ω-limit nuqta bo'ladi. $\Omega(\Gamma^+)$ yopiq to'plam bo'lganligi uchun $\Omega(\tilde{\Gamma}) \subset \Omega(\Gamma^+)$. Demak, $\Omega(\tilde{\Gamma})$ ω-limit to'plamda ham muvozanat nuqtalar yo'q. Bu holat, agar $\tilde{\Gamma}$ traektoriya yopiq bo'lmasa, hozirgina isbotlangan jumlaga zid bo'ladi. Demak, $\tilde{\Gamma}$ – yopiq traektoriya. Yuqoridagilardan ravshanki, Γ^+ traektoriya $\tilde{\Gamma}$ ga spiralsimon o'raladi va, demak, $\Omega(\Gamma^+)$ to'plam bir dona $\tilde{\Gamma}$ yopiq traektoriyadan iborat bo'ladi. ♤

Misol. *Avtotebranishlar generatori.* Ba'zi fizik sistemalarda tashqi ta'sirsiz doimiy ravishda takrorlanib turuvchi (davriy) o'zgarishlar (harakatlar) kuzatiladi. Masalan, mayatnikli soat, yuqori chastotali elektr tebranishlar generatori bunaqa sistemalarga misol bo'la oladi. Elektr tebranishlar generatori triodlar yoki tranzistorlar asosida tuzilishi mumkin.

Eng oddiy tranzistorli elektr tebranishlar generatori LCR tebranishlar konturi, L ga induktiv bog'liq bo'lgan va tranzistorga ulangan L^* g'altak (M – induksiya koeffitsienti) hamda elektr manbasidan iborat (V.18-rasm).



V.18-rasm.

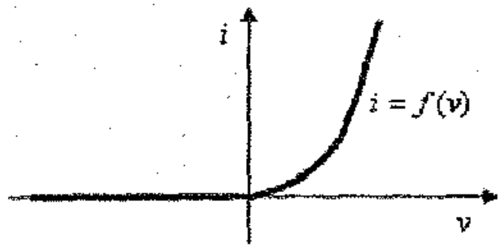
Kirxgof qonuniga ko`ra $i = i_L + i_C$. Ma'lumki, kondensatordagi tok $i_C = C\frac{du}{dt}$. Yana Kirxgof qonuniniga ko`ra $u = Ri_L + L\frac{di_L}{dt}$. Demak,

$$i = i_L + CR\frac{di_L}{dt} + CL\frac{d^2 i_L}{dt^2}. \quad (V.4.6)$$

Bundan tashqari,

$$u_{us} = M\frac{di_L}{dt},\ i = f(u_{us}) = f(M\frac{di_L}{dt}); \quad (V.4.7)$$

bu yerda $i = f(v)$ – tranzistorning xarakteristikasi , u i tokning v kuchlanishiga bog`lanish qonuniyatini ifodalaydi. Bu bog`lanishning tipik grafigi V.19- rasmda keltirilgan.



V.19-rasm.

(V.4.7) ni (V.4.6) ga qo`yib, i_L tok uchun quyidagi ikkinchi tartibli differensial tenglamani topamiz:

$$CL\frac{d^2 i_L}{dt^2} + CR\frac{di_L}{dt} - f(M\frac{di_L}{dt}) + i_L = 0. \quad (V.4.8)$$

Bu yerda shuni e'tirof etaylikki, uch elektrodli elektron lampali generatordagi anod toki ham (V.4.8) ko`rinishdagi tenglamani qanoatlantiradi [9].

Qulaylik uchun (V.4.8) tenglamada $t\sqrt{LC} = \tau$, $i_L(t) - f(0) = x(\tau)$ almashtirish bajaraylik. U holda ushbu

$$\frac{d^2x}{d\tau^2} + F(\frac{dx}{d\tau}) + x = 0, \quad (V.4.9)$$

tenglamaga kelamiz; bu yerda

$$F(y) = R\sqrt{\frac{C}{L}}y - f(\frac{M}{\sqrt{LC}}y) + f(0),\ F(0) = 0. \quad (V.4.10)$$

Odatdagicha (V.4.9) tenglamadan

$$\frac{dx}{d\tau} = y,\ \frac{dy}{d\tau} = -x - F(y) \quad (V.4.11)$$

sistemaga o`tamiz. Ravshanki, (V.4.11) sistema yagona kritik (muvozanat) nuqtaga ega: $x = 0, y = 0$.

Jumla. *Faraz qilaylik,* $F \in C^1$, $F(0) = 0, F'(0) < 0$, $y \geq b$ $(b > 0)$ *bo`lganda* $F(y) > m$, $y \leq -b$ *bo`lganda esa* $F(y) < k$ $(k < m)$ *bo`lsin. U holda* (V.4.11) *sistema yopiq traektoriyaga ega. (Agar* $f$ – *chegaralangan,* $\in C^1$ *hamda* $Mf'(0) > RC$ *bo`lsa,* (V.4.10)*dagi* F *funksiya keltirilgan faraz shartlarini qanoarlantiradi.*)

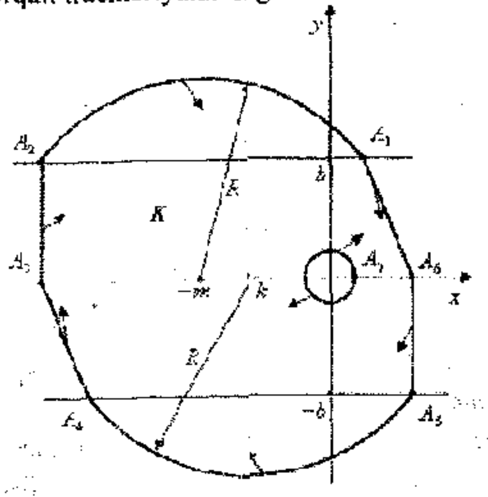
⊶ x, y tekislikda shunday yopiq xalqasimon soha K ni quramizki, har qanday yechim uning ichidan tashqariga chiqib ketmaydi.

K ning ichki chegarasini $x^2 + y^2 = r^2$ aylana ko`rinishida tanlaymiz. $r > 0$ ni kichik tanlaymizki, uning uchun $0 < |y| \leq r$ bo`lganda $yF(y) < 0$ bo`lsin. $|y| \leq r$ bo`lganda (V.4.11) sistemaning yechimi uchun $\frac{d}{d\tau}(x^2 + y^2) = -2yF(y) \geq 0$, demak, yechim $x^2 + y^2 = r^2$ aylanadan uning ichiga kirolmaydi.

K ning tashqi chegarasini bir necha qismdan iborat qilib tuzamiz. Uning $y \geq b$ yarim tekislikdagi chegarasi $(x+m)^2 + y^2 = R^2$ aylananing $\overset{\cup}{A_1A_2}$ yoyidan iborat bo`lsin, R ni keyinroq tanlaymiz, V.20-rasmga qarang. Bu yoyda (V.4.11) sistemaning yechimi uchun

$$\frac{d}{d\tau}((x+m)^2+y^2)=2y(m-F(y))<0,$$

chunki $y\ge b$ bo'lganda $F(y)>m$. $y\le -b$ yarim tekislikda $(x+k)^2+y^2=R^2$ aylananing $\overset{\cup}{A_4A_5}$ yoyini olamiz. A_2A_3 va A_5A_6 vertikal kesmalarda mos ravishda $x'=y>0$ va $x'=y<0$ va, demak, ular orqali traektoriyalar K ga kiradi.



V.20-rasm.

A_6A_1 va A_5A_4 kesmalarning burchak koeffitsienti $-b/(m-k)$. Bu kesmalarni kesuvchi traektoriyalarda $\dfrac{dy}{dx}=-\dfrac{x+F(y)}{y}$. $R>0$ ni yetarlicha katta tanlash evaziga $|x|$ ni kattalashtiramiz va $\dfrac{dy}{dx}<-\dfrac{b}{m-k}$ tengsizlikning bajarilishini ta'minlaymiz. U holda traektoriyalar A_6A_1 va A_5A_4 kesmalar orqali K ga kiradi.

Shunday qilib, qurilgan yopiq xalqasimon K sohada maxsus nuqta yo'q va (V.4.11) sistemaning traektoriyalari K dan

chiqib ketmaydi. Puankare-Bendikson teoremasidan K da (V.4.11) sistemaning yopiq traektoriyasi mavjud ekanligi kelib chiqadi.

Izoh. Hilbertning 16- muammosi tekislikda

$$\begin{cases} x' = A(x,y) \\ y' = B(x,y) \end{cases},$$ bunda $A(x,y)$ va $B(x,y)$ ko'phadlar,

polinomial sistemaning maksimal limit davralar sonini va ularning o'zaro joylashuvini aniqlash bilan bog'liq. $A(x,y)$ va $B(x,y)$ ko'phadlar darajalarining kattasini n bilan, sistemaning maksimal limit davralar sonini H_n bilan belgilaylik. Ma'lumki, $H_0=0$, $H_1=0$, $H_2\ge 4$, $H_3\ge 8$, toq n lar uchun $H_n\ge(n-1)/2$, hamda $H_n<+\infty$. Lekin hatto $H_2=4$ degan (gipoteza) taxmin ham hanuzgacha to'la isbotlanmagan [Ilyashenko, Y. and S. Yakovenko, Eds. (1995). Concerning the Hilbert 16th Problem. Providence, AMS].

Masalalar

1. Bir o'lchamli

$$x'=f(x)$$

avtonom sistema uchun $f(x)\in C^1(\mathbb{R})$ va $f(x)$ ikkita nolga ega bo'lsin: $f(a)=0$, $f(b)=0$ $(a<b)$. Bu sistemaning har qanday traektoriyasi $(-\infty;a),\{a\},(a;b),\{b\},(b;+\infty)$ to'plamlarning biridan iborat bo'lishini isbotlang.

2. Faraz qilaylik, berilgan $x'=f(x)$ avtonom sistemaning o'ng tomoni $x\in\mathbb{R}^n$ da aniqlangan bo'lsin. Bu sistemaning $x=x(t)$ traektoriyasidagi t parametr o'rniga $\tau=\tau(t)$ parametrni ushbu $\dfrac{d\tau}{dt}=\sqrt{1+\|f(x)\|^2}$ tenglamaning yechimi sifatida kiritaylik. U holda $\dfrac{dx}{d\tau}=\dfrac{f(x)}{\sqrt{1+\|f(x)\|^2}}$. Oxirgi sistemaning yechimlari $\tau\in(-\infty;+\infty)$ oraliqda aniqlangan va uning fazaviy tasviri berilgan avtonom sistemaning fazaviy tasviri bilan bir xil ekanligini ko'rsating.

3. $A-3\times 3$ o'lchamli haqiqiy matritsa bo'lsin. Teskarilanuvchi

shunday S matritsa topish mumkinki, uning uchun quyidagi tengliklarning biri o'rinli bo'ladi:

$$SAS^{-1}=\begin{pmatrix}\alpha & \beta & 0\\ -\beta & \alpha & 0\\ 0 & 0 & \lambda\end{pmatrix},\quad SAS^{-1}=\begin{pmatrix}\lambda & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \nu\end{pmatrix},\quad SAS^{-1}=\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mu\end{pmatrix},$$

$$SAS^{-1}=\begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix}\quad (\alpha,\beta,\lambda,\mu,\nu-\text{haqiqiy sonlar va } \beta\neq 0).$$

Shu tasdiqni isbotlang. Undan foydalanib ushbu [illegible]

$$\begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}=A\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

uch o'lchamli avtonom sistemaning traektoriyalarini tekshiring.

4. Faraz qilaylik, $f\in C(\mathbb{R},(0,+\infty))$ funksiya $\tau>0$ davrga ega bo'lsin. Agar $x=x(t)$ funksiya $x'=f(x)$ tenglamaning yechimi va

$$T=\int_0^{\tau}\frac{1}{f(x)}dx$$

bo'lsa, u holda har qanday $t\in\mathbb{R}$ uchun $x(T+t)-x(t)=\tau$ bo'lishini isbotlang. f funksiya davriy va ishorasini almashtiruvchi bo'lgan holni ham tekshiring.

5. Ushbu

$$\begin{cases}x'=y\\ y'=-p(y)y-x\end{cases},\quad p(y)\in C(\mathbb{R}),\ p(y)>0,$$

sistema limit siklga ega emasligini isbotlang.

6. Ushbu

$$\begin{cases}x'=f(x,y)\\ y'=g(x,y)\end{cases},\quad \{f,g\}\subset C^1(G;\mathbb{R}),$$

sistemaning $x(0)=\xi$, $y(0)=\eta$ nuqtadan o'tuvchi $\Gamma_{\xi\eta}$ traektoriyasi

$$\begin{cases}x=\varphi(t,\xi,\eta)\\ y=\psi(t,\xi,\eta)\end{cases}$$

parametrik ko'rinishda (t – parametr) hamda (α_1,α_2) nuqtadan o'tuvchi

va $(\beta_1,\beta_2)\neq 0$ yo'naltiruvchi vektorga ega bo'lgan L to'g'ri chiziq

$$\begin{cases}x=\alpha_1+\beta_1 u\\ y=\alpha_2+\beta_2 u\end{cases}\quad(-\infty<u<+\infty)$$

berilgan bo'lsin. u parametrni L to'g'ri chiziqdagi koordinata deb hisoblaymiz. Faraz qilaylik, $\Gamma_{\xi_0\eta_0}$ traektoriya t_0 paytda L to'g'ri chiziqni uning u_0 koordinatali nuqtasida kessin:

$$\begin{cases}\varphi(t_0,\xi_0,\eta_0)=\alpha_1+\beta_1 u_0\\ \psi(t_0,\xi_0,\eta_0)=\alpha_2+\beta_2 u_0\end{cases},$$

lekin urinmasin:

$$\begin{vmatrix}\dfrac{\partial\varphi(t_0,\xi_0,\eta_0)}{\partial t} & \dfrac{\partial\psi(t_0,\xi_0,\eta_0)}{\partial t}\\ \beta_1 & \beta_2\end{vmatrix}\neq 0.$$

Quyidagilarni isbotlang:

1^0. Shunday $\delta>0$ va $\varepsilon>0$ sonlar topiladiki, $\{|\xi-\xi_0|<\delta,\ |\eta-\eta_0|<\delta\}$ sohada C^1 sinfga tegishli va

$$\begin{cases}\varphi(t(\xi,\eta),\xi,\eta)=\alpha_1+\beta_1 u(\xi,\eta)\\ \psi(t(\xi,\eta),\xi,\eta)=\alpha_2+\beta_2 u(\xi,\eta)\end{cases};\quad t(\xi_0,\eta_0)=t_0,\ u(\xi_0,\eta_0)=u_0;$$

$|t(\xi,\eta)-t_0|<\varepsilon$

shartlarni qanoatlantiruvchi $t(\xi,\eta)$ va $u(\xi,\eta)$ funksiyalar mavjud.

2^0. Mavjudligi tasdiqlangan $t(\xi,\eta)$ va $u(\xi,\eta)$ funksiyalar yagona, ya'ni, agar $|\xi-\xi_0|<\delta$, $|\eta-\eta_0|<\delta$, $|t-t_0|<\varepsilon$ bo'lganda ushbu

$$\begin{cases}\varphi(t,\xi,\eta)=\alpha_1+\beta_1 u\\ \psi(t,\xi,\eta)=\alpha_2+\beta_2 u\end{cases}$$

tengliklar qanoatlansa, u holda, albatta,

$$\begin{cases}t=t(\xi,\eta)\\ u=u(\xi,\eta)\end{cases}$$

bo'ladi.

Bu yerdagi 1^0 va 2^0 xossalar geometrik nuqtai nazaridan quyidagini anglatadi. $t=0$ paytda (ξ_0,η_0) nuqtaga yetarlicha yaqin bo'lgan ixtiyoriy

(ξ,η) nuqtadan chiquvchi $\Gamma_{\xi\eta}$ traektoriya L to'g'ri chiziqni u_0 nuqtaga yaqin $u=u(\xi,\eta)$ nuqtada t_0 ga yaqin $t=t(\xi,\eta)$ paytda kesadi, bunda biror $|t-t_0|<\varepsilon$ paytlar uchun bu kesishish nuqtasi yagona hamda $u=u(\xi,\eta)$ va $t=t(\xi,\eta)$ funksiyalar C^1 sinfga tegishli bo'ladi.

3^0. Yuqoridagi shartlarga qo'shimcha

$$\begin{cases} x=\varphi(t,\xi_0,\eta_0) \\ y=\psi(t,\xi_0,\eta_0) \end{cases}$$

yechim eng kichik musbat davrga ega, ya'ni $\Gamma_{\xi_0\eta_0}$ – yopiq traektoriya bo'lsin. U holda shunday $\gamma>0$ son topiladiki, $|u-u_0|<\gamma$ bo'lganda

$$\begin{cases} x=\varphi(t,\alpha_1+\beta_1u,\alpha_2+\beta_2u)=\varphi(t,u) \\ y=\psi(t,\alpha_1+\beta_1u,\alpha_2+\beta_2u)=\psi(t,u) \end{cases}$$

traektoriya L to'g'ri chiziqni $t>0$ va $t<0$ paytlarda kesadi. $t>0$ bo'lgandagi birinchi kesishish paytini $t_1(u)$, L dagi $\chi_1(u)$ koordinatasini bilan, $t<0$ bo'lganda teskari yo'nalishdagi birinchi kesishish paytini $t_1(u)$, L dagi koordinatasini $\chi_{-1}(u)$ bilan belgilaylik. $u_0\in L$ nuqtaning yetarlicha kichik atrofida ushbu

$$t_1(u),\ \chi_1(u),\ t_{-1}(u),\ \chi_{-1}(u)$$

funksiyalar aniqlangan va C^1 sinfga tegishli,

$$t_1(u_0)=\tau,\ \chi_1(u_0)=u_0,\ t_{-1}(u_0)=-\tau,\ \chi_{-1}(u_0)=u_0$$

shartlarni qanoatlantiradi. Bundan tashqari, shu atrofda

$$\chi_1(\chi_{-1}(u))=u,\ \chi_{-1}(\chi_1(u))=u$$

ayniyatlar ham o'rinli bo'ladi (χ_1 va χ_{-1} funksiyalar o'zaro teskari).

7. Ushbu

$$\begin{cases} x'=(x-1)(y-x) \\ y'=x^2+y^2-2 \end{cases}$$

sistemaning traektoriyalar portretini quring.

8. Ushbu

$$\begin{cases} x'=2x+2y-xy-3 \\ y'=x^2-y^2 \end{cases}$$

sistemaning traektoriyalar portretini quring.

9. Ushbu

$$\begin{cases} x'=(1-y)(y-2) \\ y'=\dfrac{1}{2}(x-1)(x-3) \end{cases}$$

sistemaning fazaviy portretini quring.

10. Ushbu

$$\begin{cases} x'=y(7-x^2-y^2) \\ y'=6-x(7-x^2-y^2) \end{cases}$$

sistemaning fazaviy portretini quring.

11. Sistemaning fazaviy portretini quring:

$$\begin{cases} x'=\sin y \\ y'=\sin x \end{cases}$$

12. Quyidagi sistemalar davriy yechimga ega emasligini isbotlang:

1). $\begin{cases} x'=y, \\ y'=-ax-by+\alpha x^2+\beta y^2. \end{cases}$ 2). $\begin{cases} x'=x(a_1x+b_1y+c_1), \\ y'=y(a_2x+b_2y+c_2). \end{cases}$

3). $\begin{cases} x'=y+x(1+\beta y)(x^2+y^2+1), \\ y'=-x+(y-\beta x^2)(x^2+y^2+1). \end{cases}$

13. ... betdagi jumlani Puankare-Bendikson teoremasidan foydalanmay, ya'ni to`g`ridan-to`g`ri isbotlang.

VI BOB. LYAPUNOV BO'YICHA TURG'UNLIK

VI.1. Turg'unlik tushunchasi

Ko'plab jarayonlar, shu jumladan mashinalarning, asboblarning va boshqa qurilmalarning ishlash jarayoni, harakati differensial tenglamalar bilan tavsiflanadi. Bu tenglamalar cheksiz ko'p yechimga ega bo'lsa-da, tegishli jarayon bitta yechim bilan aniqlanadi va u ma'lum bir boshlang'ich qiymatlarga mos keladi. Boshlang'ich qiymatlar sal o'zgarganda hosil bo'luvchi yechim vaqt o'tishi bilan dastlabki yechimga yaqinligicha qoladimi (turg'un yechim) yoki undan uzoqlashib ketadimi (noturg'un yechim)? degan savolning javobini bilish juda katta amaliy ahamiyatga ega. Chunki odatda boshlang'ich qiymatlar xatolikka ega bo'luvchi o'lchashlar, taqribiy hisoblashlar orqali aniqlanadi va bu qiymatlarning sal o'zgarishining yechimga ta'sirini bilish nihoyatda muhimdir. Agar yechim vaqt t o'tishi bilan dastlabkisidan uzoqlashib ketsa, o'rganilayotgan jarayonning tabiatini katta t larda oldindan aytib bo'lmaydi.

Turg'unlikning turli ta'riflarini Puasson, Lagranj, Lyapunov va boshqalar kiritishgan. Biz Lyapunovga[1] ko'ra turg'unlik bilan tanishamiz.

Turg'unlik nazariyasida differensial tenglamalar sistemasi yechimlarining $t \to +\infty$ dagi tabiati o'rganiladi.

Differensial tenglamalarning quyidagi normal sistemasini qaraylik:

$$x' = f(t,x); \qquad \text{(VI.1.1)}$$

bu yerda $f \in C(\mathbb{R}_+ \times D; \mathbb{R}^n)$ ($\mathbb{R}_+ = [0,+\infty)$, $D \subset \mathbb{R}^n$ – soha) va $f(t,x)$ vektor-funksiya x bo'yicha lokal Lipshits shartini qanoatlantiradi deb hisoblanadi. Bu shartlarda $\forall (t_0, x^0) \in \mathbb{R}_+ \times D$ uchun ushbu

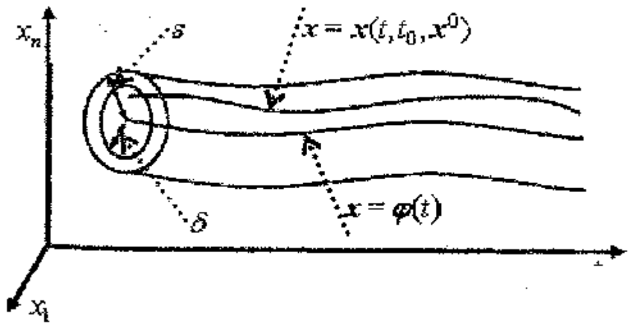
$$\begin{cases} x' = f(t,x) \\ x(t_0) = x^0 \end{cases}$$

masala $t \in [t_0, T)$ oraliqda aniqlangan (o'ngga davomsiz) yagona

$x = x(t, t_0, x^0)$ yechimga ega. Biz o'ngga cheksiz davom etgan ($T = +\infty$), ya'ni $t \in [t_0, +\infty)$ oraliqda aniqlangan yechimlarning turg'unlik xossalarini o'rganamiz.

Bizga (VI.1.1) tenglamaning $\mathbb{R}_+$ da aniqlangan $x = \varphi(t)$, $\varphi : \mathbb{R}_+ \to D$, yechimi berilgan bo'lsin. Agar ixtiyoriy $t_0 \in \mathbb{R}_+$ va $\varepsilon > 0$ sonlariga ko'ra shunday $\delta > 0$ soni topilsaki, (VI.1.1) tenglamaning $x(t_0) = x^0$ boshlang'ich qiymatli $x = x(t, t_0, x^0)$ yechimlari $\left\| x^0 - \varphi(t_0) \right\| < \delta$ bo'lganda mavjud va o'ngga $+\infty$ gacha davom ettirilib, barcha $t \in [t_0, +\infty)$ paytlarda $\| x(t, t_0, x^0) - \varphi(t) \| < \varepsilon$ bo'lsa, u holda $x = \varphi(t)$ yechim Lyapunov ma'nosida (yoki Lyapunovga ko'ra) turg'un yechim deb ataladi.

Keltirilgan shart boshlangich qiymatlarning yaqinligidan ($\left\| x^0 - \varphi(t_0) \right\| < \delta$) barcha keyingi paytlarda ham yechimlarning yaqinligi ($\| x(t, t_0, x^0) - \varphi(t) \| < \varepsilon$, $t \in [t_0, +\infty)$) kelib chiqishini anglatadi (VI.1-rasm).



VI.1-rasm.

Turg'un yechim ta'rifidagi "barcha $t \in [t_0, +\infty)$ paytlarda $\| x(t, t_0, x^0) - \varphi(t) \| < \varepsilon$ bo'lsa" shartni ushbu

“$\sup\limits_{t \ge t_0} \| x(t,t_0,x^0) - \varphi(t) \| < \varepsilon$ bo'lsa” shart bilan almashtirish mumkin.

Oldindan beriladigan ixtiyoriy $\varepsilon > 0$ sonni yetarlicha kichik deb hisoblasa bo'ladi, chunki biror $\varepsilon_0 > 0$ ga ko'ra topilgan $\delta = \delta_0 > 0$ soni har qanday $\varepsilon \ge \varepsilon_0$ son uchun ham δ bo'lib xizmat qiladi. Demak, barcha $\varepsilon \in (0;\varepsilon_0]$ sonlar, ya'ni yetarlicha kichik ε lar uchun ularga mos δ larni topish kerak xolos.

Umumiy holda topiladigan $\delta > 0$ soni tayinlangan $t_0 \in \mathbb{R}_+$ va berilgan $\varepsilon > 0$ sonlarga bog'liq bo'ladi, ya'ni $\delta = \delta(t_0, \varepsilon)$.

Agar turg'unlik ta'rifidagi $\delta > 0$ sonini $t_0 \in [0,+\infty)$ ga bog'liqsiz holda tanlash mumkin, ya'ni $\delta = \delta(\varepsilon)$ bo'lsa, u holda yechim ($\mathbb{R}_+$da) (yoki $t_0 \in [0,+\infty)$ ga nisbatan) **tekis turg'un yechim** deb ataladi.

Turg'un bo'lmagan yechim **noturg'un yechim** deyiladi.

Agar

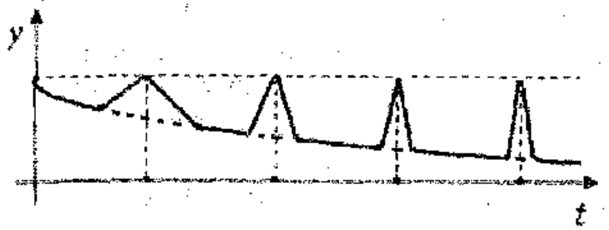
1) $x = \varphi(t)$ yechim turg'un va

2) shunday $\delta_0 > 0$ mavjud bo'lib, $\|x^0 - \varphi(t_0)\| < \delta_0$ ekanliligidan $\lim\limits_{t \to +\infty} \| x(t,t_0,x^0) - \varphi(t) \| = 0$ bo'lishi ham kelib chiqsa, u holda $x = \varphi(t)$ yechim ***asimtotik turg'un yechim*** deb ataladi.

Umumiy holda turg'unlikdan tekis turg'unlik kelib chiqmaydi; bundan tashqari, turg'unlikdan asimptotik turg'unlik ham kelib chiqmaydi. Bu fikrlarni quyidagi misol asoslaydi.

Misol (X. L. Massera). $f(t) \in C^1([0;+\infty);\mathbb{R})$ funksiyani quyidagicha aniqlaylik. Har qanday $n \in \mathbb{N}$ uchun funksiya $[n - 2^{-n}; n]$ oraliqda $f(n - 2^{-n}) = \exp(2^{-n} - n)$ qiymatdan $f(n) = 1$ qiymatgacha o'sadi, $[n; n + 2^{-n}]$ oraliqda u $f(n) = 1$ dan $f(n + 2^{-n}) = \exp(-2^{-n} - n)$ gacha kamayadi hamda

$[n - 2^{-n}; n + 2^{-n}]$ ko'rinishdagi oraliqlarning tashqarisida $f(t)$ funksiya $\exp(-t)$ funksiya bilan ustma-ust tushadi (VI.2-rasm; uchlar silliqlashtirilgan, funksiya $\in C^1$).



VI.2- rasm.

Ushbu

$$x' = \frac{f'(t)}{f(t)} x, \quad t \ge 0, \qquad \text{(VI.1.2)}$$

tenglamani qaraylik. Uning umumiy yechimi

$$x = x(t; t_0, x_0) = \frac{x_0}{f(t_0)} f(t) \qquad \text{(VI.1.3)}$$

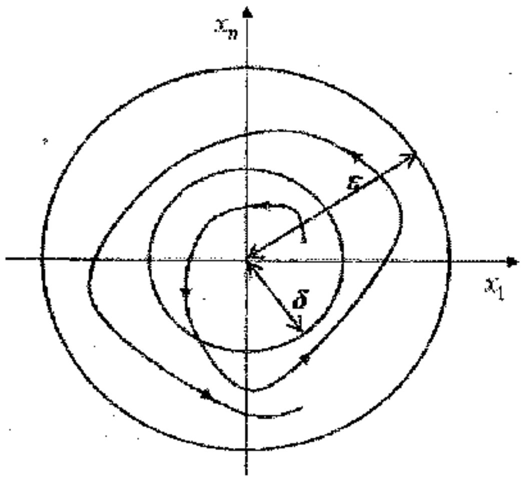
ko'rinishga ega. (VI.1.2) tenglamaning $x(t) \equiv 0$ yechimi turg'un, chunki $|x(t;t_0,x_0)| \le \sup\limits_{t \ge t_0} |x(t;t_0,x_0)| = |x_0| / f(t_0) < \varepsilon$ bo'lishi uchun δ sifatida $\delta = \delta(t_0, \varepsilon) = \varepsilon \cdot f(t_0)$ ni tanlash kifoya. Tushunarliki, bu δ ni yaxshilab, ya'ni kattalashtirib bo'lmaydi. Lekin $x(t) \equiv 0$ yechim $t_0 \ge 0$ ga nisbatan tekis turg'un emas, chunki $\inf\limits_{t_0 \ge 0} \delta(t_0, \varepsilon) = \varepsilon \cdot \inf\limits_{t_0 \ge 0} f(t_0) = 0$. Umumiy yechim uchun (VI.1.3) formuladan $x(t) \equiv 0$ yechimning asimptotik turg'un emasligi ham kelib chiqadi. Haqiqatan ham, agar u asimptotik turg'un bo'lganda edi, u holda yetarli kichik $|x_0|$ lar uchun $|x(t;t_0,x_0)| = \frac{|x_0|}{f(t_0)} f(t) \xrightarrow[t \to +\infty]{} 0$ bo'lardi. Lekin $t = n$ da

$|x(n;t_0,x_0)| = \frac{|x_0|}{f(t_0)} f(n) = \frac{|x_0|}{f(t_0)}$ va ziddiyat hosil bo'lardi. ♣

Berilgan tenglamaning ixtiyoriy tayinlangan $x = \varphi(t)$, $t \geq 0$, yechimini turg'unlikka tekshirishni boshqa bir tenglamaning trivial, ya'ni nolga teng yechimini turg'unlikka tekshirishga keltirish mumkin. Buning uchun (VI.1.1) tenglamada $y = x - \varphi(t)$ almashtirish bajarish kerak. Yangi noma'lum y quyidagi tenglamani qanoatlantiradi:

$$y' = f(t, y + \varphi(t)) - f(t, \varphi(t)) \equiv g(t, y), \quad g(t, 0) = 0.$$

Oxirgi differensial tenglama $y = 0$ trivial yechimga ega. Bu yechimning turg'unligi (asimptotik turg'unligi) (VI.1.1) tenglama $x = \varphi(t)$ yechimining turg'unligiga (asimptotik turg'unligiga) teng kuchlidir.



VI.3-rasm.

Nol yechimning turg'unligi quyidagini anglatadi: Fazalar fazosida ixtiyoriy $B(0,\varepsilon)$ sharni olaylik ; yetarlicha kichik $\delta > 0$ radiusli shunday $B(0,\delta)$ shar topiladiki, $t = t_0$ da bu shar ichidan chiquvchi ixtiyoriy traektoriya $t \geq t_0$ paytlarda to'laligicha $B(0,\varepsilon)$

shar ichida qoladi (VI.3-rasm).

(VI.1.1) ning o'ng tomonidan talab qilingan shartlarda $x = \varphi(t)$ yechim boshlang'ich qiymatlarga uzluksiz bog'liq, ya'ni $\forall[\alpha,\beta] \subset [t_0,+\infty)$ segment va $\forall \varepsilon > 0$ son uchun shunday $\delta > 0$ topiladiki, $\gamma \in [\alpha,\beta]$ paytda z^0 qiymat qabul qiluvchi $x = x(t,\gamma,z^0)$ yechim, agar $\| x(\gamma,\gamma,z^0) - \varphi(\gamma) \| = \| z^0 - \varphi(\gamma) \| < \delta$ bo'lsa, barcha $t \in [\alpha,\beta]$ larda aniqlangan va $\forall t \in [\alpha,\beta]$ uchun $\|x(t,\gamma,z^0) - \varphi(t)\| < \varepsilon$ tengsizlikni qanoatlantiradi. Bu xossadan ravshanki, $x = \varphi(t)$ yechimning turg'unligi (asimptotik turg'unligi) boshlang'ich payt $t_0 \in [0,+\infty)$ ning tanlanishiga bog'liq emas, ya'ni agar biror $t_0 \in [0,+\infty)$ uchun $x = x(t,t_0,\varphi(t_0))$ yechim turg'un (asimptotik turg'un) bo'lsa, u holda ixtiyoriy $\tilde{t}_0 \in [0,+\infty)$ uchun ham mos $x = x(t,\tilde{t}_0,\varphi(\tilde{t}_0))$ yechim turg'un (asimptotik turg'un) bo'ladi.

VI.2. Chiziqli sistemalarning turg'unligi

Bu bandda ushbu

$$x' = A(t)x + b(t) \tag{VI.2.1}$$

chiziqli sistema yechimlarining turg'unligini tekshiramiz; bunda $A(t) \in C([0;+\infty); M_{n\times n}(\mathbb{R}))$ va $b(t) \in C([0;+\infty); \mathbb{R}^n)$ deb hisoblanadi. Demak, ixtiyoriy $x|_{t_0} = x^0 \in \mathbb{R}^n$, $t_0 \in [0;+\infty)$, boshlang'ich shart uchun bira-to'la $[0;+\infty)$ oraliqda aniqlangan $x = x(t) = x(t;t_0,x^0)$ yagona yechim mavjud. (VI.2.1) ga mos bir jinsli sistema

$$y' = A(t)y \tag{VI.2.2}$$

ko'rinishda bo'ladi.

Teorema 1. (VI.2.1) *chiziqli sistemaning har qanday* *yechimining turg'unligi (asimptotik turg'unligi) mos bir jinsli sistema* (VI.2.2)*ning bitta* $y = 0$ *trival yechimining turg'unligiga (mos ravishda asimptotik turg'unligiga) teng kuchli.*

⊶ Teoremaning turg'unlikka oid qismini isbotlaymiz. Uning asimptotik turg'unlikka oid qismi shunga o'xshash isbotlanadi. (VI.2.1) sistemaning ixtiyoriy bir $x = \varphi(t)$ turg'un yechimini olaylik. Demak, $t \in [t_0, +\infty)$ uchun

$$\varphi'(t) = A(t)\varphi(t) + b(t),$$

$$\left\|x^0 - \varphi(t_0)\right\| < \delta \implies \| x(t,t_0,x^0) - \varphi(t) \| < \varepsilon,\ t \ge t_0; \quad \text{(VI.2.3)}$$

bu yerda $\delta > 0$ son oldindan berilgan ixtiyoriy $\varepsilon > 0$ songa ko'ra turg'unlik ta'rifidan topilgan. Biz (VI.2.2) sistemaning $y = y(t;t_0,y^0)$ yechimi uchun

$$\| y^0 \| < \delta \implies \| y(t;t_0,y^0) \| < \varepsilon,\ t \ge t_0 \quad \text{(VI.2.4)}$$

implikatsiyaning o'rinliligini ko'rsatishimiz kifoya, chunki bu holda (VI.2.2) ning $y = 0$ yechimi turg'un bo'ladi. Agar $y = x - \varphi(t)$ desak, u holda bu yerdagi $x = y + \varphi(t)$ funksiya (VI.2.1) sistemaning yechimi bo'ladi va (VI.2.4) implikatsiya (VI.2.3) dan bevosita kelib chiqadi. Endi faraz qilaylik, (VI.2.2)ning $y = 0$ yechimi turg'un, ya'ni (VI.2.4) implikatsiya o'rinli bo'lsin. (VI.2.1) ning ixtiyoriy $x = \varphi(t)$ yechimi turg'un ekanligini isbotlash kerak. Bu esa yana o'sha $y = x - \varphi(t)$ almashtirish yordamida yuqoridagiga o'xshash asoslanadi. ♧

Shunday qilib, quyidagi alternativa o'rinli:
yo (VI.2.1) chiziqli sistemaning barcha yechimlari turg'un (asimptotik turg'un); bu holda (VI.2.1) sistema ***turg'un sistema*** (mos ravishda ***asimptotik turg'un sistema***) deb ataladi,
yoki uning barcha yechimlari noturg'un; bu holda esa (VI.2.1) sistema ***noturg'un sistema*** deb ataladi.

Biz endi (VI.2.2) sistemaning trivial yechimini turg'unlikka tekshiramiz. Bu turg'unlik (VI.2.1) va (VI.2.2) sistemalarning turg'unligiga teng kuchli. (VI.2.2) sistemada noma'lumni odatdagidek $x = x(t)$ bilan belgilab, uni

$$x' = A(t)x \quad \text{(VI.2.5)}$$

ko'rinishda yozib olamiz.

Ma'lumki, (VI.2.5) sistemaning $t = t_0$ paytda x^0 ga aylanuvchi yechimi $x = x(t;t_0,x^0)$ ushbu

$$x(t;t_0,x^0) = \Phi(t,t_0)x^0 \quad \text{(VI.2.6)}$$

formula bilan ifodalanadi. Bu yerdagi $\Phi(t,t_0)$ matritsa (VI.2.5) sistemaning normalangan fundamental matritsasi, ya'ni uning ustunlari (VI.2.5)ning n dona chiziqli erkli yechimlaridan tashkil topgan va $\Phi(t_0,t_0) = E$ – birlik matritsa.

(VI.2.6) formuladan ravshanki, ixtiyoriy λ va μ sonlar va ixtiyoriy $a \in \mathbb{R}^n$ va $b \in \mathbb{R}^n$ vektorlar uchun

$$x(t;t_0,\lambda a + \mu b) = \lambda x(t;t_0,a) + \mu x(t;t_0,b).$$

Bu formula (VI.2.5) sistemaning $x = x(t;t_0,x^0)$ yechimi x^0 boshlang'ich qiymatga nisbatan chiziqli funksiya ekanligini anglatadi.

Teorema 2. *Chiziqli bir jinsli tenglamalar sistemasi* (VI.2.5) *ning turg'un bo'lishi uchun uning biror (va, demak, har qanday) fundamental matritsasining ixtiyoriy* $[t_0;+\infty)$ *oraliqda chegaralangan bo'lishi yetarli va zarurdir.*

⊶ (VI.2.5)ning ixtiyoriy fundamental matritsasi normalangan $\Phi(t,t_0)$ fundamental matritsani biror teskarilanuvchi o'zgarmas matritsaga ko'paytirishdan hosil bo'ladi. Shuning uchun (VI.2.5) ning fundamental matritsalarining barchasi bir vaqtda yo chegaralangan yoki chegaralanmagan.

Aytaylik, $\Phi(t,t_0)$ fundamental matritsa chegaralangan bo'lsin, ya'ni

$$\exists m > 0\ \forall t \ge t_0 \quad \| \Phi(t,t_0) \| \le m. \quad \text{(VI.2.7)}$$

$x(t;t_0,x^0)$ yechim uchun (VI.2.6) formuladan (VI.2.7) ga ko'ra

$$\| x(t;t_0,x^0) \| = \| \Phi(t,t_0)x^0 \| \le \| \Phi(t,t_0) \| \cdot \| x^0 \| \le m \| x^0 \|.$$

Demak, $\forall \varepsilon > 0$ soni uchun $\delta = \varepsilon / m$ desak, u holda

$\| x^0 \| < \delta$ ekanligidan $\forall t \ge t_0$ uchun

$$\| x(t;t_0,x^0) \| \le m \| x^0 \| < m\delta = \varepsilon$$

bo'lishi kelib chiqadi. Bu esa (VI.2.5) ning trivial yechimining turg'unligini anglatadi. Demak, (VI.2.5) sistema ham turg'un.

Endi faraz qilaylik, (VI.2.5) sistema turg'un bo'lsin. Demak, xususan, uning trivial yechimi ham turg'un. Shuning uchun $\varepsilon = 1$ songa ko'ra shunday $\delta_0 > 0$ topamizki, $\| x^0 \| < \delta_0$ bo'lganda (VI.2.5) ning $x(t;t_0,x^0)$ yechimi uchun

$$\forall t \ge t_0 \text{ paytda } \| x(t;t_0,x^0) \| < 1 \qquad \text{(VI.2.8)}$$

bo'ladi. $e^1, e^2, \ldots, e^n$ vektorlar $\mathbb{R}^n$ ning standart bazisi bo'lsin. Ularga ko'ra qurilgan $x(t;t_0,e^j)$, $j = \overline{1,n}$, yechimlar (VI.2.5) ning fundamental sistemasini tashkil etadi. Yechimning boshlang'ich qiymatga nisbatan chiziqlilik xossasidan

$$x(t;t_0,(\delta_0/2)e^j) = (\delta_0/2)x(t;t_0,e^j), \ j = \overline{1,n}.$$

Bundan $\| (\delta_0/2)e^j \| = \delta_0/2 < \delta_0$ bo'lgani uchun (VI.2.8) ga ko'ra

$$\| (\delta_0/2)x(t;t_0,e^j) \| < 1,$$

ya'ni

$$\| x(t;t_0,e^j) \| < 2/\delta_0, \ t \ge t_0, \ j = \overline{1,n}.$$

Shunday qilib, qurilgan bazis yechimlar chegaralangan. Demak, ulardan tuzilgan fundamental matritsa ham chegaralangan. ♘

Teorema 3. *Chiziqli bir jinsli differensial tenglamalar sistemasi* (VI.2.5) *ning asimptotik turg'un bo'lishi uchun uning biror (va, demak ixtiyoriy) fundamental matritsasining $t \to +\infty$ dagi limiti nol-matritsadan iborat bo'lishi yetarli va zarurdir.*

⊸ Faraz qilaylik, nol yechim asimptotik turg'un bo'lsin. Demak, $\| x^0 \| < \delta_0$ boshlang'ich qiymatlar uchun

$$x(t;t_0,x^0) \underset{t\to+\infty}{\to} 0. \qquad \text{(VI.2.9)}$$

Biz biror fundamental matritsaning nolga intilishini ko'rsatishimiz kifoya. Ushbu

$$x(t;t_0,(\delta_0/2)e^j) = (\delta_0/2)x(t;t_0,e^j), \ j = \overline{1,n},$$

yechimlarni qaraylik. Ularning boshlangich qiymatlari $\| (\delta_0/2)e^j \| < \delta_0$ bo'lgani uchun (VI.2.9) farazimizga ko'ra

$$x(t;t_0,(\delta_0/2)e^j) \underset{t\to+\infty}{\to} 0$$

Demak, ana shu $x(t;t_0,(\delta_0/2)e^j)$, $j = \overline{1,n}$, yechimlardan tuzilgan fundamental matritsa nolga intiladi.

Endi faraz qilaylik, biror fundamental matritsa $t \to +\infty$ da nolga intilsin. Demak, $\Phi(t,t_0)$ normalangan fundamental matritsa ham nolga intiladi:

$$\| \Phi(t,t_0) \| \underset{t\to+\infty}{\to} 0.$$

Ixtiyoriy $x(t;t_0,x^0)$ yechim uchun

$$\| x(t;t_0,x^0) \| = \| \Phi(t,t_0)x^0 \| \le \| \Phi(t,t_0) \| \cdot \| x^0 \| \underset{t\to+\infty}{\to} 0.$$

Demak, $x(t;t_0,x^0) \underset{t\to+\infty}{\to} 0$. Bundan esa nol-yechimning asimptotik turg'unligi kelib chiqadi. ♘

Endi chiziqli o'zgarmas koeffitsientli sistemalarning turg'unligini o'rganamiz.

O'zgarmas koeffitsientli ushbu

$$x' = Ax \qquad \text{(VI.2.10)}$$

sistemani qaraylik, bunda A – haqiqiy sonlardan tuzilgan $n \times n$ matritsa, ya'ni $A \in M_{n\times n}(\mathbb{R})$. Bu sistemaning turg'unligi (noturg'unligi) A maritsaning xos sonlari bilan aniqlanadi.

Teorema 4.

1^0. *Agar xos sonlarning hammasi manfiy haqiqiy qismlarga ega bo'lsa, u holda* (VI.2.10) *sistema asimptotik turg'un bo'ladi.*

2^0. *Agar xos sonlarning barchasi nomusbat haqiqiy qismlarga ega bo'lib, haqiqiy qismi nol bo'lgan xos sonlarga faqat 1- tartibli Jordan kataklari mos kelsa, u holda* (VI.2.10) *sistema turg'un bo'ladi.*

3^0. *Agar xos sonlarning birortasi musbat haqiqiy qismga ega* *bo'lsa, yoki haqiqiy qismi nol bo'lgan xos sonlarning birortasiga kamida ikkinchi tartibli Jordan katagi mos kelsa, u holda* (VI.2.10) *sistema noturg'un bo'ladi.*

🗝 A matritsaning (turli) xos sonlarini $\lambda_1, \lambda_2, \ldots, \lambda_s$ $(s \le n)$ bilan, $\tilde{k}_j$ bilan esa λ_j ga mos kelgan Jordan kataklarining eng katta tartibini belgilaylik. λ_j $(j = \overline{1,s})$ xos sonlarning haqiqiy va mavhum qismlarini ajrataylik: $\lambda_j = \alpha_j + i\beta_j$. U holda fundamental matritsaning elementlari

$$\sum_{j=1}^{s} e^{\alpha_j t}\left(p_j(t)\cos(\beta_j t) + q_j(t)\sin(\beta_j t)\right) \qquad \text{(VI.2.11)}$$

ko'rinishda yoziladi, bunda $p_j(t)$ va $q_j(t)$ ko'phadlarning darajalari $\tilde{k}_j - 1$ dan kichik yoki unga teng.

Analizdan ma'lumki, ixtiyoriy $\alpha < 0$ son va ixtiyoriy $p(t)$ ko'phad uchun $\lim\limits_{t \to +\infty} p(t)e^{\alpha t} = 0$. Demak, 1^0 holda barcha $\alpha_j = \mathrm{Re}\lambda_j < 0$ bo'lgani uchun

$$\left|\sum_{j=1}^{s} e^{\alpha_j t}\left(p_j(t)\cos(\beta_j t) + q_j(t)\sin(\beta_j t)\right)\right| \le$$

$$\le \sum_{j=1}^{s} |p_j(t)|\, e^{\alpha_j t} + \sum_{j=1}^{s} |q_j(t)|\, e^{\alpha_j t} \xrightarrow[t \to +\infty]{} 0,$$

ya'ni fundamental matritsaning hamma elementlari $t \to +\infty$ da nolga intiladi. Shuning uchun 1^0 holda (VI.2.10) sistema asimptotik turg'un.

2^0 holda (VI.2.11) dagi $\alpha_j = \mathrm{Re}\lambda_j < 0$ sonlarga mos keluvchi qo'shiluvchilar ixtiyoriy $[t_0; +\infty)$ oraliqda chegaralangan (1^0 holdagi singari), $\alpha_j = \mathrm{Re}\lambda_j = 0$ sonlarga mos keluvchi qo'shiluvchilar ham chegaralangan, chunki ular nolinchi darajali ko'phadlardan (o'zgarmaslardan) iborat. Demak, fundamental

matritsaning barcha elementlari $[t_0; +\infty)$ da chegaralangan va (VI.2.10) sistema turg'un.

Endi 3^0 holni qaraylik. Agar biror $\alpha_j = \mathrm{Re}\lambda_j > 0$ bo'lsa, u holda (VI.2.11) dagi shu songa mos kelgan qo'shiluvchilar va, demak, fundamental matritsa ham $[t_0; +\infty)$ da chegaralanmagan. Agar $\alpha_j = \mathrm{Re}\lambda_j = 0$ va $\tilde{k}_j \ge 2$ bo'lsa, u holda (VI.2.11) dagi shu songa mos kelgan

$$e^{\alpha_j t}\left(p_j(t)\cos(\beta_j t) + q_j(t)\sin(\beta_j t)\right) = p_j(t)\cos(\beta_j t) + q_j(t)\sin(\beta_j t)$$

qo'shiluvchida $\deg p_j(t) \ge 1$, $\deg q_j(t) \ge 1$ bo'lgani uchun u $[t_0; +\infty)$ da chegaralanmagan. Demak, fundamental matritsa ham chegaralanmagan. Shuning uchun (VI.2.10) sistema turg'un emas. 🖐

Izoh. Teoremada A matritsaning o'zgarmas ekanligi muhim. Quyidagi misol bu tasdiqni asoslaydi. Ushbu

$$\begin{cases} x' = (-3 + 4\cos^2 3t)x + (-3 + 2\sin 6t)y \\ y' = (3 + 2\sin 6t)x + (-3 + 4\sin^2 3t)y \end{cases}$$

o'zgaruvchi koeffitsientli sistemani qaraylik. Bu sistemaning matritsasi:

$$A(t) = \begin{pmatrix} -3 + 4\cos^2 3t & -3 + 2\sin 6t \\ 3 + 2\sin 6t & -3 + 4\sin^2 3t \end{pmatrix}.$$

Uning xos sonlari:

$$\lambda_{1,2} = -1 \pm i\sqrt{5} \quad (t \text{ ga bog'liq emas}).$$

Osongina tekshirib ko'rish mumkinki, qaralayotgan sistema

$$\begin{cases} x = ce^t \cos 3t \\ y = ce^t \sin 3t \end{cases} \quad (c \ne 0)$$

ko'rinishdagi yechimga ega. Bu yechim $\mathrm{Re}\lambda_{1,2} < 0$ bo'lishiga qaramasdan $t \to +\infty$ da chegaralanmagan. Bundan qaralayotgan sistema trivial yechimining turg'un emasligi kelib chiqadi. Shunday qilib, umumiy holda koeffitsientlari o'zgaruvchi bo'lgan chiziqli sistema matritsasining xos sonlari uning turg'unligini aniqlamaydi.

Misol 1. Ushbu

$$\begin{cases} \dot{x} = 2y - z \\ \dot{y} = 3x - 2z \\ \dot{z} = 5x - 4y \end{cases}$$

sistemani turg'unlikka tekshiraylik.

🔑 Sistemaning xarakteristik tenglamasi

$$\begin{vmatrix} -\lambda & 2 & -1 \\ 3 & -\lambda & -2 \\ 5 & -4 & -\lambda \end{vmatrix} = -\lambda^3 + 9\lambda - 8 = 0.$$

Ravshanki, $\lambda = 1 > 0$ bu tenglamaning ildizi. Demak, (boshqa xarakteristik sonlarning qiymatlaridan qat'iy nazar) berilgan sistema noturg'un. ♘

Misol 2. Ushbu

$$x' = -x + y - z,\;\; y' = 2x - \frac{1}{3}y - \frac{7}{3}z,\;\; z' = x + \frac{4}{3}y - \frac{8}{3}z$$

sistemani turg'unlikka tekshiring.

🔑 Sistemaning xarakteristik sonlari

$$\begin{vmatrix} -1-\lambda & 1 & -1 \\ 2 & -\frac{1}{3}-\lambda & -\frac{7}{3} \\ 1 & \frac{4}{3} & -\frac{8}{3}-\lambda \end{vmatrix} = 0 \;\Rightarrow\; \lambda_1 = -2, \lambda_{2,3} = -1 \pm i.$$

Barcha xarakteristik sonlarning haqiqiy qismi manfiy bo'lgani uchun sistema asimptotik turg'un. ♘

Ushbu

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda^1 + a_0 = 0,\; a_n > 0, \qquad \text{(VI.2.12)}$$

haqiqiy koeffitsientli algebraik tenglama ildizlarining haqiqiy qismi manfiy bo'lishini aniqlash uchun foydalaniladigan mezonni isbotsiz

keltiramiz. Dastlab ushbu

$$\begin{vmatrix} a_n & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{n-2} & a_{n-1} & a_n & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & 0 & \cdots & 0 & 0 \\ a_{n-6} & a_{n-5} & a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} & a_n & \cdots & 0 & 0 \\ & & & & \ddots & & & & & \\ & & & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_0 \end{vmatrix}$$

determinantni tuzaylik. Uning bosh diagonalida (VI.2.12) ko'phadning $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ koeffitsientlari, satrlarida esa a_j lar indeksning o'sish tartibida joylashgan bo'lib, bunda $j < 0$ yoki $j > n$ indekslar uchun $a_j = 0$ deb hisoblanadi. Bu determinantning bosh diagonal minorlarini

$$\Delta_n = a_n,\; \Delta_{n-1} = \begin{vmatrix} a_n & 0 \\ a_{n-2} & a_{n-1} \end{vmatrix},\; \Delta_{n-2} = \begin{vmatrix} a_n & 0 & 0 \\ a_{n-2} & a_{n-1} & a_n \\ a_{n-4} & a_{n-3} & a_{n-2} \end{vmatrix},$$

$$\Delta_{n-3} = \begin{vmatrix} a_n & 0 & 0 & 0 \\ a_{n-2} & a_{n-1} & a_n & 0 \\ a_{n-4} & a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-6} & a_{n-5} & a_{n-4} & a_{n-3} \end{vmatrix}, \;\ldots$$

bilan belgilaylik.

Teorema (L'enar–Shipar mezoni). (VI.2.12) *tenglama barcha ildizlarining mavhum qismlari manfiy bo'lishi uchun ushbu*

1) *barcha a_j lar musbat, ya'ni*

2) $a_n > 0,\; a_{n-1} > 0,\; a_{n-2} > 0,\; \ldots,\; a_1 > 0,\; a_0 > 0$;

1) $\Delta_{n-2} > 0,\; \Delta_{n-4} > 0,\; \Delta_{n-6} > 0,\; \ldots$

shartlarning bir vaqtda bajarilishi yetarli va zarurdir.

VI.3. Lyapunov funksiyalari yordamida turg'unlikka tekshirish

Bu bandda (paragrafda) ushbu

$$x' = f(t,x),\ f(t,0) \equiv 0,\ t \ge 0, \qquad \text{(VI.3.1)}$$

sistemaning $x(t) \equiv 0$ yechimini (muvozanat holatini) turg'unlikka tekshirishda Lyapunovning to'g'ri metodi, ya'ni Lyapunov funksiyalaridan foydalanish bilan tanishamiz. (VI.3.1) sistemada $f \in C(\mathbb{R}_+ \times B_\rho; \mathbb{R}^n)$, bunda $\mathbb{R}_+ = [0,+\infty)$, $B_\rho = \{x \in \mathbb{R}^n \mid \| x \| < \rho\})\}$ – nolning ρ radiusli atrofi ($\rho > 0$) va $f(t,x)$ vektor-funksiya x bo'yicha lokal Lipshits shartini qanoatlantiradi deb hisoblanadi.

Bir misoldan boshlaylik. m massali ($m > 0$) moddiy nuqta x lar o'qida harakat qilsin va u x nuqtada bo'lganda unga uni koordinatalar boshiga qaytaruvchi $F_{el} = -kx$ ($k = \text{const} > 0$) elastiklik kuchi ta'sir etsin. Nuqtaning harakat tenglamasi, Nyutonning ikkinchi qonuniga ko'ra,

$$mx'' + kx = 0$$

ko'rinishdagi garmonik ossilyator tenglamasidan iborat bo'ladi. Harakatdagi nuqtaning to'la mexanik energiyasi v uning $k\dfrac{x^2}{2}$ potensial va $m\dfrac{x'^2}{2}$ kinetik energiyalarining yig'indisidan iborat, y'ani $v = k\dfrac{x^2}{2} + m\dfrac{x'^2}{2}$. Harakat tenglamasini normal sistema ko'rinishiga o'tkazsak,

$$x' = y,\ y' = -\frac{k}{m}x$$

hosil bo'ladi. Harakat davomida $v = v(x,y) = k\dfrac{x^2}{2} + m\dfrac{y^2}{2}$ to'la energiya, ma'lumki, saqlanadi. Endi faraz qilaylik, harakatlanuvchi

nuqtaga $F_q = -\mu(x,x')x'$ ($\mu = \mu(x,x') \ge 0,\ \mu \in C^1$) qarshilik kuchi ta'sir etsin. U holda harakat tenglamasi

$$\begin{cases} x' = y \\ y' = -\dfrac{k}{m}x - \dfrac{\mu(x,y)}{m}y \end{cases}$$

ko'rinishda ifodalanadi. Ixtiyoriy $x = x(t)$, $y = y(t)$ harakatni qaraylik. Shu harakat davomida to'la mexanik energiya $v(t) = v(x(t),y(t)) = k\dfrac{x^2(t)}{2} + m\dfrac{y^2(t)}{2}$ bog'lanish bo'yicha o'zgaradi. Uning o'zgarish tezligi

$$\frac{dv(t)}{dt} = \frac{\partial v(x(t),y(t))}{\partial x}\frac{dx(t)}{dt} + \frac{\partial v(x(t),y(t))}{\partial y}\frac{dy(t)}{dt} =$$

$$= kx(t)x'(t) + my(t)y'(t) =$$

$$= kx(t)y(t) + my(t)\Big(-\frac{k}{m}x(t) - \frac{\mu(x(t),y(t))}{m}y(t)\Big) =$$

$$= -\mu(x(t),y(t))\cdot y^2(t) \le$$

$$\le 0.$$

Demak, harakat davomida v to'la energiya ortmaydi, ya'ni barcha $t \ge 0$ paytlarda $v(t) = k\dfrac{x^2(t)}{2} + m\dfrac{y^2(t)}{2} \le v|_{t=0}$ bo'ladi. Bundan $x = x(t)$, $y = y(t)$ yechimning (harakatning) chegaralanganligi va barcha $t \ge 0$ paytlarda mavjudligi kelib chiqadi. v energiyaning ortmaganligi va quyidan nol bilan chegaralanganligi uchun $\lim\limits_{t\to+\infty} v(t) = r \ge 0$ mavjud. Agar $r > 0$ bo'lsa $x = x(t)$, $y = y(t)$ harakat vaqt o'tishi bilan $k\dfrac{x^2}{2} + m\dfrac{y^2}{2} = r$ ellipsga yaqinlashadi ($\lim\limits_{t\to+\infty} x(t)$ va $\lim\limits_{t\to+\infty} y(t)$ limitlar mavjud, chunki $x'(t)$ va $y'(t)$ lar chegaralangan) $r = 0$ bo'lganda esa $x(t)$ va $y(t)$ lar nolga intiladi. Bu yerda shuni ta'kidlaylikki, qaralgan $v = v(x,y)$

funksiya (to'la mexanik energiya) yechimlarning tabiatini ochishga yordam berdi. Tekshirilgan misolning juda ham uzoqqa boruvchi umumlashishi Lyapunovning ikkinchi metodini tashkil etadi. Bu metodda yechimlarning xususiyatlari Lyapunov funksiyalari deb ataluvchi (misoldagi $v = v(x, y)$ ga o'xshash) funksiyalar orqali o'rganiladi. Lyapunov bu metodini o'zining 1982 yilda yozgan doktorlik dissertatsiyasida bayon qilgan. Hozirgi zamon turg'unlik nazariyasi ana shu ishdan boshlangan deb hisoblanadi.

Biror $v(t, x) = v(t, x_1, x_2, \ldots, x_n)$, $v(t, x) \in C^1(\mathbb{R}_+ \times B_\rho)$, funksiya berilgan bo'lsin. Bu funksiya (VI.3.1) sistemaning $x = x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ yechimida t o'zgaruvchining ushbu $v(t, x(t)) = v(t, x_1(t), x_2(t), \ldots, x_n(t))$ funksiyasiga aylanadi. Uning hosilasi

$$\frac{d}{dt} v(t, x_1(t), x_2(t), \ldots, x_n(t)) = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x_1} \cdot \frac{dx_1}{dt} + \cdots + \frac{\partial v}{\partial x_n} \cdot \frac{dx_n}{dt} =$$

$$= \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x_1} \cdot f_1 + \cdots + \frac{\partial v}{\partial x_n} \cdot f_n .$$

formula bilan hisoblanadi. Shundan kelib chiqib, $v(t, x)$ **funksiyaning** (VI.3.1) **sistemaga ko'ra hosilasi** deb ushbu

$$\left.\frac{dv}{dt}\right|_{(1)} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x_1} \cdot f_1 + \frac{\partial v}{\partial x_2} \cdot f_2 + \cdots + \frac{\partial v}{\partial x_n} \cdot f_n \quad \text{(VI.3.2)}$$

funksiyaga aytiladi. Agar v funksiya t ga bog'liq bo'lmay, faqat x ga bog'liq, ya'ni $v = v(x)$ bo'lsa, bu funksiyaning (VI.3.1) sistemaga ko'ra hosilasi

$$\left.\frac{dv}{dt}\right|_{(1)} = \frac{\partial v}{\partial x_1} \cdot f_1 + \frac{\partial v}{\partial x_2} \cdot f_2 + \cdots + \frac{\partial v}{\partial x_n} \cdot f_n$$

yoki

$$\left.\frac{dv}{dt}\right|_{(1)} = \text{grad} v \cdot f, \quad \text{grad} v = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \ldots, \frac{\partial v}{\partial x_n}\right)$$

formula bilan aniqlanadi.

Agar $v(x)$ (t ga bog'liq bo'lmagan) funksiya $x = 0$ nuqtaning biror $B_{\tilde\rho}$ ($\tilde\rho > 0$) atrofida C^1 sinfga tegishli, $v(0) = 0$ va shu atrofdagi barcha $x \neq 0$ nuqtalarda $v(x) > 0$ bo'lsa, u holda bu $v(x)$ funksiyani **aniq musbat funksiya** deymiz va buni $v(x) \succ 0$ kabi ifodalaymiz.

Masalan, $v(x) = \|x\|^2$ yoki umumiyroq $v(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2$, bunda $\lambda_1 > 0, \lambda_2 > 0, \ldots, \lambda_n > 0$, funksiya (kvadratik forma) aniq musbatdir. Lekin $v(x, y) = (x - y)^2$ funksiya nomanfiy bo'lsada, aniq musbat emas.

Tushunarliki, $v(x)$ aniq musbat funksiya $x = 0$ nuqtada minimumga ega. Demak, uning shu nuqtadagi xususiy hosilalari nolga teng:

$$\frac{\partial v(0)}{\partial x_1} = \frac{\partial v(0)}{\partial x_2} = \cdots = \frac{\partial v(0)}{\partial x_n} = 0 .$$

Faraz qilaylik, $v(x)$ funksiya $x = 0$ nuqtadaning biror atrofida C^2 sinfga tegishli, hamda

$$v(0) = 0, \text{ va } \frac{\partial v(0)}{\partial x_1} = \frac{\partial v(0)}{\partial x_2} = \cdots = \frac{\partial v(0)}{\partial x_n} = 0$$

bo'lsin. U holda Teylor formulasiga ko'ra

$$v(x) = \frac{1}{2}\sum_{k,j=1}^{n} a_{kj} x_k x_j + \alpha(x)\|x\|^2, \quad a_{kj} = \frac{\partial^2 v(0)}{\partial x_k \partial x_j}, \quad \alpha(x) \xrightarrow[x \to 0]{} 0.$$

Bu formuladan ravshanki, agar $\frac{1}{2}\sum_{k,j=1}^{n} a_{kj} x_k x_j$ kvadratik forma aniq musbat, ya'ni $\frac{1}{2}\sum_{k,j=1}^{n} a_{kj} x_k x_j \geq \lambda_0 \|x\|^2$ ($\lambda_0 > 0$) bo'lsa, u holda $v(x)$ funksiya ham aniq musbat bo'ladi. Agar bu kvadratik forma nomanfiy bo'lsa, $v(x)$ning aniq musbatligini yoyilmadagi yuqori tartibli had, ya'ni $\alpha(x)\|x\|^2$ aniqlaydi.

Kvadratik formani aniq musbatlikka tekshirish uchun algebradan ma'lum bo'lgan **Sil'vestr mezonidan** foydalanish mumkin. Bu mezonga ko'ra $\sum_{k,j=1}^{n} a_{kj}x_k x_j$ kvadratik forma aniq musbat bo'lishi uchun uning matritsasining ushbu

$$\Delta_1 = a_{11},\ \Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},\ \ldots,\ \Delta_n = \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix}$$

barcha bosh diagonal minorlari musbat bo'lishi yetarli va zarurdir.

Misol 1. Ushbu $v = v(x,y) = 1 + x^2 + x^3 - \cos(x-y)$ funksiyani aniq musbatlikka tekshiraylik.

Teylor formulasiga ko'ra

$$v = \frac{1}{2}(3x^2 - 2xy + y^2) + \ldots,$$

bunda ... bilan yuqori tartibli hadlar belgilangan. En $3x^2 - 2xy + y^2$ kvadratik formaning matritsasini tuzib, uning bo diagonal minorlarini hisoblaymiz:

$$\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix},\ \Delta_1 = 3 > 0,\ \ \Delta_2 = \begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} = 2 > 0.$$

Demak, Sil'vestr mezoniga ko'ra $3x^2 - 2xy + y^2$ kvadratik forma, va, demak, berilgan $v(x,y) = 1 + x^2 + x^3 - \cos(x-y)$ funksiya ham aniq musbat.

Izoh. $3x^2 - 2xy + y^2$ kvadratik formaning aniq musbatligini Sil'vestr mezonisiz ham asoslash mumkin. Buning uchun bu kvadratik formadan to'la kvadrat ajratish kifoya: $3x^2 - 2xy + y^2 = (x-y)^2 + 2x^2$.

Endi (VI.3.1) sistemaning $x(t) \equiv 0$ yechimini turg'unlikka tekshirishda ishlatiladigan ba'zi teoremalar bilan tanishamiz.

Teorema (Lyapunovning turg'unlik haqidagi teoremasi). *Agar* (VI.3.1) *sistema uchun ushbu*

$$v(x) \succ 0 \ \ va \ \ \left.\frac{dv}{dt}\right|_{(1)} \le 0$$

shartlarni qanoatlantiruvchi $v(x)$ *(aniq musbat va* (VI.3.1) *sistemaga ko'ra hosilasi noldan kichik yoki unga teng) funksiya mavjud bo'lsa, u holda* (VI.3.1) *sistemaning* $x(t) \equiv 0$ *yechimi turg'un bo'ladi.*

Yetarlicha kichik ixtiyoriy $\varepsilon > 0$ ($\varepsilon < \rho$, $\varepsilon < \tilde{\rho}$) son berilgan bo'lsin. $S = \{x \in \mathbb{R}^n \mid \|x\| = \varepsilon\}$ sferada (S – kompakt) $v(x)$ funksiya uzluksiz. Demak, u S da o'zining eng kichik qiymatiga biror $x_* \in S$ nuqtada erishadi: $\min_{x\in S} v(x) = v(x_*) = m$. $v(x) \succ 0$ bo'lganligi uchun $m = v(x_*) > 0$. $v(x)$ funksiya uzluksiz va $v(0) = 0$ bo'lganligi uchun esa uzluksizlik ta'rifiga ko'ra shunday $\delta > 0$ ($\delta < \varepsilon$) topamizki, $\|x\| < \delta$ ekanligidan $v(x) < m$ tengsizlik kelib chiqadi. Endi topilgan δ ning turg'unlik ta'rifidagi δ bo'lib xizmat qilishini ko'rsatamiz. Buning uchun (VI.3.1) sistemaning boshlang'ich qiymati $|x^0| = |x(t_0)| < \delta$ shartni qanoatlantiruvchi ixtiyoriy $x = x(t) = x(t; t_0, x^0)$ yechimini qaraylik. Bu yechim bo'ylab $v(x)$ funksiya o'smaydi, chunki berilganga ko'ra

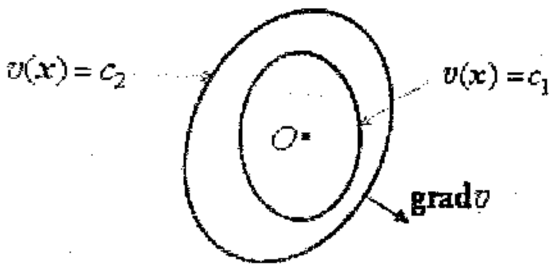
$$\frac{dv(x(t))}{dt} = \left.\frac{dv}{dt}\right|_{(1)} \le 0.$$

Demak, barcha $t \ge t_0$ paytlar uchun

$$v(x(t)) \le v(x(t_0)) = v(x^0) < m,$$

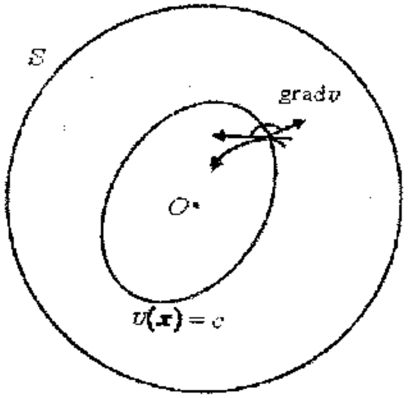
chunki $\|x^0\| < \delta \Rightarrow v(x^0) < m$. Shunday qilib, $x = x(t)$ yechim hech qachon S sferaga yetib borolmaydi, chunki bu sferada $v(x) \ge m$. Demak, $x = x(t)$ yechim o'ngga cheksiz davom etadi va barcha $t \geq t_0$ lar uchun $\| x(t) \| < \varepsilon$ ham bo'ladi. ♁

Turg'unlik haqidagi teoremaning geometrik ma'nosini ochaylik. $v(x)$ aniq musbat funksiyaning $v(x) = c$ ($c > 0$ va yetarli kichik) sath to'plami $x = 0$ nuqtani qurshab oluvchi yopiq sirtdan iborat. $0 < c_1 < c_2$ bo'lganda $v(x) = c_1$ sirt $v(x) = c_2$ sirt ichida joylashadi (VI.4- rasm).



VI.4-rasm. $v(x)$ aniq musbat funksiyaning sath to'plamlari

Ma'lumki, grad$v$ vektori $v(x) = c$ sirtga perpendikulyar va $v(x)$ funksiyaning o'sish yo'nalishini ko'rsatadi. $\left.\frac{dv}{dt}\right|_{(1)} = \text{grad}v \cdot f \leq 0$ shart grad$v$ normal vektor va f tezlik vektorining orasidagi burchak o'tmas yoki $\pi/2$ ga teng ekanligini anglatadi. Bu esa traektoriya $v(x) = c$ sirtning ichiga qarab kirishini yoki shu sirtda qolishini bildiradi. Demak, $S = \{x \in \mathbb{R}^n \mid \| x \| = \varepsilon\}$ sferaning ichida joylashgan $v(x) = c$ sirtdan boshlangan yechim shu sirtdan tashqariga chiqib ketolmaydi, ya'ni u $S = \{x \in \mathbb{R}^n \mid \| x \| = \varepsilon\}$ sferanining ichida qoladi (VI.5-rasm). Boshqacha qilib aytsak, (VI.3.1) sistemaning nol-yechimi turg'un.



VI.5-rasm. Harakat $v(x) = c$ sirtning ichi tomonga yo'nalgan

Isbotlangan teoremadagi ikkinchi shartni kuchaytirib, asimptotik turg'unlik haqidagi teoremani hosil qilish mumkin.

Teorema (Lyapunovning asimptotik turg'unlik haqidagi teoremasi). *Aytaylik,* (VI.3.1) *sistema uchun nol nuqtaning biror atrofida ushbu*

$$v(x) \succ 0 \quad va \quad \left.\frac{dv}{dt}\right|_{(1)} \leq -w(x) < 0, \; x \neq 0,$$

shartlarni qanoatlantiruvchi $v(x)$ *va uzluksiz* $w(x)$ *funksiyalar mavjud bo'lsin. U holda* (VI.3.1) *sistemaning* $x(t) \equiv 0$ *yechimi asimptotik turg'un bo'ladi.*

⊶ Bundan oldingi teoremaning isbotida yetarlicha kichik ixtiyoriy $\varepsilon > 0$ son uchun shunday $\delta > 0$ topdikki, boshlang'ich qiymati $|x^0| = |x(t_0)| < \delta$ shartni qanoatlantiruvchi ixtiyoriy $x = x(t)$ yechim uchun barcha $t \geq t_0$ paytlarda $\| x(t) \| < \varepsilon$ bo'ldi, ya'ni (VI.3.1) sistemaning $x(t) \equiv 0$ yechimi turg'un. Endi biror ε va unga mos δ ni tayinlab, boshlang'ich qiymati $|x^0| = |x(t_0)| < \delta$ shartni qanoatlantiruvchi barcha $x = x(t)$ yechimlar uchun $\lim_{t \to +\infty} x(t) = 0$ ham bo'lishini ko'rsatamiz va $x(t) \equiv 0$ yechimning asimptotik turg'unligini isbotlaymiz.

Dastlab aytilgan har qanday $x = x(t)$, $|x(t_0)| < \delta$, yechim uchun $\lim_{t\to+\infty} v(x(t)) = 0$ ekanligini ko'rsatamiz. Berilganga ko'ra

$$v(x(t)) \geq 0, \quad \frac{dv(x(t))}{dt} = \frac{dv}{dt}\bigg|_{(1)} \leq 0 \quad \text{bo'lgani uchun} \quad v(x(t))$$

funksiya keng ma'noda kamayuvchi va chekli $\lim_{t\to+\infty} v(x(t)) = r$, $r \geq 0$, limitga ega. Biz $r > 0$ bo'lolmasligini isbotlaymiz. Teskarisini faraz qilaylik, ya'ni $r > 0$ bo'lsin. U holda limitning ta'rifiga ko'ra shunday t_* topamizki, barcha $t \geq t_*$ lar uchun $v(x(t)) > r/2$ bo'ladi. Endi $v(x)$ ning uzluksizligiga ko'ra shunday $\delta_* > 0$ topamizki, $\|x\| < \delta_*$ bo'lganda $v(x) < r/2$ tengsizlik bajariladi. Demak, $t \geq t_*$ bo'lganda $\|x(t)\| \geq \delta_*$ tengsizlik o'rinli bo'ladi. Shunday qilib, $t \geq t_*$ paytlarda $x = x(t)$ yechim $\delta_* \leq \|x\| \leq \varepsilon$ kompaktda joylashdi. Berilganga ko'ra $w(x)$ uzluksiz funksiya uchun $x \neq 0$ da $w(x) > 0$. Demak, $\delta_* \leq \|x\| \leq \varepsilon$ kompaktda $w(x) \geq \min_{\delta_* \leq \|x\| \leq \varepsilon} w(x) \geq \beta > 0$. Yana berilganga ko'ra

$$v(x(t)) - v(x(t_*)) = \int_{t_*}^{t} \frac{dv(x(s))}{ds} ds \leq -\beta(t - t_*) \xrightarrow[t\to+\infty]{} -\infty,$$

ya'ni $v(x(t)) \xrightarrow[t\to+\infty]{} -\infty$; bu esa barcha $t \geq t_*$ lar uchun o'rinli bo'lgan $v(x(t)) > r/2$ tengsizlikka zid. Shunday qilib, farazimiz noto'g'ri va $r = 0$, ya'ni $\lim_{t\to+\infty} v(x(t)) = 0$.

Endi $\lim_{t\to+\infty} v(x(t)) = 0$ munosabatdan $\lim_{t\to+\infty} x(t) = 0$ ekanligini keltirib chiqarish qoldi. Teskarisini faraz qilaylik, ya'ni $x = x(t)$, $|x(t_0)| < \delta$, yechim uchun shunday $\varepsilon_0 > 0$ va $t_1, t_2, \ldots, t_k, \ldots \to +\infty$ topilib, ular uchun $\|x(t_k)\| \geq \varepsilon_0$, $k \in \mathbb{N}$,

tengsizliklar o'rinli bo'lsin. Bunda $\varepsilon_0 < \varepsilon$ bo'lishi kerak, chunki $|x(t_k)\| < \varepsilon$. Ravshanki, $\varepsilon_0 \leq \|x\| \leq \varepsilon$ kompaktda $v(x) \geq \operatorname{minimum}_{\varepsilon_0 \leq \|x\| \leq \varepsilon} v(x) \geq \tilde{m} > 0$. Demak,

$$v(x(t_k)) \geq \tilde{m} > 0, \; k \in \mathbb{N}.$$

Bu esa $\lim_{k\to+\infty} v(x(t_k)) = 0$ bo'lishi kerakligiga zid. ♦

Lyapunov $v(x) \succ 0$ funksiyalar o'rniga umumiyroq $v(t, x)$ funksiyalarni ishlatib, keltirilgan teoremalarga qaraganda kuchliroq teoremalarni isbotlagan. Yechimlarni turg'unlikka tekshirishda ishlatiladigan funksiyalar Lyapunov funksiyalari deb ataladi. Lyapunov funksiyalarini qurishning umumiy metodi mavjud emas. Konkret sistemalar uchun uning tuzilishidan kelib chiqib, Lyapunov funksiyalarini u yoki bu ko'rinishda tanlashga harakat qilish mumkin. Ba'zan Lyapunov funksiyasini kvadratik forma ko'rinishida qurish mumkin bo'ladi.

Misol 2. Ushbu

$$\begin{cases} x' = y + 4x^2y^2 - 4x^5 \\ y' = -x - 2y^3 - 4x^3 y \end{cases} \tag{VI.3.3}$$

sistemaning $x(t) \equiv 0, y(t) \equiv 0$ yechimini (muvozanat holatini) turg'unlikka tekshiraylik.

⊖→ Lyapunov funksiyasi sifatida ushbu

$v = v(x, y) = \frac{1}{4}(x^2 + y^2)$ kvadratik formani tanlaymiz. Uning aniq musbat ekanligi ravshan. v ning berilgan sistemaga ko'ra hosilasi

$$\frac{dv}{dt}\bigg|_{(VI.3.3)} = \frac{1}{4}(2xx' + 2yy') =$$

$$= \frac{1}{2}(x(y + 4x^2y^2 - 4x^5) + y(-x - 2y^3 - 4x^3y)) =$$

$$= -(2x^6 + y^4).$$

Demak, $w(x,y) = 2x^6 + y^4$ deb tanlasak, Lyapunovning asimptotik turg'unlik haqidagi teoremasining barcha shartlari bajariladi. Bu teoremaga ko'ra (VI.3.3) sistemaning $x(t) \equiv 0, y(t) \equiv 0$ yechimi asimptotik turg'un. ♁

Chetayevning quyidagi teoremasi Lyapunovning noturg'un yechim haqidagi teoremasining **muhim** umumlashishidir.

Teorema (Chetayevning noturg'unlik haqidagi teoremasi). *Aytaylik, (VI.3.1) sistema uchun quyidagi shartlarni qanoatlantiruvchi* U *soha va* $v = v(\boldsymbol{x})$ *funksiya (Lyapunov funksiyasi) mavjud bo'lsin:*

1^0. U *soha* $0 \in \mathbb{R}^n$ *nuqtaning biror* B_{ε_0} *atrofida yotadi, ya'ni* $U \subset B_{\varepsilon_0}$ *va* $0 \in \partial U$;

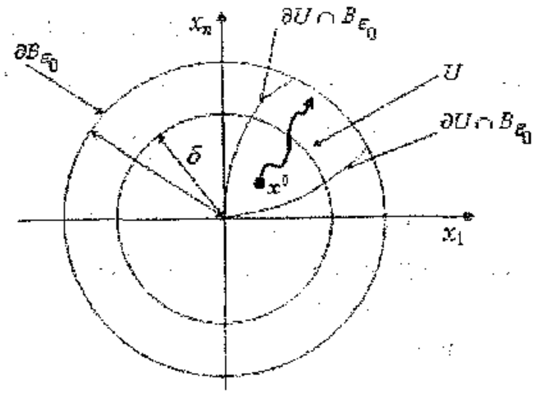
2^0. $v \in C(U \cup \partial U)$, U *sohada* $v > 0$, *lekin* ∂U *ning* B_{ε_0} *dagi qismida* $v(\boldsymbol{x})|_{\partial U \cap B_{\varepsilon_0}} = 0$;

3^0. $v \in C^1(U)$ *va biror* $w \in C(U \cup \partial U)$ *funksiya uchun* $t \in [0;+\infty)$, $x \in U$ *bo'lganda*

$$\left.\frac{dv}{dt}\right|_{(1)} \geq w(\boldsymbol{x}) > 0.$$

U holda (VI.3.1) *sistemaning* $\boldsymbol{x}(t) \equiv 0$ *yechimi noturg'un bo'ladi.*

⊶ Teskarisini faraz qilaylik, ya'ni $\boldsymbol{x}(t) \equiv 0$ yechim turg'un bo'lsin. U holda ta'rifga asosan $\varepsilon_0 > 0$ songa ko'ra shunday $\delta > 0$ topiladiki, boshlang'ich qiymati $\|\boldsymbol{x}^0\| = \|\boldsymbol{x}(t_0)\| < \delta$ shartni qanoatlantiruvchi har qanday $\boldsymbol{x} = \boldsymbol{x}(t) = \boldsymbol{x}(t;t_0,\boldsymbol{x}^0)$ yechim uchun barcha $t \geq t_0$ paytlarda $\|\boldsymbol{x}(t)\| < \varepsilon_0$ ($\boldsymbol{x}(t) \in B_{\varepsilon_0}$) bo'ladi.



VI.6-rasm.

$0 \in \partial U$ bo'lganligi uchun $\boldsymbol{x}^0 = \boldsymbol{x}(t_0) \in U, \|\boldsymbol{x}^0\| < \delta$, tanlab, shunaqa boshlang'ich qiymatli $\boldsymbol{x} = \boldsymbol{x}(t)$ yechimlar uchun barcha $t \geq t_0$ larda ham $\|\boldsymbol{x}(t)\| < \varepsilon_0$ bo'lavermasligini ko'rsatamiz. $\boldsymbol{x}(t) \in U$ tegishlilik saqlangunga qadar teorema shartiga ko'ra $\frac{dv(\boldsymbol{x}(t))}{dt} = \left.\frac{dv}{dt}\right|_{(1)} > 0$, ya'ni $v(\boldsymbol{x}(t))$ o'suvchi bo'ladi. Demak, bunday t lar uchun $v(\boldsymbol{x}(t)) > v(\boldsymbol{x}(t_0)) = v_0 > 0$. Endi $\tilde{U} = \{\boldsymbol{x} \in U \cup \partial U \mid v(\boldsymbol{x}) \geq v_0\}$ to'plamni qaraylik. Bu $\tilde{U}$ to'plam yopiq, chunki $U \cup \partial U$ yopiq, $v(\boldsymbol{x})$ esa uzluksiz bo'lganligi uchun $\tilde{U}$ ning ixtiyoriy y limit nuqtasi uchun $y \in U \cup \partial U$ va $v(y) \geq v_0$, ya'ni $y \in \tilde{U}$. $\tilde{U}$ to'plam chegaralangan hamdir, chunki uning nuqtalari uchun $\|\boldsymbol{x}\| \leq \varepsilon_0$. Demak, $\tilde{U} \subset U \cup \partial U$ – kompakt va $\tilde{U}$ da $w(\boldsymbol{x}) \geq \beta > 0$ hamda $v(\boldsymbol{x})$ yuqoridan chegaralangan. Qaralayotgan $\boldsymbol{x}(t)$ yechim $\tilde{U}$ to'plamning chegarasigacha yetib kelolmaydi: $\partial\tilde{U}$ ning B_{ε_0} dagi qismida $v(\boldsymbol{x}) \geq v_0 > 0$, lekin $v(\boldsymbol{x})|_{\partial U \cap B_{\varepsilon_0}} = 0$; $\partial\tilde{U}$ ning

$\partial B_{\varepsilon_0}$ dagi qismida $\|x\| = \varepsilon_0$, yechim uchun esa barcha $t \geq t_0$ paytlarda $\|x(t)\| < \varepsilon_0$. Teoremaning shartiga ko'ra $x(t)$ yechim uchun $\frac{dv(x(t))}{dt} \geq w(x(t)) \geq \beta > 0$. Bundan $v(x(t)) \geq v(x(t_0)) + \beta(t - t_0) \xrightarrow[t \to +\infty]{} +\infty$. Bu munosaba $v(x)$ ning $\tilde{U}$ da chegaralanganligiga zid. Demak, farazimiz noto'g'ri va teorema isbot bo'ldi. ♘

Misol 3. Ushbu

$$\begin{cases} x' = y - 2x^2 \\ y' = 2xy + y^3 \end{cases}$$

sistemaning $x(t) \equiv 0, y(t) \equiv 0$ yechimini turg'unlikka tekshiraylik.

🗝 U soha sifatida birinchi chorakni olib, $v(x, y) = xy$ funksiyani qaraylik. Ravshanki, $(0,0) \in \partial U$ va U sohaning chegarasida $v(x,0) = v(0, y) = 0$. $v(x, y) = xy$ funksiyaning berilgan sistemaga ko'ra hosilasi U sohada

$$\frac{dv}{dt} = x'y + xy' = (y - 2x^2)y + x(2xy + y^3) = y^2 + xy^3,$$

$$\frac{dv}{dt} = w(x, y) > 0, \ \ w(x, y) = y^2 + xy^3, \ x > 0, \ y > 0.$$

Chetayev teoremasining barcha shartlari bajarildi. Demak, berilgan sistemaning $x(t) \equiv 0, y(t) \equiv 0$ yechimi noturg'un. ♘

VI.4. Birinchi yaqinlashishga ko'ra turg'unlik

(VI.3.1) sistema ushbu

$$x' = Ax + g(t, x) \qquad \text{(VI.4.1)}$$

ko'rinishda bo'lsin, bunda $A \in \mathbb{M}_{n\times n}(\mathbb{R})$, $g \in C(\mathbb{R}_+ \times B_p; \mathbb{R}^n)$, $g(t, x)$ vektor-funksiya x bo'yicha lokal Lipshits shartini qanoatlantiradi va $\|g(t, x)\| \leq \alpha(x)\|x\|$, $\alpha(x) \xrightarrow[x \to 0]{} 0$, deb

hisoblanadi. Xususan, $g(t,0) \equiv 0$ va (VI.4.1) sistema $x(t) \equiv 0$ yechimga ega.

Agar (VI.4.1) sistemada $x \to 0$ da yuqori tartibli cheksiz kichik miqdor $g(t, x)$ ni tashlab yuborsak, **birinchi yaqinlashish sistemasi** deb ataluvchi

$$x' = Ax \qquad \text{(VI.4.2)}$$

sistemani hosil qilamiz. Oxirgi chiziqli sistema (VI.4.1) ning chiziqlilashtirilishi deb ham yuritiladi.

Dastlab (VI.4.2) birinchi yaqinlashish sistemasi uchun Lyapunov funksiyasini barcha xarakteristik sonlarining haqiqiy qismi manfiy bo'lganda quraylik.

Lemma. *Agar A matritsaning barcha λ_j xarakteristik sonlari uchun $\mathrm{Re}\lambda_j < 0$ bo'lsa, u holda Lyapunovning asimptotik turg'unlik haqidagi teoremasi shartlarini qanoatlantiruvchi Lyapunov funksiyasi mavjud.*

🗝 Lyapunov funksiyasini

$v(x) = (x, Qx) = x^T Qx$ $\quad (Q \in \mathbb{M}_{n\times n}(\mathbb{R})$ – simmetrik matritsa) kvadratik forma ko'rinishida izlaymiz. Uning (VI.4.2) sistemaga ko'ra hosilasi

$$\left.\frac{dv}{dt}\right|_{(VI.4.2)} = \frac{dx^T}{dt}Qx + x^T Q\frac{dx}{dt} = (Ax)^T Qx + x^T QAx =$$

$$= x^T A^T Qx + x^T QAx = x^T(A^T Q + QA)x .$$

Demak, agar Q matritsani ushbu

$$A^T Q + QA = -E \qquad \text{(VI.4.3)}$$

shartdan tanlasak, u holda

$$\left.\frac{dv}{dt}\right|_{(VI.4.2)} = -x^T x = -\|x\|^2 \quad (w(x) = \|x\|^2)$$

bo'ladi. (VI.4.3) tenglamani qanoatlantiruvchi Q matritsani topish uchun ushbu

$$\frac{dX}{dt} = A^T X + XA, \ \ X(0) = E \qquad \text{(VI.4.4)}$$

matritsaviy Koshi masalasini qaraylik. Bu masala

$$X(t) = E + A^T \cdot \int_0^t X(\tau)d\tau + \int_0^t X(\tau)d\tau \cdot A \qquad \text{(VI.4.5)}$$

integral tenglamaga ekvivalent. Agar uning $t \to +\infty$ da nol-matritsaga intiluvchi yechimi mavjud va mos xosmas integrallar yaqinlashuvch bo'lsa, u holda

$$A^T \cdot \int_0^{+\infty} X(\tau)d\tau + \int_0^{+\infty} X(\tau)d\tau \cdot A = -E$$

tenglik o'rinli bo'ladi, ya'ni (VI.4.3) tenglamaning

$$Q = \int_0^{+\infty} X(\tau)d\tau$$

yechimi topiladi. (VI.4.4) Koshi masalasining yechimini

$$X(t) = Y(t)Z(t)$$

ko'rinishda izlaymiz. Buni (VI.4.4) ga qo'yib,

$$Y'(t)Z(t) + Y(t)Z'(t) = A^T Y(t)Z(t) + Y(t)Z(t)A,$$
$$Y(0)Z(0) = E.$$

munosabatlar qanoatlanishi uchun

$$Y'(t) = A^T Y(t),\ Z'(t) = Z(t)A,\ Y(0) = E,\ Z(0) = E,$$

deymiz. Bularni yechib,

$$Y(t) = e^{tA^T},\ Z(t) = e^{tA}$$

ekanligini topamiz. Demak, (VI.4.4) Koshi masalasining yechimi ushbu

$$X(t) = e^{tA^T} e^{tA}$$

simmetrik matritsaviy funksiyadan iborat.

Endi e^{tA} matritsaning elementlarini baholaymiz. $\alpha_j = \operatorname{Re}\lambda_j$, $\beta_j = \operatorname{Im}\lambda_j$ deb, e^{tA} matritsaning elementlarini

$$\sum_{j=1}^{s} e^{\alpha_j t}\left(p_j(t)\cos(\beta_j t) + q_j(t)\sin(\beta_j t)\right)$$

ko'rinishda yozamiz, bunda $p_j(t),\ q_j(t)$ – haqiqiy koeffitsientli

ko'phadlar. Shartga ko'ra $\max_j \alpha_j < 0$. Ushbu $\max_j \alpha_j < -\alpha < 0$ shartni qanoatlantiruvchi ixtiyoriy $\alpha > 0$ sonni olaylik. $\max_j \alpha_j + \alpha < 0$ bo'lgani uchun shunday $c > 0$ son topiladiki, uning uchun

$$e^{(\alpha_j+\alpha)t}|p_j(t)| \le c,\ e^{(\alpha_j+\alpha)t}|q_j(t)| \le c \quad (t \ge 0,\ j = 1,2,\dots,s)$$

bo'ladi. Endi $e^{tA} = \|\varphi_{kl}(t)\|$ matritsaning elementlari quyidagicha baholanadi:

$$|\varphi_{kl}(t)| = \left|\sum_{j=1}^{s} e^{\alpha_j t}\left(p_j(t)\cos(\beta_j t) + q_j(t)\sin(\beta_j t)\right)\right| \le$$
$$\le e^{-\alpha t}\left(\sum_{j=1}^{s} e^{(\alpha_j+\alpha)t}|p_j(t)| + \sum_{j=1}^{s} e^{(\alpha_j+\alpha)t}|q_j(t)|\right) \le$$
$$\le 2sce^{-\alpha t}\ (t \ge 0). \qquad \text{(VI.4.6)}$$

$(e^{tA})^T = e^{tA^T}$ bo'lgani uchun e^{tA^T} matritsaning elementlari uchun ham shu (VI.4.6) baholashlar o'rinli. Demak, $X(t) = e^{tA^T}e^{tA}$ matritsaning $x_{kl}(t)$ elementlari uchun

$$|x_{kl}(t)| \le \text{const} \cdot e^{-2\alpha t},\ \int_0^{+\infty} |x_{kl}(t)|dt \le \text{const} \cdot \int_0^{+\infty} e^{-2\alpha t}dt < +\infty.$$

Shuning uchun (VI.4.5) tenglikda $t \to +\infty$ deb limitga o'tish mumkin. Natijada

$$Q = \int_0^{+\infty} X(t)dt = \int_0^{+\infty} e^{tA^T}e^{tA}dt \quad (Q^T = Q)$$

deb

$$v(x) = x^T Q x = \int_0^{+\infty} x^T e^{tA^T} e^{tA} x\, dt = \int_0^{+\infty} (e^{tA}x, e^{tA}x)dt = \int_0^{+\infty} \| e^{tA}x \|^2 dt$$

funksiyani topamiz. Oxirgi formuladan ravshanki, $v(x) \succ 0$. Qurilishiga ko'ra

$$\left.\frac{dv}{dt}\right|_{(\&)} = -w(x),\ w(x) = \|x\|^2.$$

Shunday qilib, qurilgan $v(x) = \int_0^{+\infty} \| e^{tA} x \|^2 dt$ funksiya izlangan Lyapunov funksiyasidir. ♧

Teorema (Birinchi yaqinlashishga ko'ra asimptotik turg'unlik haqidagi). *Agar A matritsaning barcha λ_j xarakteristik sonlari uchun $\operatorname{Re}\lambda_j < 0$ bo'lsa,* (VI.4.1) *sistemaning $x(t) \equiv 0$ yechimi asimptotik turg'un bo'ladi.*

⊶ Asimptotik turg'unlik haqidagi Lyapunov teoremasidan foydalanamiz. Lyapunov funksiyasi sifatida birinchi yaqinlashish sistemasi uchun qurilgan

$$v(x) = \sum_{k,l=1}^{n} q_{kl} x_k x_l = x^T Q x = \int_0^{+\infty} (e^{tA} x, e^{tA} x) dt,$$

$$Q = \int_0^{+\infty} e^{tA^T} e^{tA} dt = \| q_{kl} \|$$

Lyapunov funksiyasini olamiz. Uning (VI.4.1) sistemaga ko'ra $\frac{dv}{dt}$ hosilasini hisoblaymiz

$$\left.\frac{dv}{dt}\right|_{(VI.4.1)} = \operatorname{grad}v \cdot (Ax + g(t,x)) = \operatorname{grad}v \cdot Ax + \operatorname{grad}v \cdot g(t,x) \qquad \text{(VI.4.7)}$$

Bu yerdagi birinchi qo'shiluvchi uchun lemmaning isbotida

$$\left.\frac{dv}{dt}\right|_{(\&)} = \operatorname{grad}v \cdot Ax = -\|x\|^2 \qquad \text{(VI.4.8)}$$

ekanligi ko'rsatilgan edi. Ikkinchi qo'shiluvchini baholaymiz. Tushunarliki,

$$\frac{\partial v}{\partial x_j} = \frac{\partial}{\partial x_j} \sum_{k,l=1}^{n} q_{kl} x_k x_l = 2\sum_{l=1}^{n} q_{jl} x_l \ (j = 1,2,\dots,n).$$

Demak, Koshi-Bunyakovskiy tengsizligiga ko'ra

$$\left|\frac{\partial v}{\partial x_j}\right| = 2\left|\sum_{l=1}^{n} q_{jl} x_l\right| \le 2\sqrt{\sum_{l=1}^{n} q_{jl}^2}\ \|x\| \le c\|x\|,\ j = 1,2,\dots,n;$$

bunda $c = 2\max_j \sqrt{\sum_{l=1}^{n} q_{jl}^2}$. Demak,

$$\|\operatorname{grad}v\| = \sqrt{\sum_{j=1}^{n} \left|\frac{\partial v}{\partial x_j}\right|^2} \le nc\|x\|.$$

Yana Koshi-Bunyakovskiy tengsizligiga ko'ra

$$\begin{aligned} \operatorname{grad}v \cdot g(t,x) &\le \|\operatorname{grad}v\| \cdot \|g(t,x)\| \le \\ &\le nc\|x\| \cdot \alpha(x)\|x\| = \\ &= nc\alpha(x)\|x\|^2 \end{aligned} \qquad \text{(VI.4.9)}$$

Endi (VI.4.9) va (VI.4.8) munosabatlardan foydalanib, (VI.4.7) dan $\alpha(x) \le \frac{1}{2nc}$ shartni qanoatlantiruvchi barcha x lar va ixtiyoriy $t \ge 0$ uchun

$$\left.\frac{dv}{dt}\right|_{(*)} \le -\|x\|^2 + nc\alpha(x)\|x\|^2 \le -\frac{1}{2}\|x\|^2.$$

ekanligini topamiz. Demak, (VI.4.1) sistema uchun asimptotik turg'unlik haqidagi Lyapunov teoremasiga ko'ra uning $x(t) \equiv 0$ yechimi asimptotik turg'un. ♧

Misol 1. Ushbu

$$x'' + h'(x)x' + x = 0$$

Lyenard tenglamasini qaraylik, bunda $h \in C^1(\mathbb{R})$ va $h(0) = 0$. Bu tenglama $x(t) \equiv 0$ nol yechimga ega. Uning turg'unligi deganda os

$$\begin{cases} x' = y - h(x) \\ y' = -x \end{cases}$$

normal sistema nol yechimining turg'unligi tushuniladi. Oxirgi sistemani chiziqlilashtiramiz:

$$\begin{cases} x' = -h'(0)x + y \\ y' = -x \end{cases}$$

Bu chiziqli sistemaning xarakteristik tenglamasi

$$\lambda^2 + h'(0)\lambda + 1 = 0.$$

Ravshanki, agar $h'(0) > 0$ bo'lsa, xarakteristik sonlarning haqiqiy qismi manfiy. Shuning uchun yuqoridagi sistemaning va, demak, L'yenard tenglamasining nol yechimi asimptotik turg'un.

$h(x) = x - x^3/3$ bo'lganda L'yenard tenglamasi ushbu

$$x'' + (1 - x^2)x' + x = 0$$

Van der Pol tenglamasiga aylanadi. Van der Pol tenglamasining nol yechimi turg'undir. ♁

Teorema (Birinchi yaqinlashishga ko'ra noturg'unlik haqidagi). *Agar A matritsaning biror λ_j xarakteristik soni uchun* $\mathrm{Re}\lambda_j > 0$ *bo'lsa,* (VI.4.1) *sistemaning* $x(t) \equiv 0$ *yechimi noturg'un bo'ladi.*

Bu teoremani isbotsiz qabul qilamiz.

Agar $\max_j \alpha_j = \max_j \mathrm{Re}\lambda_j = 0$ bo'lsa, turg'unlik yoki noturg'unlik birinchi yaqinlashishga ko'ra hal qilinmaydi. Bu holda $x(t) \equiv 0$ yechimning turg'unligi yoki noturg'unligi (VI.4.1) sistemadagi yuqori tartibli had $g(t,x)$ ($\|g(t,x)\| \le \alpha(x)\|x\|$, $\alpha(x) \xrightarrow[x\to 0]{} 0$) bilan aniqlanadi.

Misol 2. Ushbu

$$\begin{cases} x' = y + 4x^2y^2 - 4x^5 \\ y' = -x - 2y^3 - 4x^3y \end{cases}$$

sistemaning nol yechimi asimptotik turg'un. (VI.3. banddagi misol 2 ga qarang). Uning chiziqli qismi

$$\begin{cases} x' = y \\ y' = -x \end{cases}$$

sistemadan iborat. Xarakteristik sonlar $\lambda_{1,2} = \pm i$. Demak, birinchi yaqinlashishga ko'ra nol yechimni turg'unlikka tekshirib bo'lmaydi.

Misol 3. Ushbu

$$\begin{cases} x' = y - 2x^2 \\ y' = 2xy + y^3 \end{cases}$$

sistemaning nol yechimi, bizga ma'lumki, noturg'un (VI.3. banddagi misol 3 ga qarang).

Sistemaning birinchi yaqinlashishi

$$\begin{cases} x' = y \\ y' = 0 \end{cases}$$

uchin xatakteristik sonlar $\lambda_1 = \lambda_2 = 0$. Demak, birinchi yaqinlashishga ko'ra nol yechimning turg'unligi haqida hech narsa deb bo'lmaydi. ♁

VI.5. Lorens sistemasining muvozanat holatlarini turg'unlikka tekshirish

Ushbu

$$\begin{cases} x' = -\sigma x + \sigma y \\ y' = rx - y - xz \\ z' = -bz + xy \end{cases}$$

Lorens differensial tenglamalar sistemasini qaraylik, bunda σ, r, b – musbat o'zgarmaslar. Bu sistemani Lorens (Edward N. Lorentz - Massachusets texnologiya institutida metereolog olim) atmosferadagi havo oqimlarining matematik modeli sifatida hosil qilgan va $\sigma = 10$, $b = 8/3$, $r = 28$ holida o'rgangan. Lorens sistemasining muvozanat holatlari

$$\begin{cases}-\sigma x+\sigma y=0\\ rx-y-xz=0\\ -bz+xy=0\end{cases}$$

sistemadan topiladi. Ixtiyoriy $r>0$ uchun bu sistema $x=0, y=0, z=0$ yechimga ega. Agar $r>1$ bo'lsa, yana ikkita muvozanat nuqtasi hosil bo'ladi:

$x=x_0, y=y_0, z=z_0$ va $x=-x_0, y=-y_0, z=z_0$

($x_0=y_0=\sqrt{b(r-1)}$, $z_0=r-1$).

$x=0, y=0, z=0$ muvozanat nuqtani turg'unlikka tekshiraylik. Bu nuqta atrofida Lorens sistemasining chiziqlashtirilishi

$$\begin{cases}x'=-\sigma x+\sigma y\\ y'=rx-y\\ z'=-bz\end{cases}$$

ko'rinishga ega. Xarakteristik tenglama

$$\begin{vmatrix}-\sigma-\lambda & \sigma & 0\\ r & -1-\lambda & 0\\ 0 & 0 & -b-\lambda\end{vmatrix}=0.$$

Xarakteristik sonlar

$$\lambda_1=-b,\ \lambda_{2,3}=\frac{1}{2}\left(-1-\sigma\pm\sqrt{(1+\sigma)^2-4\sigma(1-r)}\right).$$

Agar $0<r<1$ bo'lsa, hamma xususiy sonlar manfiy va, Lorens sistemasining nol-yechimi asimptotik turg'un.

Agar $r>1$ bo'lsa, $\lambda_2=\frac{1}{2}\left(-1-\sigma+\sqrt{(1+\sigma)^2-4\sigma(1-r)}\right)$ xarakteristik son musbat va Lorens sistemasining nol-yechimi noturg'un. Demak, $r=1$ da nol-yechim turg'unligi almashinadi.

$r=1$ bo'lganda xarakteristik sonlar $\lambda_1=-b,\ \lambda_2=0,\ \lambda_3=-(1+\sigma)$ va chiziqlashtirilgan sistema Lorens sistemasi nol-yechimining turg'unligi haqida hech narsa

deya olmaydi. Lekin bu holda Lyapunov funksiyasini qurishga harakat qilish mumkin. Kvadratik forma ko'rinishidagi ushbu

$$v=v(x,y,z)=x^2/\sigma+y^2+z^2$$

aniq musbat funksiyani qaraylik. Bu funksiyaning Lorens sistemasiga ko'ra hosilasi

$$\frac{dv}{dt}=2\frac{x}{\sigma}(-\sigma x+\sigma y)+2y(rx-y-xz)+2z(-bz+xy)=$$

$$=-2\left(\left(x-\frac{r+1}{2}y\right)^2+\left(1-\frac{(r+1)^2}{4}\right)y^2+bz^2\right).$$

Agar $r<1$ bo'lsa, qurilgan v funksiya Lyapunovning asimptotik turg'unlik haqidagi teoremasi shartlarini qanoatlantiradi; demak, yana nol-yechim asimptotik turg'un.

$r=1$ holini qaraylik. Bu holda v funksiya Lyapunovning turg'unlik haqidagi teoremasi shartlarini qanoatlantiradi; demak, bu holda nol-yechim turg'un. $\frac{dv}{dt}=-2\left((x-y)^2+bz^2\right)$ hosila $x=y, z=0$ to'g'ri chiziqda nolga aylanib, boshqa nuqtalarda qat'iy manfiy. Noldan farqli har qanday yechim bu to'g'ri chiziq bilan uchrashgach, undan albatta chiqib ketadi, chunki bunda $z'=-bz+xy=x^2\neq 0$. Shuning uchun vaqt o'tishi bilan yechim $v=x^2/\sigma+y^2+z^2$ funksiyaning $x^2/\sigma+y^2+z^2=c$ $(c>0)$ sath to'plamlarini (ellipsoidlarni) c ning kamayish yo'nalishida kesib boradi va koordinatalar boshiga intiladi, ya'ni $r=1$ holida nol-yechim asimptotik turg'un hamdir.

Endi $r>1$ holida $x=x_0, y=y_0, z=z_0$ va $x=-x_0, y=-y_0, z=z_0$ muvozanat nuqtalarni turg'unlikka tekshiramiz. Buning uchun Lorens sistemasida

$$x=u+x_0,\ y=v+y_0,\ z=w+z_0$$

almashtirishni bajaramiz, bunda u, v, w – yangi noma'lum funksiyalar. Natijada

$$\begin{cases} u' = -\sigma u + \sigma v \\ v' = u - v - x_0 w - uw \\ w' = x_0 u + x_0 v - bw + uv \end{cases} \quad (x_0 = \sqrt{b(r-1)})$$

sistemaga kelamiz. Bu sistemaning birinchi yaqinlashishi uchun xarakteristik tenglama

$$\begin{vmatrix} -\sigma - \lambda & \sigma & 0 \\ 1 & -1-\lambda & -x_0 \\ x_0 & x_0 & -b-\lambda \end{vmatrix} = 0.$$

Bu tenglama $x_0$ning ishorasi o'zgarganda o'zgarmaydi, ya'ni $x = -x_0, y = -y_0, z = z_0$ muvozanat nuqta uchun ham shu xarakteristik tenglama hosil bo'ladi. Xarakteristik tenglamani

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + r)\lambda + 2\sigma b(r-1) = 0$$

$(\sigma > 0,\ b > 0,\ r > 1)$.

ko'rinishda yozish mumkin. Bu tenglama ildizlarining haqiqiy qismi manfiy bo'lishi uchun L'enar-Shipar mezoniga ko'ra

$$(\sigma + b + 1)(\sigma + r)b - 2\sigma b(r-1) > 0$$

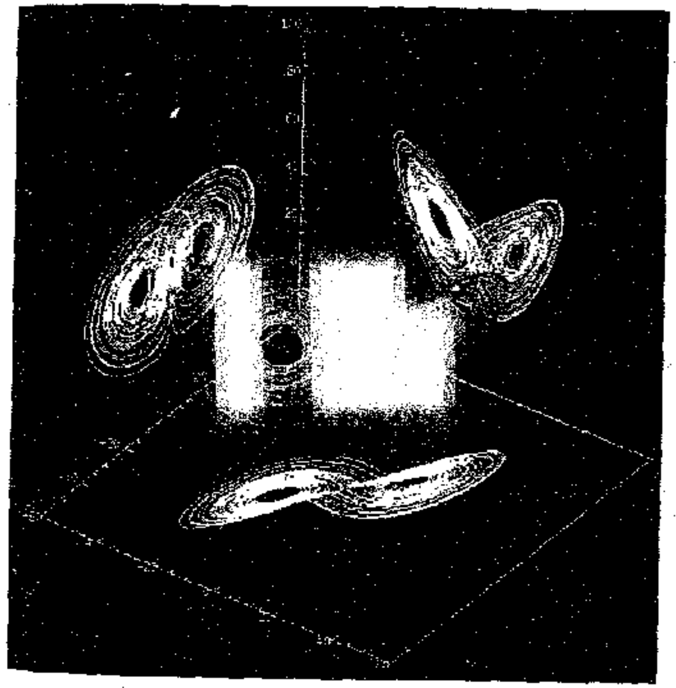
shart bajarilishi kerak. Oxirgi shart

$$(\sigma - b - 1)r < \sigma(\sigma + b + 3)$$

tengsizlikka teng kuchli. $\sigma = 10,\ b = 8/3$ holini qaraylik. Oxirgi tengsizlikdan $r < r_0$, $r_0 \approx 22{,}74$, ni topamiz. Demak, $1 < r < r_0$ bo'lganda qaralayotgan muvozanat nuqtalar asimptotik turg'un, $r > r_0$ bo'lganda esa ular noturg'un (Lorens tekshirgan holda $r = 28 > r_0$ bo'lgan). $r = r_0$ da bitta manfiy haqiqiy qismli va ikkita sof mavhum xarakteristik sonlar mavjud. Bu kritik holni tekshirmaymiz.

Qaralayotgan sistemaning yechimlarini $\sigma = 10,\ b = 8/3$, $r = 28$ holida Lorens sonli usullar yordamida o'rgangan. U

yechimlarning to'satdan betartib ravishda to $(\sqrt{72}; \sqrt{72}; 27)$, to $(-\sqrt{72}; -\sqrt{72}; 27)$ noturg'un muvozanat nuqtalari atrofida buralib boshlashini aniqlagan (VI.1.7-rasm). Bunda yechimlarning necha marta bir muvozanat nuqtasi atrofida buralib so'ngra ikkinchisi atrofidga o'tib buralishi ham betartib bo'lgan. Yechimlarning tartibsiz o'zgarishi boshlang'ich qiymatga kuchli bog'liq bo'lgan. Yechimning bunday betartib tabiati xaos deb ataladi. Xaos nazariyasida nochiziqli dinamik sistemalarning turg'un bo'lmagan davrsiz yechimlari tabiati o'rganiladi.



VI.7-rasm.

Masalalar

1. Chiziqli o'zgarmas koeffitsientli sistemaning turg'unligi to'g'risidagi teorema shartlarining zarur ekanlgini isbotlang.

2. Agar A matritsaning barcha λ_j xarakteristik sonlari uchun $\operatorname{Re}\lambda_j<0$ bo'lsa, ushbu

$v(x)=\int_0^{+\infty}(e^{tA}x,e^{tA}x)dt$ funksiya ma'noga ega va uning $x'=Ax$ sistemaga ko'ra hosilasi uchun $\left.\frac{dv}{dt}\right|_{x'=Ax}=-\|x\|^2$ bo'lishini bevosita isbotlang.

3. Faraz qilaylik, $\{A,B,C\}\subset \mathrm{M}_{n\times n}(\mathbb{R})$ bo'lsin. Agar A va B matritsalarning barcha λ_j xarakteristik sonlari uchun $\operatorname{Re}\lambda_j<0$ bo'lsa, u holda

$$X=\int_0^{+\infty}e^{tA}Ce^{tB}dt$$

matritsa aniqlangan (xosmas integral yaqinlashuvchi) hamda u

$$AX+XB=-C$$

tenglamaning yechimi bo'lishini ko'rsating.

4. Ushbu

$$\begin{cases}x'=-y+x^3\\ y'=x+y^3\end{cases}$$

sistemaning nol-yechimini turg'unlikka tekshiring.

5. Ushbu

$$\begin{cases}x'=y+xy\\ y'=-x-xy\end{cases}$$

sistemaning nol-yechimini turg'unlikka tekshiring.

6. Ushbu

$$\begin{cases}x'=y-x^3\\ y'=-x^3-y^3\end{cases}$$

sistemaning nol-yechimini turg'unlikka tekshiring.

7. Ushbu

$$\begin{cases}x'=y+kx(x^2+y^2)\\ y'=-x+ky(x^2+y^2)\end{cases}$$

sistemaning nol-yechimi $k>0$ holida turg'un, $k<0$ holida esa noturg'un ekanligini ko'rsating.

8. Quyidagi sistemalarning fazaviy portretlarini quring:

1). $\begin{cases}x'=xy,\\ y'=x^2+y^2.\end{cases}$ 2). $\begin{cases}x'=xy,\\ y'=y^2-x^2.\end{cases}$

3). $\begin{cases}x'=xy,\\ y'=y^2-6x^2y+x^4.\end{cases}$ 4). $\begin{cases}x'=-xy,\\ y'=x/2-y^2.\end{cases}$

5). $\begin{cases}x'=x^2(y-1)(4-x^2),\\ y'=y^2(x-1)(y^2-x).\end{cases}$ 6). $\begin{cases}x'=(1-x^2)(x-\mu y),\\ y'=(1-y^2)(y+\mu x).\end{cases}$ $(\mu>1)$

7). $\begin{cases}x'=(1-x^2)(y+x(1-x^2)),\\ y'=-x+(1-x^2)y.\end{cases}$ 8). $\begin{cases}x'=-y^3(x^2-1)(2+xy),\\ y'=x^3(y^2-1)(2-xy).\end{cases}$

9). $\begin{cases}x'=y+x^7+4x^4y^2-y,\\ y'=\mu x^n y-x^7+y^2.\end{cases}$ $(\mu\neq 0,\ n\geq 0)$

10). $\begin{cases}x'=(-3x^4+6x^2y^2+y^4)(x^2+y^2)^{-3/2},\\ y'=8xy^3(x^2+y^2)^{-3/2}.\end{cases}$

9. Fazaviy portretni quring:

1) $\begin{cases}x'=x(1-x+y)\\ y'=y(2-x-y)\end{cases}$ 2) $\begin{cases}x'=x-y\\ y'=\dfrac{4x^2}{1+3x^2}-y\end{cases}$

3) $\begin{cases}x'=y\\ y'=x^3-x-y\end{cases}$ $(x''+x'+x-x^3=0)$

4) $\begin{cases} x' = y \\ y' = -x^3 - x - y \end{cases}$ $(x'' + x' + x + x^3 = 0)$ 5) $\begin{cases} x' = -\sin y \\ y' = \sin x \end{cases}$

10. Ushbu

$$\begin{cases} x' = -\sigma x + \sigma y \\ y' = rx - y - xz \\ z' = -bz + xy \end{cases} \quad (\sigma, r, b - \text{musbat sonlar})$$

Lorens sistemasining yechimlari $(-\infty, +\infty)$ oralig'ida aniqlanganligini va uning barcha traektoriyalari nol hajmli to'plamga intilishini isbotlang.

VII BOB. YECHIMNING PARAMETRGA SILLIQ BOG'LIQLIGI VA UNING TATBIQLARI

VII.1. Yechimning boshlang'ich ma'lumotlar va parametr bo'yicha differensiallanuvchligi

Ushbu

$$\begin{cases} x' = f(t, x, \mu) \\ x|_{t_0} = x^0 \end{cases} \qquad \text{(VII.1.1)}$$

$\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in M$ $(M \subset \mathbb{R}^m - \text{soha})$ μ parametr(lar)ga bo'gliq bo'lgan Koshi masalasini qaraylik, bunda $(t, x) \in D \subset \mathbb{R}^{1+n}$. Faraz qilaylik, har bir $(t_0, x^0, \mu) \in D \times M$ uchun (VII.1.1) masala $t \in I$ intervalda aniqlangan yagona davomsiz yechimga ega bo'lsin. Bu yechim na faqat t ga, balki tayinlangan $(t_0, x^0, \mu) \in D \times M$ qiymatlarga ham bog'liq bo'ladi va uni biz $x = \varphi(t; t_0, x^0, \mu)$ ko'rinishda belgilaymiz. Davomsiz yechimning aniqlanish intervali tayinlangan (t_0, x^0, μ) qiymatlarga bog'liq bo'lgani ($I = I(t_0, x^0, \mu)$) uchun $x = \varphi(t; t_0, x^0, \mu)$ yechim $(t; t_0, x^0, \mu) \in I \times D \times M \subset R^{2+n+m}$ sohada aniqlangan. Agar (t_0, x^0) tayinlangan bo'lib, faqat μ parametr turli qiymatlar qabul qilsa, u holda $x = \varphi(t; t_0, x^0, \mu)$ yozuv o'rniga qisqaroq $x = \varphi(t; \mu)$ yozuvni ishlatamiz. Biz yechimning parametrlarga uzluksiz bog'liqligini III.6 bandda o'rgangan edik. Endi uning differensiallanuvchiligini o'rganamiz.

Soddalik uchun parametrlar sonini birga teng deb hisoblaymiz. Bu holda $m = 1$, M – sonli interval, $\mu = \mu \in M$.

Teorema 1 (yechimning parametr bo'yicha differensiallanuvchiligi). *Aytaylik,* $f(t, x, \mu)$,

$\frac{\partial f(t,x,\mu)}{\partial x_j}$ $(j=1,\dots,n)$ *va* $\frac{\partial f(t,x,\mu)}{\partial \mu}$ *funksiyalar* $(t,x,\mu)\in D\times M$ *sohada uzluksiz,* (VII.1.1) *masalaning* $x=\varphi(t;\mu)$ *yechimi esa har bir* $\mu\in M$ *uchun* $t\in[t_1,t_2]$ $(t_0\in[t_1,t_2])$ *segmentda aniqlangan bo'lsin. U holda bu yechimning* $u\equiv\frac{\partial\varphi(t;\mu)}{\partial\mu}$ $(u=u(t;\mu))$ *hosilasi* $(t,\mu)\in[t_1,t_2]\times M$ *bo'lganda uzluksiz va u variatsiya uchun tenglama deb ataluvchi ushbu*

$$\frac{du}{dt}=\frac{\partial f}{\partial x}u+\frac{\partial f}{\partial\mu},\quad u\big|_{t=t_0}=0, \qquad \text{(VII.1.2)}$$

chiziqli tenglamani qanoatlantiradi, bunda xususiy hosilalar $x=\varphi(t;\mu)$ *bo'lganda hisoblangan, ya'ni*

$$\frac{\partial f}{\partial x}=\left.\frac{\partial f(t,x,\mu)}{\partial x}\right|_{x=\varphi(t;\mu)},\quad \frac{\partial f}{\partial\mu}=\left.\frac{\partial f(t,x,\mu)}{\partial\mu}\right|_{x=\varphi(t;\mu)}.$$

Variatsiya uchun (VII.1.2) *vektorli tenglamaning skalyar ko'rinishi quyidagi variatsiyalar uchun tenglamalar sistemasidan iborat:*

$$\frac{du_i}{dt}=\sum_{j=1}^{n}\frac{\partial f_i}{\partial x_j}u_j+\frac{\partial f_i}{\partial\mu},\quad u_j\big|_{t=t_0}=0\ \ (i=1,\dots,n). \qquad \text{(VII.1.3)}$$

Teoremaning shartlariga ko'ra har qanday $\mu\in M$ uchun (VII.1.2) chiziqli masalaning $t\in[t_1,t_2]$ segmentda aniqlangan yagona $u=u(t;\mu)$ yechimi mavjud va III.6 banddagi teoremaga ko'ra $u(t;\mu)\in C([t_1,t_2]\times M)$. Bu yechim, ravshanki, ushbu

$$u(t;\mu)=\int_{t_0}^{t}\frac{\partial f(s,\varphi(s;\mu),\mu)}{\partial x}u(s;\mu)\,ds+\int_{t_0}^{t}\frac{\partial f(s,\varphi(s;\mu),\mu)}{\partial\mu}ds \qquad \text{(VII.1.4)}$$

tenglamani qanoatlantiradi. Teoremani isbot qilish uchun hosilaning ta'rifiga ko'ra $\mu\in M$ tayinlanganda

$$\psi(t;\bar\mu)\stackrel{def}{=}\varphi(t;\bar\mu)-\varphi(t;\mu)-u(t;\mu)(\bar\mu-\mu)\ \ (\bar\mu\in M) \qquad \text{(VII.1.5)}$$

funksiya uchun

$$\|\psi(t;\bar\mu)\|=o(\bar\mu-\mu),\ \bar\mu\to\mu,$$

asimptotik tenglikning o'rinli ekanligini ko'rsatamiz. Bunda $x=\varphi(t;\bar\mu)$ vektor-funksiya (VII.1.1) masalaning parametr $\bar\mu$ ga teng bo'lgandagi yechimi, ya'ni

$$\begin{cases}\dfrac{d\varphi(t;\bar\mu)}{dt}=f(t,\varphi(t;\bar\mu),\bar\mu)\\ \varphi(t;\bar\mu)\big|_{t=t_0}=x^0.\end{cases} \qquad \text{(VII.1.6)}$$

Demak,

$$\varphi(t;\bar\mu)=x^0+\int_{t_0}^{t}f(s,\varphi(s;\bar\mu),\bar\mu)\,ds. \qquad \text{(VII.1.7)}$$

Shunga o'xshash

$$\varphi(t;\mu)=x^0+\int_{t_0}^{t}f(s,\varphi(s;\mu),\mu)\,ds. \qquad \text{(VII.1.8)}$$

Endi (VII.1.7), (VII.1.8) va (VII.1.4) formulalarga ko'ra (VII.1.5) dan quyidagini hosil qilamiz:

$$\psi(t;\bar\mu)=\int_{t_0}^{t}\Psi(s;\bar\mu)\,ds, \qquad \text{(VII.1.9)}$$

bu yerda qisqalik uchun

$$\Psi(s;\bar\mu)\stackrel{def}{=}F(s,\bar\mu)-(\bar\mu-\mu)\frac{\partial f(s,\varphi(s;\mu),\mu)}{\partial x}u(s;\mu)-$$
$$-(\bar\mu-\mu)\frac{\partial f(s,\varphi(s;\mu),\mu)}{\partial\mu}, \qquad \text{(VII.1.10)}$$

$$F(s,\bar\mu)\stackrel{def}{=}f(s,\varphi(s;\bar\mu),\bar\mu)-f(s,\varphi(s;\mu),\mu) \qquad \text{(VII.1.11)}$$

deb belgilangan. $F(s)=(F_1(s),F_2(s),...,F_n(s))^T$ vektor-funksiya koordinatalarining ko'rinishini Lagranj formulasi va (VII.1.5) dan topilgan $\varphi(t;\bar\mu)-\varphi(t;\mu)=\psi(t;\bar\mu)+(\bar\mu-\mu)u(t;\mu)$ tenglikka ko'ra quyidagicha almashtiramiz:

$$F_j(s,\bar\mu)=(f_j(s,\varphi(s;\bar\mu),\bar\mu)-f_j(s,\varphi(s;\bar\mu),\mu))+$$
$$+(f_j(s,\varphi(s;\bar\mu),\mu)-f_j(s,\varphi(s;\mu),\mu))=$$
$$=\frac{\partial f_j(s,\varphi(s;\bar\mu),\mu^{j*})}{\partial\mu}(\bar\mu-\mu)+\frac{\partial f_j(s,x^{j*},\mu)}{\partial x}(\varphi(s;\bar\mu)-\varphi(s;\mu))=$$
$$=\frac{\partial f_j(s,\varphi(s;\bar\mu),\mu^{j*})}{\partial\mu}(\bar\mu-\mu)+$$
$$+\frac{\partial f_j(s,x^{j*},\mu)}{\partial x}(\psi(s;\bar\mu)+(\bar\mu-\mu)u(s;\mu))\ ;$$

bu yerda

$$\mu^{j*}=\mu+\theta_{1j}(\bar\mu-\mu),\ x^{j*}=\varphi(s;\mu)+\theta_{2j}(\varphi(s;\bar\mu)-\varphi(s;\mu)),$$
$$0<\theta_{1j},\theta_{2j}<1,\ j=1,...,n. \qquad \text{(VII.1.12)}$$

Shunday qilib,

$$F_j(s,\bar\mu)=\left(\frac{\partial f_j(s,\varphi(s;\bar\mu),\mu^{j*})}{\partial\mu}+\frac{\partial f_j(s,x^{j*},\mu)}{\partial x}u(s;\mu)\right)(\bar\mu-\mu)+$$
$$+\frac{\partial f_j(s,x^{j*},\mu)}{\partial x}\psi(s;\bar\mu). \qquad \text{(VII.1.13)}$$

Endi (VII.1.10) dan (VII.1.11) va (VII.1.13) ga ko'ra quyidagini hosil qilamiz:

$$\Psi_j(s;\bar\mu)=(\bar\mu-\mu)\Big[\Big(\frac{\partial f_j(s,\varphi(s;\mu),\mu^{j*})}{\partial\mu}-\frac{\partial f_j(s,\varphi(s;\mu),\mu)}{\partial\mu}\Big)+$$
$$+\Big(\frac{\partial f_j(s,x^{j*},\mu)}{\partial x}-\frac{\partial f_j(s,\varphi(s;\mu),\mu)}{\partial x}\Big)u(s;\mu)\Big]+$$

$$+\frac{\partial f_j(s,x^{j*},\mu)}{\partial x}\psi(s;\bar\mu). \qquad \text{(VII.1.14)}$$

$\bar\mu\in M$ o'zgaruvchining (tayinlangan) $\mu\in M$ ga yaqin qiymatlarida (VII.1.14) formulaning o'ng tomondagi o'rta qavs ichidagi birinchi va ikkinchi qo'shiluvchilarni xohlagancha kichik qilish mumkinligini ko'rsatamiz.

Ixtiyoriy $\varepsilon>0$ soni berilgan bo'lsin. Uzluksiz funksiyalar kompozitsiyasi sifatida $g_j(s,\bar\mu)=\dfrac{\partial f_j(s,\varphi(s;\mu),\bar\mu)}{\partial\mu}$ $(j=1,...,n)$ funksiya $s\in[t_1,t_2]$ va μ ga yetarlicha yaqin $\bar\mu$ lar uchun tekis uzluksiz bo'ladi (Kantor teoremasiga ko'ra). Demak, $\varepsilon>0$ soniga ko'ra shunday $\delta=\delta(\varepsilon)>0$ topish mumkinki, $|\bar\mu-\mu|<\delta$ ekanligidan barcha $s\in[t_1,t_2]$ lar uchun

$$\left|\frac{\partial f_j(s,\varphi(s;\mu),\bar\mu)}{\partial\mu}-\frac{\partial f_j(s,\varphi(s;\mu),\mu)}{\partial\mu}\right|<\varepsilon$$

bo'lishi kelib chiqadi. $|\bar\mu-\mu|<\delta$ bo'lganda (VII.1.12) ga ko'ra $|\mu^{j*}-\mu|<|\bar\mu-\mu|<\delta$ va, demak, $s\in[t_1,t_2]$ lar uchun

$$\left|\frac{\partial f_j(s,\varphi(s;\mu),\mu^{j*})}{\partial\mu}-\frac{\partial f_j(s,\varphi(s;\mu),\mu)}{\partial\mu}\right|<\varepsilon \qquad \text{(VII.1.15)}$$

va,demak, vektorning normasi uchun

$$\left\|\frac{\partial f(s,\varphi(s;\mu),\mu^{j*})}{\partial\mu}-\frac{\partial f(s,\varphi(s;\mu),\mu)}{\partial\mu}\right\|<n\varepsilon \qquad \text{(VII.1.16)}$$

tengsizlik ham o'rinli bo'ladi.

Endi $h_k^j(s,x)=\dfrac{\partial f_j(s,x,\mu)}{\partial x_k}$ $(j,k=1,...,n)$ funksiyani qaraylik. U D sohada joylashgan ixtiyoriy kompaktda tekis uzluksiz. Uni D da yotuvchi va $\varphi(s;\mu)$, $s\in[t_1,t_2]$, funksiyaning grafigini o'z ichiga oluvchi kompaktda qaraymiz. Demak, shunday

$\sigma=\sigma(\varepsilon)>0$ mavjudki, $\|\bar{x}-x\|<\sigma$ tengsizlikdan barcha $s\in[t_1,t_2]$ lar uchun

$$\left|\frac{\partial f_j(s,\bar{x},\mu)}{\partial x_k}-\frac{\partial f_j(s,x,\mu)}{\partial x_k}\right|<\varepsilon \qquad \text{(VII.1.17)}$$

ekanligi kelib chiqadi. $\varphi(s;\bar{\mu})$ funksiya $s\in[t_1,t_2]$, $\bar{\mu}\in B_{\delta_0}(\mu)$ (δ_0 – yetarlicha kichik musbat son) bo'lganda tekis uzluksiz. Demak, shunday $\delta_1=\delta_1(\varepsilon)>0$ topiladiki, $|\bar{\mu}-\mu|<\delta_1$ bo'lishidan barcha $s\in[t_1,t_2]$ lar uchun

$$\|\varphi(s;\bar{\mu})-\varphi(s;\mu)\|<\sigma \qquad \text{(VII.1.18)}$$

tengsizlik kelib chiqadi. $\delta_1=\delta$ deb hisoblaymiz (har doim ularni kichiklashtirish mumkin). Shunday qilib, $|\bar{\mu}-\mu|<\delta$ bo'lganda barcha $s\in[t_1,t_2]$ lar uchun (VII.1.12) ga ko'ra $\|x^{\mu}-\varphi(s;\mu)\|<\|\varphi(s;\bar{\mu})-\varphi(s;\mu)\|<\sigma$ va (VII.1.18) va (VII.1.17) tengsizliklarga asosan

$$\left|\frac{\partial f_j(s,x^{\mu},\mu)}{\partial x_k}-\frac{\partial f_j(s,\varphi(s;\mu),\mu)}{\partial x_k}\right|<\varepsilon$$

va, demak, matritsaning normasi uchun

$$\left\|\frac{\partial f(s,x^{\mu},\mu)}{\partial x}-\frac{\partial f(s,\varphi(s;\mu),\mu)}{\partial x}\right\|<n\varepsilon \qquad \text{(VII.1.19)}$$

bo'ladi. $u(s;\mu)$ funksiya $s\in[t_1,t_2]$, $\bar{\mu}\in B_{\delta_0}(\mu)$ bo'lganda chegaralangan, ya'ni biror $\tilde{L}>0$ soni uchun

$$\|u(s;\mu)\|\le\tilde{L} \qquad \text{(VII.1.20)}$$

baholash o'rinli; shunga o'xshash biror $L>0$ va barcha $s\in[t_1,t_2]$, $\bar{\mu}\in B_{\delta_0}(\mu)$ lar uchun

$$\left\|\frac{\partial f(s,x^{\mu},\mu)}{\partial x}\right\|\le L \qquad \text{(VII.1.21)}$$

tengsizlik ham o'rinli bo'ladi.

Nihoyat, ixtiyoriy $\bar{\mu}\in B_\delta(\mu)$ va ixtiyoriy $s\in[t_1,t_2]$ uchun (VII.1.16), (VII.1.19), (VII.1.20) va (VII.1.21) tengsizliklarga ko'ra (VII.1.14) formuladan Koshi-Bunyakovskiy tengsizligidan foydalanib, quyidagi baholashlarni amalga oshiramiz:

$$\|\Psi(s;\bar{\mu})\|\le|\bar{\mu}-\mu|\cdot\left(\left\|\frac{\partial f(s,\varphi(s;\mu),\mu^{\mu})}{\partial\mu}-\frac{\partial f(s,\varphi(s;\mu),\mu)}{\partial\mu}\right\|+\right.$$

$$\left.+\left\|\frac{\partial f(s,x^{\mu},\mu)}{\partial x}-\frac{\partial f(s,\varphi(s;\mu),\mu)}{\partial x}\right\|\cdot\|u(s;\mu)\|\right)+$$

$$+\left\|\frac{\partial f(s,x^{\mu},\mu)}{\partial x}\right\|\cdot\|\psi(s;\bar{\mu})\|\le$$

$$\le|\bar{\mu}-\mu|\cdot(n\varepsilon+n\varepsilon\tilde{L})+L\|\psi(s;\bar{\mu})\|.$$

Oxirgi tengsizlikka ko'ra (VII.1.9) formuladan barcha $\bar{\mu}\in B_\delta(\mu)$ va $t\in[t_1,t_2]$ lar uchun ushbu

$$\|\psi(t;\bar{\mu})\|\le|\bar{\mu}-\mu|\cdot(1+\tilde{L})\cdot n\cdot|t-t_0|\varepsilon+L\left|\int_{t}^{t}\|\psi(s;\bar{\mu})\|ds\right|$$

baholashni topamiz. Bundan Gronuoll-Bellman tengsizligiga ko'ra ((III.5.8) formulaga qarang):

$$\|\psi(t;\bar{\mu})\|\le|\bar{\mu}-\mu|n\frac{1+\tilde{L}}{L}(e^{L|t-t_0|}-1)\varepsilon.$$

Bu tengsizlikdan $t\in[t_1,t_2]$ ga nisbatan tekis baho(lash)ni ham topish mumkin:

$$\|\psi(t;\bar{\mu})\|\le|\bar{\mu}-\mu|n\frac{1+\tilde{L}}{L}\sup_{t\in[t_1,t_2]}(e^{L|t-t_0|}-1)\varepsilon\le$$

$$\leq |\bar{\mu}-\mu| n \frac{1+\tilde{L}}{L}(e^{L(t_2-t_0)}-1)\cdot\varepsilon . \quad \text{(VII.1.22)}$$

Shunday qilib, ixtiyoriy $\varepsilon > 0$ soniga ko'ra shunday $\delta > 0$ soni topildiki, $|\bar{\mu}-\mu| < \delta$ tengsizlikdan (VII.1.22) tengsizlik kelib chiqdi. Bu $\|\psi(t;\bar{\mu})\| = o(\bar{\mu}-\mu)$, $\bar{\mu}\to\mu$, ya'ni

$$\psi(t;\bar{\mu}) = \psi(t;\mu) + u(t;\mu)(\bar{\mu}-\mu) + o(\bar{\mu}-\mu)$$

$$(o(\bar{\mu}-\mu) \overset{t\in[t_1,t_2]}{\rightrightarrows} 0,\ \bar{\mu}\to\mu).$$

ekanligini anglatadi. ♤

Teorema parametrlar soni bittadan ko'p bo'lganda ham o'rinli. Bu holda $x=\varphi(t;\mu_1,\mu_2,...,\mu_m)$ yechimning har bir $u^j \equiv \frac{\varphi(t;\mu)}{\partial\mu_j}$ $(j=1,...,m)$ xususiy hosilasi variatsiya uchun mos (VII.1.2) chiziqli tenglamani qanoatlantiradi:

$$\frac{du^j}{dt} = \frac{\partial f}{\partial x}u^j + \frac{\partial f}{\partial \mu_j}, \quad u^j\big|_{t=t_0} = 0 \ (j=1,...,m).$$

Keltirilgan teoremani quyidagicha qisqaroq (lekin noaniqroq) ifodalash mumkin:

Agar $x' = f(t,x,\mu)$ sistemaning o'ng tomoni $f(t,x,\mu)\in C^1$ bo'lsa, uning $x=\varphi(t;\mu)$ yechimi ham C^1 sinfga tegishli bo'ladi.

Variatsiyalar uchun (VII.4.24) tenglamalar sistemasini (yoki uning (VII.1.2) vektor ko'rinishini) hosil qilish uchun ushbu

$$\frac{d\varphi_i(t,\mu)}{dt} = f_i(t,\varphi_1(t,\mu),...,\varphi_n(t,\mu),\mu), \quad (i=1,...,n)$$

ayniyatlarni μ bo'yicha differensiallash va aralash hosilalarda differensiallash tartibini almashtirish kerak. Agar $x=\varphi(t;\mu)$ yechim biror μ da ma'lum bo'lsa, μ ning shu qiymatida yechimning μ bo'yicha hosilasi $u=\frac{\varphi(t;\mu)}{\partial\mu}$ ni (VII.1.2) (yoki(VII.1.3)) masalani yechib aniqlash mumkin.

Misol 1. Ushbu

$$x' = x + \mu x^3,\ x(0)=1$$

masalaning $x=\varphi(t;\mu)$ yechimi uchun $\frac{\partial\varphi(t;0)}{\partial\mu}$ hosilani hisoblang.

Berilgan tenglamaning o'ng tomoni $f(t,x,\mu) = x+\mu x^3 \in C^1(\mathbb{R}^3)$, aslida $\in C^\infty(\mathbb{R}^3)$. Demak, isbotlangan teoremani qo'llash mumkin. Tenglamaga $x=\varphi(t;\mu)$ yechimni qo'yib, hosil bo'lgan ayniyatni μ bo'yicha differensiallaymiz va variatsiya uchun tenglamani topamiz $\left(u(t;\mu) = \frac{\partial\varphi(t;\mu)}{\partial\mu}\right.$ kattalik yechimning parametr o'zgarishi bilan o'zgarishini (variatsiyasini) xarakterlaydi):

$$\frac{du(t;\mu)}{dt} = u(t;\mu) + \varphi^3(t;\mu) + 3\mu\varphi^2(t;\mu)u(t;\mu),$$

$$u(0;\mu) = \frac{\partial\varphi(0;\mu)}{\partial\mu} = 0.$$

Biz $u(t;0) = \frac{\partial\varphi(t;0)}{\partial\mu}$ ni hisoblashimiz kerak. Oxirgi masalada (tenglmada) $\mu=0$ deb, topamiz:

$$\frac{du(t;0)}{dt} = u(t;0) + \varphi^3(t;0),\ u(0;0)=0.$$

Bu yerdagi $\varphi(t;0)$ funksiya berilgan masalada $\mu=0$ deb topiladi:

$$x'=x,\ x(0)=1, \text{ ya'ni } \varphi_t'(t;0)=\varphi(t;0),\ \varphi(0;0)=1.$$

Bu masalani yechib, $\varphi(t;0)=e^t$ ekanligini aniqlaymiz. Demak, $u(t;0)$ uchun

$$\frac{du(t;0)}{dt} = u(t;0) + e^{3t},\ u(0;0)=0,$$

masala hosil bo'ldi. Bu masalani yechib, $u(t;0)=\frac{1}{3}(e^{3t}-e^t)$ ni topamiz. Shunday qilib, berilgan masalaning $x=\varphi(t;\mu)$ yechimi uchun $\frac{\partial\varphi(t;0)}{\partial\mu}=u(t;0)=\frac{1}{3}(e^{3t}-e^{t})$.

Endi yechimni boshlang'ich qiymatlar bo'yicha differensiallash masalasi bilan shug'ullanamiz. Buning uchun

$$\begin{cases} x'=f(t,x) \\ x|_{t_0}=\xi \end{cases} \qquad \text{(VII.1.23)}$$

ko'rinishdagi Koshi masalasini qaraylik; bunda $(t,x)\in D$ $((t_0,\xi)\in D)$, $D-\mathbb{R}^{1+n}$ fazodagi soha. Bu masalaning yechimini $x=\varphi(t;\xi)$ $(\varphi(t_0;\xi)=\xi)$ ko'rinishda belgilaymiz (boshlang'ich payt t_0 tayinlangan).

Teorema 2 (yechimning boshlang'ich qiymatlar bo'yicha differensiallanuvchiligi). *Aytaylik, $f(t,x)$ vektor-funksiya va uning $\frac{\partial f(t,x)}{\partial x}$ xususiy hosilasi D sohada uzluksiz hamda (VII.1.23) masalaning $\xi=\xi^0$ dagi $x=\varphi(t;\xi^0)$ yechimi $t\in[t_1,t_2]$ oraliqda aniqlangan bo'lsin. U holda ξ^0 nuqtaning biror $B_{\delta_0}(\xi^0)$ atrofiga tegishli bo'lgan barcha ξ lar uchun $x=\varphi(t;\xi)$ yechimning boshlang'ich qiymatlar bo'yicha*

$$w^j=\frac{\partial\varphi(t;\xi)}{\partial\xi_j}\ (j=1,\dots,n) \quad \textit{hosilalari} \quad (t,\xi)\in[t_1,t_2]\times B_{\delta_0}(\xi^0)$$

to'plamda uzluksiz va ular quyidagi masalalar yechimlaridir:

$$\frac{dw^j}{dt}=\frac{\partial f}{\partial x}w^j\ ,\ w^j|_{t_0}=e^j$$

$$\Big(e^j=(0,\dots,0,\underbrace{1}_{j-\text{o'rin}},0,\dots0)^T;\ j=1,\dots,n\Big)$$

bunda $\frac{\partial f}{\partial x}=\frac{\partial f(t,x)}{\partial x}\Big|_{x=\varphi(t;\xi)}$.

Yangi $y=x-\xi$ noma'lumga o'tamiz. Natijada ushbu

$$\begin{cases} \frac{dy}{dt}=f(t,y+\xi) \\ y|_{t=t_0}=0 \end{cases}$$

yangi masalani hosil qilamiz. Bu masalada endi $\xi=(\xi_1,\xi_2,\dots,\xi_n)^T$ kattaliklar parametrlar rolini o'ynaydi:

$$\begin{cases} \frac{dy}{dt}=g(t,y,\xi) \\ y|_{t=t_0}=0 \end{cases} \quad (\text{ bunda } g(t,y,\xi)=f(t,y+\xi)) \qquad \text{(VII.1.24)}$$

Yechimni $y=\psi(t;\xi)$ $(\psi(t_0;\xi)=0)$ bilan belgilaymiz. Bunda ravshanki, eski $x=\varphi(t;\xi)$ va yangi $y=\psi(t;\xi)$ yechimlar orasida $\varphi(t;\xi)=\xi+\psi(t;\xi)$ bog'lanish mavjud. Yechimning parametrlar bo'yicha differensiallanuvchuligi haqidagi teoremani (VII.1.24) masalaga, ya'ni $y=\psi(t;\xi)$ yechimga qo'llab, oxirgi $\varphi(t;\xi)=\xi+\psi(t;\xi)$ munosabatga ko'ra teoremani isbotlaymiz.

Natija. *Yechimning boshlang'ich qiymatlar bo'yicha differensiallanuvchiligi haqidagi teorema shartlarida ushbu*

$$\det\frac{\partial\varphi(t;t_0,\xi^0)}{\partial\xi}=\exp\int_{t_0}^{t}\sum_{j=1}^{n}\frac{\partial f_j(s,\varphi(s;t_0,\xi^0))}{\partial x_j}ds$$

formula o'rinli.

Teoremadan ravshanki,

$$x'=\frac{\partial f(t,\varphi(t;t_0,\xi^0))}{\partial x}x$$

chiziqli sistemaning fundamental matritsasi ushbu

$$\Phi=[w^1,w^2,\dots,w^n]=\Big[\frac{\partial\varphi(t;\xi)}{\partial\xi_1},\frac{\partial\varphi(t;\xi)}{\partial\xi_2},\dots,\frac{\partial\varphi(t;\xi)}{\partial\xi_n}\Big]=$$

$$=\frac{\partial\varphi(t;\xi)}{\partial\xi}$$

matritsadan iborat. Bundan tashqari $\Phi|_{t=t_0}=E$. Endi Liuvill formulasi isbotni tugatadi. ♁

Nihoyat, yechimning boshlang'ich payt bo'yicha differensiallanuvchiligini qarab chiqamiz. Ushbu

$$\begin{cases} x'=f(t,x) \\ x|_{\tau}=x^0 \end{cases} \qquad \text{(VII.1.25)}$$

boshlang'ich masalani qaraylik; bunda $(t,x)\in D$ ($(\tau,x^0)\in D$) Bu masalaning yechimini $x=\varphi(t;\tau)$ $(\varphi(\tau;\tau)=x^0)$ ko'rinishda belgilaymiz (boshlang'ich qiymat x^0 tayinlangan).

Teorema 3 (yechimning boshlang'ich payt bo'yicha differensiallanuvchiligi). *Aytaylik, $f(t,x)$ vektor-funksiya va uning $\frac{\partial f(t,x)}{\partial x}$ xususiy hosilasi D sohada uzluksiz hamda (VII.1.25) masalaning $\tau=t_0$ bo'lgandagi $x=\varphi(t;t_0)$ yechimi $t\in[t_1,t_2]$ oraliqda aniqlangan bo'lsin. U holda t_0 nuqtaning biror yetarli kichik $(t_0-\delta_0,t_0+\delta_0)$ atrofiga tegishli bo'lgan barcha τ lar uchun $x=\varphi(t;\tau)$ yechimning boshlang'ich payt bo'yicha $w=\frac{\partial\varphi(t;\tau)}{\partial\tau}$ hosilalasi $(t,\tau)\in[t_1,t_2]\times(t_0-\delta_0,t_0+\delta_0)$ to'plamda uzluksiz va u ushbu*

$$\begin{cases} \frac{dw}{dt}=f'_x(t,\varphi(t;\tau))w \\ w|_{t=\tau}=-f(\tau,x^0) \end{cases} \qquad \text{(VII.1.26)}$$

masalaning yechimidan iborat bo'ladi..

⊸ Yangi $s=t-\tau$ erkli o'zgaruvchini kiritamiz. Natijada (VII.1.25) masala o'rniga ushbu

$$\begin{cases} \frac{dx}{ds}=f(s+\tau,x) \\ x|_{s=0}=x^0 \end{cases}$$

yangi masalani hosil qilamiz. Bu masalada endi τ kattalik parametr rolini o'ynaydi:

$$\begin{cases} \frac{dx}{ds}=g(s,x,\tau) \\ x|_{s=0}=x^0 \end{cases} \quad (\text{bunda } g(s,x,\tau)=f(s+\tau,x)) \qquad \text{(VII.1.27)}$$

Yechimni $x=\psi(s;\tau)$ $(\psi(0;\tau)=x^0)$ bilan belgilaymiz:

$$\begin{cases} \frac{d\psi(s;\tau)}{ds}=f(s+\tau,\psi(s;\tau)) \\ \psi(s;\tau)|_{s=0}=x^0 \end{cases}$$

Bunda, ravshanki, eski $x=\varphi(t;\tau)$ va yangi $x=\psi(s;\tau)$ yechimlar orasida $\varphi(t;\tau)=\psi(t-\tau,\tau)$ (yoki $\varphi(s+\tau;\tau)=\psi(s,\tau)$) bog'lanish mavjud. Yechimning parametr bo'yicha differensiallanuvchiligi haqidagi teoremani (VII.1.27) masalaga qo'llab, $x=\psi(s;\tau)$ yechim τ parametrning t_0 ga yaqin qiymatlarida ($|\tau-t_0|<\delta_0$ $(\delta_0>0)$) mavjud, uning $\tilde{w}(s,\tau)=\frac{\partial\psi(s,\tau)}{\partial\tau}$ hosilasi uzluksiz hamda $\tilde{w}(s,\tau)|_{s=0}=0$ ekanligini topamiz. $\varphi(t;\tau)=\psi(t-\tau,\tau)$ formulaga ko'ra

$$w(t;\tau)=\frac{\partial\varphi(t;\tau)}{\partial\tau}=\frac{\partial\psi(t-\tau,\tau)}{\partial s}\cdot(-1)+\frac{\partial\psi(t-\tau,\tau)}{\partial\tau}=$$
$$=-f(t,\psi(t-\tau,\tau))+\tilde{w}(t-\tau,\tau) \qquad \text{(VII.1.28)}$$

hosila ham uzluksiz ($t\in[t_1,t_2]$, $|\tau-t_0|<\delta_0$) va

$$w|_{t=\tau}=-f(\tau,\varphi(\tau,\tau))+\tilde{w}(0,\tau)=-f(\tau,x^0).$$

Bu (VII.1.26) dagi ikkinchi munosabat. $x=\varphi(t;\tau)$ yechim bo'lgani uchun, u (VII.1.25) dagi differensial tenglamani qanoatlantiradi: $\varphi'_t(t;\tau)=f(t,\varphi(t;\tau))$. Bu tenglikni τ bo'yicha differensiallaymiz: $\varphi''_{t\tau}(t;\tau)=f'_x(t,\varphi(t;\tau))\varphi'_\tau(t;\tau)$. Bu tenglikning o'ng tomoni uzluksiz. Demak, uning chap tomonidagi $\varphi''_{t\tau}(t;\tau)$ aralash hosila ham uzluksiz. Differensiallash tartibini o'zgartirib, (VII.1.26) dagi birinchi tenglikni hosil qilamiz. ♁

Eslatma. $\xi\to x=\varphi(t;t_0,\xi)$ akslantirish ξ^0 nuqta atrofida teskarilanuvchi va $\xi=\varphi(t_0;t,x)$; bu yechimning yagonaligidan ravshan. Yuqoridagi teorema shartlarida to'g'ri va teskari akslantirishlar barcha o'zgaruvchilar bo'yicha lokal C^1 sinfga tegishli bo'ladi.

Agar $x'=f(t,x,\mu)$ tenglamaning o'ng tomoni x va μ bo'yicha m marta uzluksiz differensiallanuvchi bo'lsa, uning $x=\varphi(t;\mu)$ yechimi ham μ bo'yicha m marta uzluksiz differensiallanuvchi bo'ladi. Bu tasdiqning aniq ifodalanishi quyidagi teoremada keltirilgan.

Teorema 4. *Yechimning parametr bo'yicha differensiallanuvchiligi haqidagi teorema 1 shartlariga qo'shimcha holda $f(t,x,\mu)$ funksiya $x_1,\dots,x_n,\mu$ lar bo'yicha C^m sinfga tegishli bo'lsin. U holda $x=\varphi(t;\mu)$ yechimning t,μ bo'yicha birinchi tartibli, μ bo'yicha esa $m-$ tartibgacha hosilalari uzluksiz bo'ladi.*

⊶ m bo'yicha matematik induksiya metodini qo'llaymiz. $m=1$ holi teorema 1da qaralgan. Faraz qilaylik, teorema $m=1,2,\dots,k-1$ $(k\ge 2)$ qiymatlar uchun o'rinli bo'lsin. Teoremani $m=k$ uchun isbotlash kerak. Ravshanki,
$$\frac{\partial^k\varphi(t;\mu)}{\partial\mu^k}=\frac{\partial^{k-1}u}{\partial\mu^{k-1}}\quad\left(u=\frac{\partial\varphi(t;\mu)}{\partial\mu}\right)$$
va u funksiya (VII.1.2) masalaning yechimi, ya'ni
$$u'_t=f'_x(t,\varphi(t;\mu))u+f'_\mu(t,\varphi(t;\mu)),\quad u|_{t=t_0}=0.$$
Bu yerdagi differensial tenglamaning o'ng tomoni $u_1,\dots,u_n,\mu$ lar bo'yicha C^{k-1} sinfga tegishli ekanligini ko'rsatish kifoya, chunki u holda $m=k-1$ uchun induksiya farazini yuqoridagi masalaga qo'llab, u yechimning μ bo'yicha $m=k-1$ tartibli
$$\frac{\partial^{k-1}u}{\partial\mu^{k-1}}=\frac{\partial^k\varphi(t;\mu)}{\partial\mu^k}$$
hosilasi uzluksiz ekanligini topamiz. Birinchidan, teoremaning shartiga ko'ra f funksiya $x_1,\dots,x_n,\mu$ lar bo'yicha C^k sinfga tegishli. Demak, f'_x va f'_μ xususiy hosilalar $x_1,\dots,x_n,\mu$ lar bo'yicha $k-1$ marta uzluksiz differensiallanuvchi. Ikkinchidan, induksiya faraziga ko'ra $x=\varphi(t;\mu)$ yechim μ bo'yicha C^{k-1} sinfga tegishli. Shuning uchun $f'_x(t,\varphi(t;\mu))$ va $f'_\mu(t,\varphi(t;\mu))$ murakkab funksiyalar μ bo'yicha, $f'_x(t,\varphi(t;\mu))u+f'_\mu(t,\varphi(t;\mu))$ funksiya esa $u_1,\dots,u_n,\mu$ lar bo'yicha C^{k-1} sinfga tegishli ekanligi ravshan. ♁

Masalalar

1. Ushbu
$$\begin{cases}x'=f(t,x)\\ x|_{t_0}=x^0\end{cases}\quad\text{va}\quad\begin{cases}x'=f(t,x)+r(t,x)\\ x|_{t_0}=x^0\end{cases}$$
masalalarning yechimlarini $x=\varphi(t;t_0,x^0)$ va (mos ravishda) $x=\psi(t;t_0,x^0)$ bilan belgilaylik; bunda $\{f(t,x),\ f'_x(t,x),\ r(t,x),\ r'_x(t,x)\}\subset C(\mathbb{R}^{1+n},\mathbb{R}^n)$ deb faraz qilinadi. Quyidagi belgilashni kiritaylik:
$$\Phi(t;t_0,x^0)=\frac{\partial\varphi(t;t_0,x^0)}{\partial x^0}.$$
U holda

1) $$\psi(t;t_0,x^0)=\varphi(t;t_0,x^0)+\int_{t_0}^{t}\Phi(t;s,\psi(s;t_0,x^0))f(s,\psi(s;t_0,x^0))ds$$

(V. A. Alekseev formulasi).

2) $\varphi(t;t_0,y^0)=\varphi(t;t_0,x^0)+\int_0^1 \Phi(t;t_0,x^0+s(y^0-x^0))ds(y^0-x^0)$.

3) $\psi(t;t_0,y^0)=\varphi(t;t_0,x^0)+\int_0^1 \Phi(t;t_0,x^0+s(y^0-x^0))ds(y^0-x^0)+$

$$+\int_{t_0}^{t}\Phi(t;s,\psi(s;t_0,y^0))f(s,\psi(s;t_0,y^0))ds$$

tengliklarni isbotlang.

VII.2. Kichik parametr metodi

Differensial tenglamalarning taqribiy yechimlarini topishda kichik parametr metodi muhim o'rin tutadi. Ushbu

$$\begin{cases} x'=f(t,x,\mu) \\ x|_{t_0}=x^0 \end{cases}$$

nochiziqli masalaning μ skalyar parametr qiymati $\mu=0$ bo'lgandagi yechimi $x=\varphi^0(t)$ ma'lum bo'lsin. U holda μ parametrning 0 ga yaqin (kichik) qiymatlarida bu masalaning taqribiy yechimini kichik parametr metodi yordamida qurish mumkin.

Teorema. *Aytaylik,* VII.1 *dagi teorema 4 ning shartlari* $(t,x)\in D$, $|\mu|<\varepsilon$ $(\varepsilon>0)$ *sohada o'rinli,* $\mu=0$ *bo'lgandagi* $(t_0\in[t_1,t_2])$ *masalaning* $x=\varphi^0(t)$ *yechimi* $t\in[t_1,t_2]$ *oraliqda aniqlangan bo'lsin. U holda* (*VII.4.22*) *masalaning* $x=\varphi(t;\mu)$ $(t\in[t_1,t_2])$ *yechimi uchun*

$$\varphi(t;\mu)=\varphi^0(t)+\varphi^1(t)\mu+\varphi^2(t)\mu^2+...+\varphi^m(t)\mu^m+o(\mu^m),\ \mu\to 0. \quad \text{(VII.2.1)}$$

asimptotik yoyilma o'rinli; bundan tashqari kichik o $t\in[t_1,t_2]$ *ga nisbatan tekis ham bo'ladi.*

Bu teorema VII.1 paragrafdagi teorema 4 ning

bevosita natijasidir.

Konkret masalalar yechilganda (VII.2.1) yoyilmani, ya'ni $\varphi^0(t)$, $\varphi^1(t)$, $\varphi^2(t)$, ..., $\varphi^m(t)$ vektor-funksiyalarni aniqlash uchun bu yoyilmani qaralayotgan tenglamaga qo'yib,

$$\frac{d\varphi^0(t)}{dt}+\frac{d\varphi^1(t)}{dt}\mu+\frac{d\varphi^2(t)}{dt}\mu^2+$$

$$+...+\frac{d\varphi^m(t)}{dt}\mu^m+o(\mu^m)=f(t,\varphi(t;\mu),\mu),\ \mu\to 0,$$

o'ng tomonni μ ning darajalari bo'yicha yoyib,

$$\frac{d\varphi^0(t)}{dt}+\frac{d\varphi^1(t)}{dt}\mu+\frac{d\varphi^2(t)}{dt}\mu^2+...+\frac{d\varphi^m(t)}{dt}\mu^m+o(\mu^m)=$$

$$=f(t,\varphi(t;0),0)+\left(\frac{\partial f(t,\varphi^0(t;0),0)}{\partial x}\varphi^1(t)+\frac{\partial f(t,\varphi^0(t;0),0)}{\partial \mu}\right)\mu+$$

$$+...+\left(\frac{\partial f(t,\varphi^0(t;0),0)}{\partial x}m!\varphi^m(t)+...\right)\mu^m+o(\mu^m),\ \mu\to 0,$$

hosil bo'lgan tenglikning chap va o'ng tomonlaridagi μ ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirish kerak:

$$\mu^0:\ \frac{d\varphi^0(t)}{dt}=f(t,\varphi^0(t;0),0)$$

$$\mu^1:\ \frac{d\varphi^1(t)}{dt}=\frac{\partial f(t,\varphi^0(t;0),0)}{\partial x}\varphi^1(t)+\frac{\partial f(t,\varphi^0(t;0),0)}{\partial \mu}$$

....

$$\mu^m:\ \frac{d\varphi^m(t)}{dt}=\frac{\partial f(t,\varphi^0(t;0),0)}{\partial x}m!\varphi^m(t)+...$$

Bunda $\varphi^1(t)$, $\varphi^2(t)$,..., $\varphi^m(t)$ funksiyalar uchun chiziqli tenglamalar hosil bo'ladi.

Boshlang'ich shartdan

$$\varphi(t;\mu)|_{t=t_0}=x^0=\varphi^0(t)|_{t=t_0}+\varphi^1(t)|_{t=t_0}\mu+\varphi^2(t)|_{t=t_0}\mu^2+$$

$$+...+\varphi^m(t)|_{t=t_0}\mu^m+o(\mu^m)|_{t=t_0},\ \mu\to 0,$$

ya'ni

$$\boldsymbol{\varphi}^0(t)|_{t=t_0}=\boldsymbol{x}^0,\ \boldsymbol{\varphi}^1(t)|_{t=t_0}=0,\ \boldsymbol{\varphi}^2(t)|_{t=t_0}=0,\dots,\boldsymbol{\varphi}^m(t)|_{t=t_0}=0$$

shartlar hosil bo'ladi. Hosil qilingan tenglamalardan $\varphi^0(t)$ dan boshlab ketma-ket $\varphi^1(t),\varphi^2(t),\dots,\varphi^m(t)$ yechimlarni mos boshlang'ich shartlarga ko'ra topish kerak.

Misol. Ushbu

$$\begin{cases} x'=3y+\mu x \\ y'=2t+\mu xy \\ x|_{t=1}=1,\ y|_{t=1}=1 \end{cases}$$

masala yechimining kichik μ parametr bo'yicha yoyilmasidagi dastlabki uchta hadni quring.

⊶ Berilgan sistemaning o'ng tomoni $(t,x,y)\in D=\mathbb{R}^3$, $|\mu|<+\infty$ sohada xohlagancha marta uzluksiz differensiallanuvchi. Demak, teoremaning shartlari ixtiyoriy m uchun o'rinli. Biz

$$\begin{cases} x=\varphi_0(t)+\varphi_1(t)\mu+\varphi_2(t)\mu^2+o(\mu^2) \\ y=\psi_0(t)+\psi_1(t)\mu+\psi_2(t)\mu^2+o(\mu^2) \end{cases},\ \mu\to 0,$$

yoyilmalardagi koeffitsiyentlarni topishimiz kerak. Bu yoyilmalarni berilgan sistema va boshlang'ich shartlarga qo'yamiz ($o(\mu^2)$ miqdorlar $\mu\to 0$ da tushuniladi):

$$\varphi_0'(t)+\varphi_1'(t)\mu+\varphi_2'(t)\mu^2+o(\mu^2)=$$
$$=3(\psi_0(t)+\psi_1(t)\mu+\psi_2(t)\mu^2+o(\mu^2))+$$
$$+\mu(\varphi_0(t)+\varphi_1(t)\mu+\varphi_2(t)\mu^2+o(\mu^2)),$$

$$\psi_0'(t)+\psi_1'(t)\mu+\psi_2'(t)\mu^2+o(\mu^2)=2t+$$
$$+\mu(\varphi_0(t)+\varphi_1(t)\mu+\varphi_2(t)\mu^2+o(\mu^2))\times$$
$$\times(\psi_0(t)+\psi_1(t)\mu+\psi_2(t)\mu^2+o(\mu^2)),$$

$$(\varphi_0(t)+\varphi_1(t)\mu+\varphi_2(t)\mu^2+o(\mu^2))|_{t=1}=1,$$
$$(\psi_0(t)+\psi_1(t)\mu+\psi_2(t)\mu^2+o(\mu^2))|_{t=1}=1.$$

Bu yerdagi birinchi va ikkinchi tenglamaning o'ng tomonini μ ning darajalari bo'ylab yoyamiz (qavslarni ochib, tartiblari μ^2 gacha bo'lgan hadlarni saqlaymiz):

$$\varphi_0'(t)+\varphi_1'(t)\mu+\varphi_2'(t)\mu^2+o(\mu^2)=3\psi_0(t)+$$
$$+(3\psi_1(t)+\varphi_0(t))\mu+(3\psi_2(t)+\varphi_1(t))\mu^2+o(\mu^2),$$
$$\psi_0'(t)+\psi_1'(t)\mu+\psi_2'(t)\mu^2+o(\mu^2)=2t+$$
$$+\varphi_0(t)\psi_0(t)\mu+(\varphi_0(t)\psi_1(t)+\varphi_1(t)\psi_0(t))\mu^2+o(\mu^2),$$
$$\varphi_0(1)+\varphi_1(1)\mu+\varphi_2(1)\mu^2+o(\mu^2)|_{t=1}=1,$$
$$\psi_0(1)+\psi_1(1)\mu+\psi_2(1)\mu^2+o(\mu^2))|_{t=1}=1.$$

Endi μ ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib, quyidagi masalalarni tuzamiz:

$$\mu^0:\ \begin{cases} \varphi_0'(t)=3\psi_0(t) \\ \psi_0'(t)=2t \\ \varphi_0(1)=1,\ \psi_0(1)=1 \end{cases},$$

$$\mu^1:\ \begin{cases} \varphi_1'(t)=3\psi_1(t)+\varphi_0(t) \\ \psi_1'(t)=\varphi_0(t)\psi_0(t) \\ \varphi_1(1)=0,\ \psi_1(1)=0 \end{cases},$$

$$\mu^2:\ \begin{cases} \varphi_2'(t)=3\psi_2(t)+\varphi_1(t) \\ \psi_2'(t)=\varphi_0(t)\psi_1(t)+\varphi_1(t)\psi_0(t) \\ \varphi_2(1)=0,\ \psi_2(1)=0 \end{cases}$$

Bu masalalarni birinchisidan boshlab ketma-ket yechamiz va quyidagilarni topamiz:

$$\begin{cases}\varphi_0(t)=t^3\\ \psi_0(t)=t^2\end{cases},\quad \begin{cases}\varphi_1(t)=\dfrac{t^7}{14}+\dfrac{t^4}{4}\\ \psi_1(t)=\dfrac{t^6}{6}\end{cases},\quad \begin{cases}\varphi_2(t)=\dfrac{t^{11}}{140}+\dfrac{5t^8}{192}+\dfrac{t^5}{20}\\ \psi_2(t)=\dfrac{t^{10}}{42}+\dfrac{t^7}{28}\end{cases}$$

Demak, berilgan masala yechimi uchun ushbu

$$\begin{cases}x=t^3+(\dfrac{t^7}{14}+\dfrac{t^4}{4})\mu+(\dfrac{t^{11}}{140}+\dfrac{5t^8}{192}+\dfrac{t^5}{20})\mu^2+o(\mu^2)\\ y=t^2+\dfrac{t^6}{6}\mu+(\dfrac{t^{10}}{42}+\dfrac{t^7}{28})\mu^2+o(\mu^2)\end{cases},\ \mu\to 0,$$

asimptotik yoyilmalar o'rinli. ♁

Masalalar

1. Quyidagi masalani qarang:

$$\begin{cases}x'-x+\mu x^2=0,\\ x|_{t=0}=1.\end{cases}$$

a). Masala yechimining kichik μ parametr bo'yicha yoyilmasidagi dastlabki uchta hadni quring.

b). Aniq yechimni toping (Bernulli tenglamasi).

d). Aniq yechimning kichik μ bo'yicha yoyilmasini toping va uni a) banddagi yoyilma bilan taqqoslang.

2. Ushbu

$$\begin{cases}x'=y\\ y'=-x-\mu x^2\\ x|_{t=0}=x_0,\ x'|_{t=0}=v_0\end{cases}\qquad (x''=-x-\mu x^2)$$

masala yechimining kichik μ parametr bo'yicha yoyilmasidagi dastlabki uchta hadni aniqlang.

3. Ushbu

$$\begin{cases}x'=y\\ y'=-x-\mu x^3\\ x|_{t=0}=x_0,\ x'|_{t=0}=v_0\end{cases}\qquad (x''=-x-\mu x^3 \text{— Dyuffing tenglamasi})$$

masala yechimining kichik μ parametr bo'yicha yoyilmasidagi dastlabki uchta hadni aniqlang.

4. Ushbu

$$\begin{cases}x'=y\\ y'=-x+\mu(1-x^2)y\\ x|_{t=0}=x_0,\ x'|_{t=0}=v_0\end{cases}\quad (x''=-x+\mu(1-x^2)x' \text{— Van-der-Pol tenglamasi})$$

masala yechimining kichik μ parametr bo'yicha yoyilmasidagi dastlabki uchta hadni quring.

VII.3. Birinchi integrallar

Quyidagi sistemani qaraylik:

$$\frac{dx}{dt}=f(t,x). \qquad (VII.3.1)$$

Biz bu yerda $f\in C^1(D,\mathbb{R}^n)$ deb hisoblaymiz ($D\subset\mathbb{R}^{1+n}$ – soha), $f=(f_1,\dots,f_n)^T$. Avvalgidek, (VII.3.1) sistemaning $x|_{t=t_0}=x^0$ boshlang'ich shartni qanoatlantiruvchi yechimini $x=\varphi(t,t_0,x^0)$ bilan belgilaymiz.

O'zgarmasdan farqli $u=u(t,x)\in C^1(D,\mathbb{R})$ funksiyani qaraylik. Agar (VII.3.1) sistemaning (D da joylashgan) ixtiyoriy $x=\varphi(t)$ yechimida (yechimi bo'ylab) $u(t,x)$ funksiya o'zgarmasga aylansa, ya'ni $u(t,\varphi(t))=\text{const}$ bo'lsa, u holda $u(t,x)$ funksiya (VII.3.1) sistemaning (D sohada aniqlangan) **birinchi integrali** deyiladi.

Misol 1. Ushbu

$$\begin{cases}x_1'=x_2\\ x_2'=-x_1\end{cases}$$

sistemaning birinchi integrali $u=x_1^2+x_2^2$, chunki ixtiyoriy $x_1=x_1(t)$, $x_2=x_2(t)$ yechim bo'ylab $\frac{1}{2}\frac{d}{dt}\left(x_1^2+x_2^2\right)=x_1x_1'+x_2x_2'=0$; demak, $x_1^2+x_2^2=\text{const}$. ♁

Misol 2. $H = H(p_1,\dots,p_n,q_1,\dots,q_n)\in C^2$ funksiyaga ko'ra tuzilgan ushbu

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \ \dot{q}_i = \frac{\partial H}{\partial p_i}, \ i = 1,\dots,n, \qquad \text{(VII.3.2)}$$

sistemani qaraylik. (VII.3.2) differensial tenglamalar sistemasi Hamiltonning kanonik tenglamalar sistemasi deb ataladi. Bu yerdagi H funksiya Hamilton funksiyasi deyiladi. Fizikada uchraydigan ko'p jarayonlar (VII.3.2) sistema bilan boshqariladi. ifodalanadi. Hamiltonning H funksiyasi (VII.3.2) kanonik tenglamalar sistemasi uchun birinchi integraldir.

⊶ Haqiqatdan ham, ixtiyoriy $p_i = p_i(t)$, $q_i = q_i(t)$ yechim bo'ylab

$$\frac{dH(p_1(t),\dots,p_n(t),q_1(t),\dots,q_n(t))}{dt} = \sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\dot{p}_i + \sum_{i=1}^{n}\frac{\partial H}{\partial q_i}\dot{q}_i =$$

$$= -\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\cdot\frac{\partial H}{\partial q_i} + \sum_{i=1}^{n}\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} = 0$$

va, demak, $H = \text{const}$ bo'ladi. ☝

Teorema 1. *O'zgarmasdan farqli* $u\in C^1(D,\mathbb{R})$ *funksiya* (VII.3.1) *sistemaning birinchi integrali bo'lishi uchun* D *sohada*

$$\frac{\partial u(t,x)}{\partial t} + \sum_{i=1}^{n}\frac{\partial u(t,x)}{\partial x_i}\cdot f_i(t,x) = 0 \qquad \text{(VII.3.3)}$$

tenglikning o'rinli bo'lishi yetarli va zarurdir.

⊶ **Yetarliligi.** $u\in C^1$ funksiya (VII.3.3) shartni qanoatlantirsin. (VII.3.1) sistemaning ixtiyoriy $x=\varphi(t)$ yechimida $u = u(t,x)$ funksiya o'zgarmasga aylanadi, chunki (VII.3.3) ga ko'ra uning hosilasi nolga teng:

$$\frac{du(t,\varphi(t))}{dt} = \left.\frac{\partial u}{\partial t}\right|_{x=\varphi(t)} + \sum_{i=1}^{n}\left.\frac{\partial u}{\partial x_i}\right|_{x=\varphi(t)}\cdot\dot{x}_i =$$

$$= \left.\frac{\partial u}{\partial t}\right|_{x=\varphi(t)} + \sum_{i=1}^{n}\left.\frac{\partial u}{\partial x_i}\right|_{x=\varphi(t)} = 0$$

Zarurligi. $u\in C^1$ funksiya (VII.3.1) sistemaning birinchi integrali bo'lsin. $\forall(\tau,\xi)\in D$ nuqtada (VII.3.3) munosabatning o'rinli ekanligini ko'rsatish uchun $x=\varphi(t,\tau,\xi)$ yechimni olib, $G(t) = u(t,\varphi(t,\tau,\xi))$ funksiyani tuzaylik. Bu funksiya t ga bog'liq emas (u birinchi integral bo'lgani uchun). Demak, uning t bo'yicha hosilasi nolga teng ($t=\tau$ nuqtada ham):

$$0 = \left.\frac{dG(t)}{dt}\right|_{t=\tau} = \left.\frac{\partial u}{\partial t}\right|_{\substack{x=\varphi(t,\tau,\xi)\\ t=\tau}} + \sum_{i=1}^{n}\left.\frac{\partial u}{\partial x_i}\cdot f_i\right|_{\substack{x=\varphi(t,\tau,\xi)\\ t=\tau}} =$$

$$= \frac{\partial u(\tau,\xi)}{\partial t} + \sum_{i=1}^{n}\frac{\partial u(\tau,\xi)}{\partial x_i}\cdot f_i(\tau,\xi);$$

biz bu yerda $\varphi(\tau,\tau,\xi)=\xi$ ekanligidan foydalandik. ☝

Eslatma. $u(t,x)$ funksiyaning sath to'plami (chizig'i, sirti) deb $\{(t,x)\,|\,u(t,x)=c-\text{const}\}\subset\mathbb{R}^{1+n}$ to'plamga aytiladi. Demak, $x=\varphi(t)$ yechim grafigi (integral chiziq) birinchi integralning bitta sath to'plamida to'laligicha joylashadi.

Agar $u_1(t,x),\dots,u_k(t,x)$, $k\le n$, birinchi integrallar uchun ushbu

$$\frac{\partial(u_1,\dots,u_k)}{\partial(x_1,\dots,x_n)} = \left\|\frac{\partial u_i}{\partial x_j}\right\| = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_k}{\partial x_1} & \frac{\partial u_k}{\partial x_2} & \cdots & \frac{\partial u_k}{\partial x_n} \end{pmatrix}$$

Yakobi matritsasining berilgan nuqtadagi rangi k ga teng bo'lsa, u holda $u_1,...,u_k$ birinchi integrallar qaralayotgan nuqtada erkli deb ataladi.

n – tartibli (VII.3.1) sistemaning n dona erkli $u_1,u_2,...u_n$ birinchi integrallari birinchi integrallarning to'la sistemasi deyiladi. Bu holda $\left\|\frac{\partial u_i}{\partial x_j}\right\|$ kvadrat matritsaning determinanti noldan farqli.

Birinchi integrallarning to'la sistemasi $u_1,...,u_n$ uchun (VII.3.3) shartlar

$$\frac{\partial u_i}{\partial t}+\sum_{j=1}^{n}\frac{\partial u_i}{\partial x_j}f_j=0,\ i=1,...,n,\ \ (\text{ya'ni } \frac{\partial u}{\partial t}+\frac{du}{dx}f=0)$$

(VII.3.1) sistemaning o'ng tomonini bir qiymatli aniqlaydi:

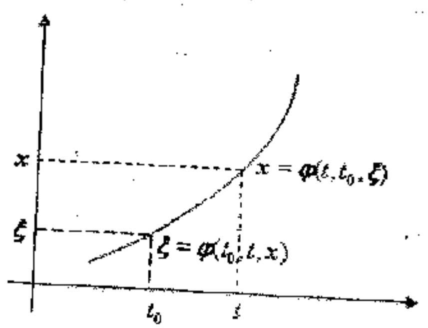
$$f=-\left(\frac{du}{dx}\right)^{-1}\frac{\partial u}{\partial t}$$

bu yerda $u=(u_1,...,u_n)^T$; $\frac{du}{dx}=\frac{d(u_1,...,u_n)}{d(x_1,...x_n)}$.

Teorema 2 (birinchi integrallarning to'la sistemasi haqida). *Agar $f\in C^1(D,\mathbb{R}^n)$ bo'lsa, (VII.3.1) sistema ixtiyoriy $(t_0,x^0)\in D$ nuqtaning yetarlicha kichik atrofida birinchi integrallarning to'la sistemasiga ega. U $u(t,x)=\varphi(t_0,t,x)$ formula bilan aniqlanadi. ((VII.3.1) sistema birinchi integrallarining to'la sistemasini aniqlash uchun uning yechimini beruvchi $x=\varphi(t,t_0,\xi)$ tenglikdan ξ ni topish kerak; yechimning yagonalik xossasiga ko'ra $\xi=\varphi(t_0,t,x)$ hosil bo'ladi).*

8→ f ga nisbatan qo'yilgan $f\in C^1(D,R^n)$ shartda $x=\varphi(t,\tau,\xi)$ yechim $(t,\tau,\xi)=(t_0,t_0,x^0)\in\mathbb{R}^{2+n}$ nuqtaning yetarlicha kichik atrofida C^1 sinfga tegishli ekanligi hamda $\left.\frac{d\varphi(t,\tau,\xi)}{d\xi}\right|_{t=\tau}=E$ bo'lishi bizga ma'lum. Demak

$\det\left.\frac{d\varphi}{d\xi}\right|_{t=\tau}=1\neq 0$ va $\det\frac{d\varphi}{d\xi}$ yakobianning qiymati (t_0,t_0,x^0) nuqtaning yetarlicha kichik atrofida ham nolga aylanmaydi.



VII.1- rasm. t_0 paytda ξ nuqtada bo'lgan yechim t paytda $x=\varphi(t,t_0,\xi)$ nuqtada bo'ladi

Teskari funksiya haqidagi teoremaga ko'ra $(t_0,x^0)\in D$ nuqtaning yetarlicha kichik atrofida $u(t,x)\equiv\varphi(t_0,t,x)\in C^1$ va $\frac{du}{dx}$ matritsaning rangi n ga teng, ya'ni u teskarilanuvchi. Endi $u(t,x)=(u_1(t,x),...,u_n(t,x))^T$ vektor-funksiya (VII.3.1) sistemaning ixtiyoriy yechimida o'zgarmasga aylanishini ko'rsatish qoldi. $x=\varphi(t)$ yechimni olaylik Aytaylik, $\varphi(t)=\varphi(t,t_0,o)$ bo'lsin. Yechimning yagonalik xossasiga ko'ra $u(t,\varphi(t))=\varphi(t_0,t,\varphi(t,t_0,o))=\xi-const$ ekanligi ravshan. ♗

Shunday qilib, agar (VII.3.1) sistema uchun birinchi integrallarning to'la sistemasi $u(t,x)=(u_1(t,x),...,u_n(t,x))^T$ topilgan bo'lsa, u holda $u(t,x)=c$ ($c\in R^n$ – o'zgarmas vektor) tenglikni x ga nisbatan yechib, (VII.3.1) sistemaning $x=\varphi(t;c)$ yechimlarini hosil qilamiz:

$$u(t,x)=c \;\Rightarrow\; \frac{\partial u}{\partial t}+\frac{du}{dx}\cdot\frac{dx}{dt}=0 \;\Rightarrow\; \frac{dx}{dt}=-\left(\frac{du}{dx}\right)^{-1}\frac{\partial u}{\partial t}\bigg|_{x=\varphi(t;c)} \qquad \text{(VII.3.4)}$$

Lekin $u(t,x)$ – to'la sistema bo'lgani uchun

$$f=-\left(\frac{du}{dx}\right)^{-1}\frac{\partial u}{\partial t}. \qquad \text{(VII.3.5)}$$

Demak, (VII.3.4), (VII.3.5) ga ko'ra $\frac{dx}{dt}=f$.

Bu yerda topilgan $x=\varphi(t;c)$ yechim umumiy yechimni ifodalaydi.

Ko'pincha $u(t,x)$ birinchi integral orqali yozilgan $u(t,x)=c$ munosabat ham birinchi integral deb ataladi.

Teorema 3. *Agar $u=(u_1,..u_n)^T$ birinchi integrallarning to'la sistemasi, $u(t,x)$ esa ixtiyoriy birinchi integral bo'lsa, u holda shunday $\varphi\in C^1$ funksiya mavjudki, uning uchun $u(t,x)=\varphi(u_1(t,x),...,u_n(t,x))$ munosabat o'rinli bo'ladi, ya'ni har qanday birinchi integral integrallarning to'la sistemasi orqali ifodalanadi.*

🗝 x o'rniga yangi noma'lum y funksiyani

$$y=u(t,x) \qquad \text{(VII.3.6)}$$

formula bilan kiritamiz. U holda (VII.3.1) sistema $y'=0$ ko'rinishga keladi. (VII.3.6) dan x ni y orqali ifodalaymiz: $x=u^{-1}(t,y)$. $u(t,x)$ funksiyani ham y orqali ifodalab, $\varphi(t,y)$ funksiyani hosil qilamiz: $u(t,x)=u(t,u^{-1}(t,y))=\varphi(t,y)$. $u(t,x)$ funksiya (VII.3.1) sistemaning birinchi integrali bo'lgani uchun $\dot{u}=\dot{\varphi}=0$ Bundan $\frac{\partial\varphi}{\partial t}+\sum\frac{\partial\varphi}{\partial y_i}\dot{y}_i=0$. Demak, $\frac{\partial\varphi}{\partial t}=0$, ya'ni φ funksiya t ga oshkor holda bog'liq emas:

$u(t,x)=\varphi(y)=\varphi(u)$. ♘

Misol 1. Ushbu

$$\frac{dx}{dt}=\frac{t-y}{y-x},\quad \frac{dy}{dt}=\frac{x-t}{y-x} \qquad \text{(VII.3.10)}$$

sistemaning umumiy integralini topaylik.

🗝 Tenglamalarni hadma-had qo'shamiz.

$$\frac{d(x+y)}{dt}=\frac{x-y}{y-x}\Rightarrow\frac{d(x+y)}{dt}=-1$$

Demak, integrallanuvchi kombinatsiya hosil bo'ldi. Uni yechib, birinchi integralni topamiz:

$$x+y=-t+c_1. \qquad \text{(VII.3.11)}$$

Birinchi tenglamaning har ikkala tomonini x ga, ikkinchisini y ga ko'paytirib, hosil bo'lgan tengliklarni hadma-had qo'shamiz:

$$\frac{d}{dt}\left(\frac{x^2}{2}+\frac{y^2}{2}\right)=\frac{xt-yt}{y-x}\Rightarrow\frac{d}{dt}\left(\frac{x^2+y^2}{2}\right)=-t$$

Oxirgi integrallanuvchi kombinatsiyadan yana bir dona

$$x^2+y^2=-t^2+c_2 \qquad \text{(VII.3.12)}$$

birinchi integralni topamiz.

(VII.3.11) va (VII.3.12) birinchi integrallar (VII.3.10) sistemaning umumiy integralini beradi (ularning erkli ekanligini tekshirish o'quvchiga havola etiladi). ♘

Endi muxtor sistema birinchi integrallarining to'la sistemasida to'xtalaylik.

Muxtor sistema

$$\dot{x}=f(x) \qquad \text{(VII.3.7)}$$

vektor ko'rinishida berilgan bo'lsin. Eslaylikki, agar $f(b)=0$ bo'lsa, $b\in R^n$ nuqta (VII.3.7) sistemaning muvozanat nuqtasi bo'ladi. Biz (VII.3.7) sistemani uning muvozanat (statsionar) nuqtasi bo'lmagan nuqta atrofida tekshiramiz.

Teorema 4. *Faraz qilaylik, $f(b)\neq 0$ va $b\in\mathbb{R}^n$ nuqtaning biror atrofida $f\in C^1$ bo'lsin. U holda b nuqtaning* *biror kichik atrofida* (VII.3.7) *sistemaning* $(n-1)$ *ta erkli birinchi integralari mavjud.*

⊶ $f(b)=(f_1(b),...,(f_n(b))^T \neq 0$ bo'lgani uchun $f_k(b)$, $k=1,...,n$, qiymatlarning birortasi noldan farqli. Aniqlik uchun $f_1(b)\neq 0$ deylik. $f_1 \in C^1$ bo'lgani uchun b nuqtaning biror atrofida ham $f_1(x)\neq 0$

Shu atrofda

$$\begin{cases} \frac{dx_1}{dt}=f_1(x_1,...,x_n) \\ \frac{dx_2}{dt}=f_2(x_1,...,x_n) \\ \dots\dots\dots\dots \\ \frac{dx_n}{dt}=f_n(x_1,...,x_n) \end{cases}$$

va $\frac{dx_1}{dt}=f_1\neq 0$ bo'lgani uchun t o'rniga x_1 erkli o'zgaruvchini kiritamiz hamda

$$\frac{dx_2}{dx_1}=\frac{f_2}{f_1},\ \frac{dx_3}{dx_1}=\frac{f_3}{f_1},\ \dots\dots\dots,\ \frac{dx_n}{dx_1}=\frac{f_n}{f_1}$$

sistemani hosil qilamiz. Oxirgi sistema uchun yuqoridagi teoremaga ko'ra birinchi integrallarning to'la sistemasi mavjud. Bu birinchi integrallar (VII.3.7) sistemaning t o'zgaruvchiga bog'liq bo'lmagan $(n-1)$ ta erkli birinchi integrallarini tashkil etadi. ⊲

Agar (VII.3.1) sistemaning k ta erkli birinchi integrallari $u_1(t,x),...,u_k(t,x)$ topilgan bo'lsa, u holda

$$\begin{cases} u_1(t,x_1,...,x_n)=c_1 \\ u_2(t,x_1,...,x_n)=c_2 \\ \dots\dots\dots\dots \\ u_k(t,x_1,...,x_n)=c_k \end{cases} \qquad \text{(VII.3.8)}$$

sistemadan $x_1,...,x_n$ noma'lumlarning k tasini qolganlari orqali ifodalash mumkin. Aniqlik uchun $x_1,...,x_k$ o'zgaruvchilar $x_{k+1},...,x_n$ ($c_1,...c_k$ hamda t) orqali ifodalansin deylik:

$$\begin{aligned} x_1 &= \varphi_1(x_{k+1},...,x_n,c_1,...,c_k,t) \\ x_2 &= \varphi_2(x_{k+1},...,x_n,c_1,...,c_k,t) \\ &\dots\dots\dots\dots \\ x_k &= \varphi_k(x_{k+1},...,x_n,c_1,...,c_k,t) \end{aligned}$$

Bu munosabatlarni (VII.3.1) sistemaning keyingi $(n-k)$ ta tenglamalariga qo'yib $x_{k+1},...,x_n$ noma'lumlarga nisbatan ushbu

$$\begin{cases} \frac{dx_{k+1}}{dt}=f_{k+1}(\varphi_1,...,\varphi_k,x_{k+1},...,x_n) \\ \frac{dx_{k+2}}{dt}=f_{k+2}(\varphi_1,...,\varphi_k,x_{k+1},...,x_n) \\ \dots\dots\dots\dots \\ \frac{dx_n}{dt}=f_n(\varphi_1,...,\varphi_k,x_{k+1},...,x_n) \end{cases} \qquad \text{(VII.3.9)}$$

sistemani hosil qilamiz.

(VII.3.9) sistema uchun birinchi integrallarning to'la sistemasi va (VII.3.8) munosabatlar birgalikda (VII.3.1) sistema birinchi integrallarining to'la sistemasini beradi, ya'ni (VII.3.1) sistemani yechish masalasini hal qiladi..

Birinchi integrallarni topishning umumiy usuli yo'q. Ko'p hollarda berilgan sistema tenglamalarini almashtirish yordamida osongina integrallanuvchi differensial tenglama hosil qilish mumkin. Bunnday tenglama integrallanuvchi kombinatsiya deb ataladi.

Berilgan differensial tenglamalar sistemasining integrallanuvchi kombinatsiyalarini tuzib, sistemaning (o'zaro bog'liq bo'lmagan) erkli birinchi integrallarini topish mumkin.

Misol 2. Ushbu

$$\begin{cases} x' = (1-y)x \\ y' = \alpha(x-1)y \end{cases} \quad (\alpha = \text{const} > 0)$$

Volterra-Lotka sistemasini qaraylik ($x > 0, y > 0$).

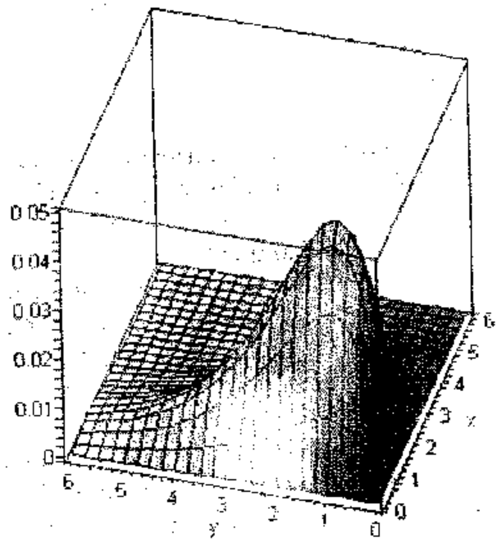
Bu sistemaning birinchi integralini topish uchun ikkinchi tenglamani birinchisiga hadma-had bo'laylik:

$$\frac{dy}{dx} = \frac{\alpha(x-1)y}{(1-y)x}.$$

Bu o'zgaruvchilari ajraluvchi tenglamni osongina yechamiz:

$$\frac{yx^{\alpha}}{e^{y}e^{\alpha x}} = c\ ,\ c \in \mathbb{R} - \text{o'zgarmas son.}$$

Demak, $u(x,y) = \dfrac{x^{\alpha}y}{e^{\alpha x}e^{y}}$ – birinchi integral. Bu $u = u(x,y) = \dfrac{x^{\alpha}y}{e^{\alpha x}e^{y}}$ funksiyaning grafigi VII.2-rasmda keltirilgan.



VII.2-rasm

Bu yerda shuni e'tirof etish mumkinki, agar $x(0) = x_0$, $y(0) = y_0$ boshlang'ich qiymatlar va x ning t paytdagi qiymati $x(t)$ ma'lum (u yoki bu usul yordanida topilgan) bo'lsa, u holda $y(t)$ qiymatni berilgan differensial tenglamalar sistemasini yechmasdan turib, birinchi integraldan foydalanib topish mumkin. Buning uchun

$$\frac{yx^{\alpha}(t)}{e^{y}e^{\alpha x(t)}} = \frac{y_0 x_0^{\alpha}}{e^{y_0}e^{\alpha x_0}}$$

transendent tenglamani $y(= y(t))$ ga nisbatan yechish kerak (biror sonli usul yordamida).

Birinchi integralning differensial tenglamani tekshirishdagi tatbig'i sifatida to'g'ri chiziq bo'ylab (inersial sanoq sistemasida) $F(x)$ kuch ta'sirida harakat qiluvchi, massasi birga teng bo'lgan moddiy nuqta harakatini o'rganamiz. Bu holda Nyutonning ikkinchi qonuni

$$\ddot{x} = F(x) \qquad \text{(VII.3.13)}$$

tenglamani beradi; bu yerda $x = x(t)$ nuqtaning t paytdagi koordinatasi, $\ddot{x} = \ddot{x}(t)$ -uning tezlanishi; biz $F(x)$ funksiyani biror intervalda differensiallanuvchi deb faraz qilamiz.

(VII.3.13) tenglamada $x_1 = x$, $x_2 = \dot{x}$ deb uni quyidagi sistemaga keltiramiz:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = F(x_1) \end{cases} \qquad \text{(VII.3.14)}$$

(VII.3.13) yoki (VII.3.14) sistema uchun

$T = \dfrac{\dot{x}^2}{2} = \dfrac{x_2^2}{2}$ – kinetik energiya $U = -\int_{x_0}^{x} F(s)ds$ potensial energiya $\left(F(x) = -\dfrac{dU}{dx}\right)$ $E = T + U$ - to'la mexanik energiya deb ataladi.

Teorema 5 (energiyaning saqlanish qonuni). *To'la energiya E (VII.3.14) sistemaning birinchi integralidir (har* *qanday harakatda to'la energiya saqlanadi).*

⊸ Isboti oson:

$$\frac{dE}{dt}=\frac{d}{dt}\left(\frac{x_2^2(t)}{2}+U(x_1(t))\right)=x_2\cdot\dot{x}_2+U'\cdot\dot{x}_1=$$

$$=x_2F(x_1)-F(x_1)\cdot x_2=0.$$

(VII.3.2) sistema traektoriyasining har biri energiyaning sath to'plamida joylashadi. Energiyaning

$$\left\{(x_1,x_2)\in\mathbb{R}^2 \;\middle|\; \frac{x_2^2}{2}+U(x_1)=E=\text{const}\right\}$$

sath to'plami (chizig'i) sistemaning muvozanat holatidan, ya'ni

$$\{(x_1,x_2)\mid F(x_1)=0,\ x_2=0\}$$

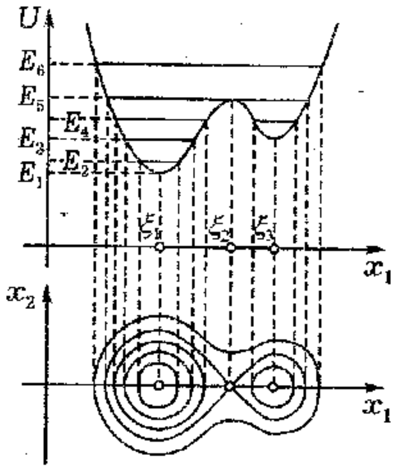
nuqta(lar)dan boshqa barcha nuqtalar atrofida silliq chiziqdan iborat bo'ladi, chunki bunday nuqtalarda

$$\frac{\partial E}{\partial x_1}=-F(x_1),\ \frac{\partial E}{\partial x_2}=x_2$$

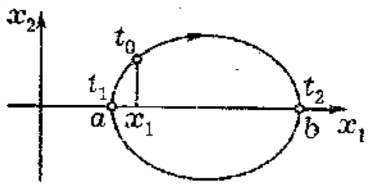
hosilalarning kamida biri 0 ga teng emas va oshkormas funksiya haqidagi teoremaga ko'ra sath to'plami bu nuqtalar atrofida $x_1=x_1(x_2)$ yoki $x_2=x_2(x_1)$ differensiallanuvchi funksiyaning grafigidan iborat bo'ladi. Energiyaning sath chizig'i (to'plami)

$$|x_2|=\sqrt{2(E-U(x_1))} \qquad \text{(VII.3.15)}$$

tenglama bilan beriladi. $U(x_1)$ funksiyaning grafigiga ko'ra (VII.3.15) chiziqni chizish qiyin emas (VII.3-rasm).



VII.3-rasm



VII.4-rasm. $U'(a)\neq 0, U'(b)\neq 0, U(a)=U(b)=E$

VII.4-rasmda ko'rsatilgan fazaviy traektoriyani qaraylik. Birinchi integraldan

$$\dot{x}=\pm\sqrt{2(E-U(x))} \qquad \text{(VII.3.16)}$$

tenglamani topamiz. Bu tenglamada o'zgaruvchilar ajraladi. (VII.3.16) ning $x(t_0)=x_1$, $\dot{x}(t_0)=x_2>0$ shartni qanoatlantiruvchi $x=x(t)$ yechimi

$$t-t_0=\int_{x_1}^{x(t)}\frac{ds}{\sqrt{2(E-U(s))}} \qquad \text{(VII.3.17)}$$

tenglikdan aniqlanadi. $U'(a)\neq 0$, $U'(b)\neq 0$ bo'lgani uchun

$$\frac{\omega}{2}=\int_a^b\frac{ds}{\sqrt{2(E-U(s))}}$$

chekli sondan iborat (integral yaqinlashuvchi).

Demak, (VII.3.17) formula (VII.3.1) ning $x = x(t)$ yechimini biror $t_1 \le t \le t_2$ oraliqda aniqlaydi, bunda $x(t_1) = a$, $x(t_2) = b$, $t_2 - t_1 = \omega/2$ bo'ladi.

Endi $x(t)$ yechimni $[t_1, t_2]$ segmentdan $\left[t_2, t_2 + \frac{\varpi}{2}\right]$ segmentgacha $x(t_2 + \tau) = x(t_2 - \tau)$, $0 \le \tau \le \omega/2$ formulaga ko'ra davom ettiramiz. So'ngra $x(t+\omega) \equiv x(t)$ munosabat bilan uni $-\infty < t < \infty$ oraliqqa davriy davom ettiramiz. Hosil bo'lgan $x = x(t)$ funksiya (VII.3.1) tenglamani $\forall t \in \mathbb{R}$ nuqtada qanoatlantiradi hamda $x(t_0) = x_1$, $\dot{x}(t_0) = x_2$ bo'ladi. Qurilgan $x = x(t)$ yechim ω davrli; uning fazaviy traektoriyasi VII.4-rasmda ko'rsatilgan silliq yopiq egri chiziqdan iborat.

Masalalar

1. Ushbu

$$\begin{cases} x' = x^2 + y^2 \\ y' = 2xy \end{cases}$$

sistemani yeching.

2. Ushbu

$$\begin{cases} x' = -x \\ y' = -y \end{cases}$$

ajralgan sistemani qarang. Quyidagi tasdiqlarni isbotlang:

1) Sistema ixtiyoriy $\delta > 0$ uchun $B_\delta = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < \delta^2\}$ doirada aniqlangan birinchi integralga ega emas.

2) Sistema $x > 0$ yarim tekislikda birinchi integalga ega.

3. Ushbu

$$\begin{cases} x' = 1 + 3y^2 \\ y' = xyz \\ z' = -xz^2 \end{cases}$$

sistemaning ikkita erkli birinchi integralini toping.

4. Ushbu

$$\begin{cases} x' = (1 - y)x \\ y' = \alpha(x - 1)y \end{cases} \quad (\alpha > 0)$$

Volterra-Lotka sistemasining $x > 0, y > 0$ sohadagi fazaviy traektoriyalari sodda yopiq chiziq (yechimlar davriy) ekanligini isbotlang.

5. Quyidagi avtonom sistemaning ikkita erkli birinchi integralini toping va ular yordamida sistemaning traektoriyalarini tekshiring:

$$\begin{cases} x_1' = x_2 - x_3 \\ x_2' = x_3 - x_1 \\ x_3' = x_1 - x_2 \end{cases}$$

VII.4. Birinchi tartibli xususiy hosilali differensial tenglamalar

Asosiy ta'riflar

Qisqalik uchun $x = (x_0, x_1, ..., x_n)^T \in \mathbb{R}^{1+n}$, $u \in \mathbb{R}$ va $p = (p_0, p_1, ..., p_n)^T \in \mathbb{R}^{1+n}$ belgilashlarni kiritaylik.Ushbu

$$F(x, u, p) \equiv F(x_0, x_1, ..., x_n, u, p_0, p_1, ..., p_n)$$

haqiqiy funksiya (x, u, p) vektor o'zgaruvchi bo'yicha biror $G \subset \mathbb{R}^{2n+3}$ sohada aniqlangan, $p_0, p_1, ..., p_n$ o'zgaruvchilar bo'yicha o'zining birinchi tartibli xusisiy hosilalari bilan birgalikda uzluksiz ($\in C^1$) bo'lsin. Shu G sohada $\sum_{i=0}^{n} \left|\frac{\partial F}{\partial p_i}\right| \ne 0$ deb ham hisoblaymiz.

Ushbu

$$F(x_0, x_1, ..., x_n, u, \frac{\partial u}{\partial x_0}, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}) = 0 \qquad \text{(VII.4.1)}$$

tenglama $u = u(x_0, x_1, ..., x_n)$ noma'lum funksiyaga nisbatan **birinchi tartibli xususiy hosilali differensial tenglama** deyiladi.

Agar $u = u(x_0, x_1, ..., x_n)$ funksiya $D \subset \mathbb{R}^{1+n}$ sohada

$$\frac{\partial u}{\partial x_0}, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}$$

uzluksiz hosilalarga ega, ya'ni $u \in C^1(D, \mathbb{R})$ bo'lib, (VII.4.1) tenglamani ayniyatga aylantirsa (qanoatlantirsa), u holda shu u funksiya (VII.4.1) tenglamaning (D sohada aniqlangan) yechimi deyiladi. Tabiiyki, bu holda

$\forall x = (x_0, x_1, ..., x_n)^T \in D$ uchun

$(x_0, x_1, ... x_n, u, \frac{\partial u}{\partial x_0}, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n})^T \in G$ bo'lishi ham kerak.

Misol 1. $u = xy + y \cdot \sqrt{x^2 + 1}$ funksiya ushbu

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = 0$$

ikki o'zgaruvchining $u = u(x, y)$ noma'lum funksiyasiga nisbatan xususiy hosilali differensial tenglamaning yechimi ekanligini asoslang.

8→ Kerakli tekshirishlarni bajaramiz:

$$\frac{\partial u}{\partial x} = y + \frac{yx}{\sqrt{x^2+1}}, \ \frac{\partial u}{\partial y} = x + \sqrt{x^2+1} \quad \left(u \in C^1(\mathbb{R}^2)\right),$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} = x\left(y + \frac{yx}{\sqrt{x^2+1}}\right) + y\left(x + \sqrt{x^2+1}\right) -$$

$$-\left(y + \frac{yx}{\sqrt{x^2+1}}\right)\left(x + \sqrt{x^2+1}\right) = 0. \ ♠$$

$u = u(x)$ yechimning $(x, u) = (x_0, x_1, ..., x_n, u)$ o'zgaruvchilar fazosidagi ($\mathbb{R}^{2+n}$ fazodagi) grafigi tenglamaning integral sirti deb ataladi.

Agar $F(x_0, x_1, ..., x_n, u, p_0, p_1, ..., p_n)$ funksiya $p_0, p_1, ..., p_n$ o'zgaruvchilarga nisbatan chiziqli (aniqrog'i, affin), ya'ni

$$F(x, u, p) = \sum_{i=0}^{n} a_i(x, u) p_i - b(x, u)$$

bo'lsa, (VII.4.1) tenglama kvazichiziqli tenglama deb ataladi Demak, birinchi tartibli xususiy hosilali kvazichiziqli differensia tenglamaning umumiy ko'rinishi quyidagicha:

$$\sum_{i=0}^{n} a_i(x, u(x)) \frac{\partial u}{\partial x_i} = b(x, u(x)) \qquad \text{(VII.4.2)}$$

Agar $F(x, u, p)$ funksiya u va $p_0, p_1, ..., p_n$ o'zgaruvchilarning chiziqli (affin) funksiyasidan iborat bo'lsa, u holda (VII.4.1) tenglama chiziqli tenglama deyiladi. Ciziqli tenglama

$$\sum_{i=0}^{n} a_i(x) \frac{\partial u}{\partial x_i} = b(x)u + c(x) \qquad \text{(VII.4.3)}$$

ko'rinishga ega.

Yechimlar majmuasi haqida umumiy ma'lumotlar

Birinchi tartibli oddiy differensial tenglamaning barcha yechimlar majmuasi umumiy holda (maxsus yechimlardan tashqari) bir parametrli yechimlar oilasidan iborat.

Birinchi tartibli xususiy hosilali tenglama holidagi vaziyat murakkabroq bo'ladi. Bu holdagi tenglamaning yechimlari ba'zi maxsus yechimlarni hisobga olmaganda erkli o'zgaruvchilardan tashqari ixtiyoriy funksiyaga ham bog'liq bo'ladi. Bu ixtiyoriy funksiyaning argumentlari soni tenglama yechimining argumentlari sonidan bittaga kam bo'ladi (umumiy holda).

Misollar qaraylik.

1. $u = u(x, y)$ ikki argumentning funksiyasiga nisbatan

$$\frac{\partial u}{\partial y} = 0 \qquad \text{(VII.4.4)}$$

tenglama berilgan bo'lsin. Bu tenglama yechimning y ga bog'liq emasligini anglatadi. Demak, berilgan (VII.4.4) tenglamaning har qanday $u = u(x, y)$ yechimi

$$u = \varphi(x)$$

ko'rinishda bo'ladi; bunda $\varphi(x)$ – bir argumentning ixtiyoriy silliq funksiyasi.

2. Endi ushbu

$$f_y'(x, y, u) \cdot \frac{\partial u}{\partial x} - f_x'(x, y, u) \cdot \frac{\partial u}{\partial y} = 0 \qquad \text{(VII.4.5)}$$

kvazichiziqli tenglamani qaraylik, bunda berilgan f funksiya nafaqat x, y erkli o'zgaruvchilarga, balki u noma'lum funksiya $u = u(x, y)$ ga ham oshkor ko'rinishda bog'liq.

Bu tenglamadan ixtiyoriy $u(x, y)$ yechim va $\tilde{f}(x, y) = f(x, y, u(x, y))$ funksiyalarning yakobiani nolga teng ekanligi kelib chiqadi:

$$\frac{\partial(u, \tilde{f})}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial \tilde{f}}{\partial x} & \frac{\partial \tilde{f}}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x} \cdot \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} \right) - \frac{\partial u}{\partial y} \cdot \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} \right) =$$

$$= \frac{\partial f}{\partial y} \cdot \frac{\partial u}{\partial x} - \frac{\partial f}{\partial x} \cdot \frac{\partial u}{\partial y} = 0 .$$

Demak, analizdan ma'lum teoremaga ko'ra, $u(x, y)$ va $f(x, y, u(x, y))$ funksiyalar bog'liq, ya'ni (VII.4.5) tenglamaning $u(x, y)$ yechimi

$$u(x, y) = \varphi(f(x, y, u(x, y)))$$

munosabat bilan beriladi; bunda $\varphi(\circ)$ – ixtiyoriy silliq funksiya. Oxirgi tenglik $u(x, y)$ yechimni oshkormas ko'rinishda aniqlaydi.

Masalan, (VII.4.5) tenglamaning xususiy holi bo'lgan ushbu

$$u_t' + u u_x' = 0 \qquad \text{(VII.4.6)}$$

tenglamaning yechimi

$$u = \varphi(x - ut)$$

formula bilan oshkormas ko'rinishda beriladi, bunda $\varphi(\circ)$ – ixtiyoriy silliq funksiya. (VII.4.6) differensial tenglamaga quyidagicha ma'no berish mumkin. Aytaylik, zarrachalar to'g'ri chiziq bo'ylab harakat qilayotgan bo'lsin. Agar $u(t, x(t))$ ni t paytda to'g'ri chiziqning $x(t)$ nuqtasidagi zarrachaning tezligi deb tushunsak, u holda (VII.4.6) differensial tenglama barcha zarrachalarning tezlanishi nolga teng ekanligini anglatadi:

$$\frac{du(t, x(t))}{dt} = u_t' + u_x' \cdot \frac{dx}{dt} = u_t' + u u_x' = 0 \, .$$

Chiziqli tenglama

Birinchi tartibli xususiy hosilali chiziqli differensial tenglama

$$a_0(\boldsymbol{x}) \frac{\partial u}{\partial x_0} + a_1(\boldsymbol{x}) \frac{\partial u}{\partial x_1} + \ldots + a_n(\boldsymbol{x}) \frac{\partial u}{\partial x_n} = b(\boldsymbol{x}) u + c(\boldsymbol{x}) \qquad \text{(VII.4.7)}$$

ni qaraylik; bu yerda $a_0(\boldsymbol{x}), a_1(\boldsymbol{x}), \ldots, a_n(\boldsymbol{x}), b(\boldsymbol{x}), c(\boldsymbol{x})$ – berilgan funksiyalar $D \subset \mathbb{R}^{1+n}$ sohada uzluksiz differensiallanuvchi hamda har bir $\boldsymbol{x} \in D$ nuqtada $a_0(\boldsymbol{x}), a_1(\boldsymbol{x}), \ldots, a_n(\boldsymbol{x})$ koeffitsiyentlarning kamida bittasi 0 dan farqli, ya'ni

$$a_0^2(\boldsymbol{x}) + a_1^2(\boldsymbol{x}) + \ldots + a_n^2(\boldsymbol{x}) > 0$$

deb faraz qilinadi. Bu shart (VII.4.7) tenglamaning har bir $\boldsymbol{x} \in D$ nuqtada differensial tenglamadan iborat bo'lishini ta'minlaydi. Biz aniqlik uchun D sohada $a_0(\boldsymbol{x})$ nolga aylanmaydi deb hisoblaymiz. Shu sohada (VII.4.7) tenglamaning har ikkala tomoni $a_0(\boldsymbol{x})$ ga bo'lib, x_0 o'zgaruvchini t bilan belgilab, uni quyidagi ko'rinishga keltiramiz:

$$\frac{\partial u}{\partial t} + f_1(t, x) \frac{\partial u}{\partial x_1} + \ldots + f_n(t, x) \frac{\partial u}{\partial x_n} = g(t, x) u + h(t, \boldsymbol{x}) \, . \qquad \text{(VII.4.8)}$$

Bu yerda endi $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ va $f_1, \ldots, f_n, g, h$ funksiyalar $\in C^1$.

Ushbu

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(t,x) \\ \dots \quad \dots \quad \dots \\ \dfrac{dx_n}{dt} = f_n(t,x) \end{cases}$$

yoki vektor ko'rinishida

$$\frac{dx}{dt} = f(t,x) \qquad \text{(VII.4.9)}$$

oddiy differensial tenglamalar sistemasi (VII.4.8) xususiy hosilali tenglamaning **xarakteristik sistemasi** deyiladi. (VII.4.9) sistema yechimlarining $(t,x) \in \mathbb{R}^{1+n}$ fazodagi grafiklari (VII.4.8) ning xarakteristikalari deb ataladi.

Farazimizga ko'ra $f(t,x) = (f_1(t,x), \dots, f_n(t,x))^T \in C^1(D)$. Demak, D sohaning har bir (t_0, ξ) nuqtasidan (VII.4.8) tenglamaning yagona xarakteristikasi o'tadi.

(VII.4.9) sistemaning yechimini ((VII.4.8) ning xarakteristikasini) ushbu

$$x = \varphi(t, t_0, \xi) \quad (\varphi(t_0, t_0, \xi) = \xi) \qquad \text{(VII.4.10)}$$

ko'rinishda yozaylik. Agar $|t_0 - t| < a$, $|\xi - x^0| < b$, ($(t_0, x^0) \in D$; $a, b-$ yetarlicha kichik musbat sonlar) bo'lsa, u holda (VII.4.10) tenglamani ξ ga nisbatan yechib,

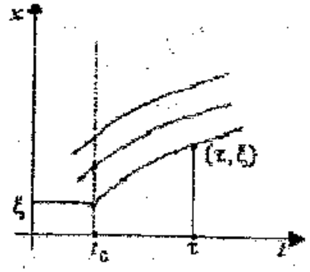
$$\xi = \varphi(t_0, t, x)$$

ekanligini topamiz. Ma'lumki,

$$\varphi(t,x) = \varphi(t_0, t, x) = (\varphi_1(t_0, t, x), \dots, \varphi_n(t_0, t, x))^T$$

vektor-funksiyaning komponentalari (VII.4.9) sistemaning erkli birinchi integrallar sistemasini aniqlaydi:

$$\frac{\partial \varphi_i}{\partial t} + \sum_{j=1}^{n} f_j \frac{\partial \varphi_i}{\partial x_j} = 0, \; i = \overline{1,n}, \qquad \text{(VII.4.11)}$$



VII.5-rasm.

yoki vektor yozuvida

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} f = 0.$$

Ravshanki,

$$\left.\frac{D\varphi(t, t_0, \xi)}{D\xi}\right|_{t=t_0} = E \quad (E - \text{birlik matritsa})$$

Demak, t_0 ga yetarlicha yaqin t lar uchun $\dfrac{D\varphi}{D\xi}$ matritsa teskarilanuvchi hamda

$$\left.\frac{D\varphi(t_0, t, \xi)}{D\xi}\right|_{t=t_0} = E$$

bo'ladi .Demak, $(t_0, x^0) \in D$ nuqtaning yetarlicha kichik atrofida $(t,x) \in D$ o'zgaruvchilari (koordinatalari) o'rniga yangi (τ, ξ) o'zgaruvchilarni (koordinatalarni)

$$\tau = t, \; \xi = \varphi(t,x) \equiv \varphi(t_0, t, x)$$

formulalar yordamida kiritish mumkin (VII.5-rasm).

Bunda

$$t = \tau, \quad x = \varphi(\tau, t_0, \xi)$$

bo'ladi. u noma'lum funksiyaning hosilalarini yangi o'zgaruvchilar orqali ifodalaymiz:

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial \tau}+\frac{\partial u}{\partial \xi}\frac{\partial \boldsymbol{\varphi}(t,\boldsymbol{x})}{\partial t}\ ,\ \ \frac{\partial u}{\partial \boldsymbol{x}}f=\frac{\partial u}{\partial \xi}\frac{\partial \boldsymbol{\varphi}(t,\boldsymbol{x})}{\partial \boldsymbol{x}}f \qquad \text{(VII.4.12)}$$

Dastlab (VII.4.8) ning xususiy holi bo'lmish ushbu

$$\frac{\partial u}{\partial t}+f_1(t,\boldsymbol{x})\frac{\partial u}{\partial x_1}+\ldots+f_n(t,\boldsymbol{x})\frac{\partial u}{\partial x_n}\equiv\frac{\partial u}{\partial t}+\frac{\partial u}{\partial \boldsymbol{x}}f=0 \qquad \text{(VII.4.13)}$$

tenglamani yechaylik.

Almashtirish formulalari (VII.4.12)ga ko'ra (VII.4.13) tenglama

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial \boldsymbol{x}}f=\frac{\partial u}{\partial \tau}+\frac{\partial u}{\partial \xi}\left(\frac{\partial \boldsymbol{\varphi}(t,\boldsymbol{x})}{\partial t}+\frac{\partial \boldsymbol{\varphi}(t,\boldsymbol{x})}{\partial \boldsymbol{x}}f\right)=\frac{\partial u}{\partial \tau}=0$$

ko'rinishni oladi. Oxirgi tenglamadan (VII.4.13) tenglamaning har qanday yechimi τ ga bog'liq bo'lmay, balki faqat $o=(o_1,\ldots o_n)^T$ ga bog'liq bo'lishi kelib chiqadi. Shunday qilib, (VII.4.13) tenglamaning har qanday yechimi (VII.4.9) xarakteristik sistema birinchi integrallari to'la sistemasi $\xi_1=\varphi_1(t,\boldsymbol{x}),\ldots,\xi_n=\varphi_n(t,\boldsymbol{x})$ ning funksiyasidan iborat, ya'ni

$$u=u(t,\boldsymbol{x})=c(\varphi_1(t,\boldsymbol{x}),\ldots,\varphi_n(t,\boldsymbol{x}))\,;$$

bu yerda $c(\xi_1,\ldots,\xi_n)-$ ixtiyoriy $\in C^1$ funksiya. Shunday qilib, (VII.4.13) tenglamaning umumiy yechimi ixtiyoriy funksiya $c(\xi_1,\ldots,\xi_n)$ ga bog'liq.

(VII.4.9) sistemaning har qanday birinchi integrali birinchi integrallarning to'la sistemasi bo'lmish $\varphi_1(t,\boldsymbol{x}),\ldots,\varphi_n(t,\boldsymbol{x})$ larning funksiyasidan iborat bo'lgani uchun qo'yidagi teorema isbot bo'ldi.

Teorema 1. *Faraz qilaylik,* $(t_0,\boldsymbol{x}^0)$ *nuqtaning atrofida* $f_1(t,\boldsymbol{x}),\ldots,f_n(t,\boldsymbol{x})$ *funksiyalar uzluksiz differensiallanuvchi bo'lsin.* $\psi_1(t,\boldsymbol{x}),\ldots,\psi_n(t,\boldsymbol{x})$ *lar bilan* (VII.4.9) *sistemaning* $(t_0,\boldsymbol{x}^0)$ *nuqta atrofida aniqlangan erkli birinchi integrallarini belgilaylik. U holda* (VII.4.13) *tenglamaning yechimi* $(t_0,\boldsymbol{x}^0)$ *nuqtaning biror atrofida mavjud va har qanday yechim*

$\psi_1(t,\boldsymbol{x}),\ldots,\psi_n(t,\boldsymbol{x})$ *larning funksiyasi sifatida ifodalanadi:*

$$u=c(\psi_1(t,\boldsymbol{x}),\ldots,\psi_n(t,\boldsymbol{x}))\,.$$

Misollar qaraylik. **1.** $\mathbb{R}^3_{(x,y,z)}$ fazoda noldan farqli $\{a;b;c\}$ o'zgarmas vektor berilgan bo'lsin. Agar $u(x,y,z)=0$ sirtning $\left\{\frac{\partial u}{\partial x};\frac{\partial u}{\partial y};\frac{\partial u}{\partial z}\right\}$ normal vektori berilgan vektorga perpendikulyar bo'lsa, u holda $u=u(x,y,z)$ funksiya uchun

$$a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}+c\frac{\partial u}{\partial z}=0 \qquad (*)$$

tenglama hosil qilamiz.

Mos xarakteristik sistemani tuzamiz.

$$\frac{dx}{a}=\frac{dy}{b}=\frac{dz}{c}$$

yoki $\qquad cdx-adz=0, \qquad cdy-bdz=0.$

Ikkita birinchi integral osongina topiladi:

$$cx-az=c_1,\qquad cy-bz=c_2$$

Xarakteristika berilgan $\{a;b;c\}$ vektorga parallel to'g'ri chiziqlardan iborat. (*) ning yechimlari ana shu xarakteristikalaridan tuziladi va u

$$\Phi(cx-az,cy-bz)=0$$

ko'rinishda beriladi; Bu yerda $\Phi-$ikki o'zgaruvchining ixtiyoriy silliq funksiyasi. Oxirgi tenglama yasovchilari $\{a;b;c\}$ vektorga parallel bo'lgan silindrik sirt tenglamasini ifodalaydi. ♨

2. (x,y,z) nuqtadagi normal vektori berilgan $A(a;b;c)$ nuqtadan shu (x,y,z) nuqtaga o'tkazilgan vektorga perpendikulyar bo'lgan $u(x,y,z)=0$ sirt uchun

$$(x-a)\frac{\partial u}{\partial x}+(y-b)\frac{\partial u}{\partial y}+(z-c)\frac{\partial u}{\partial z}=0$$

tenglama hosil bo'ladi.

Xarakteristik sistema

$$\frac{dx}{x-a}=\frac{dy}{y-b}=\frac{dz}{z-c}$$

Uning birinchi integrallari

$$\frac{x-a}{z-c}=c_1,\ \frac{y-b}{z-c}=c_2$$

Xarakteristikalari berilgan $A(a,b,c)$ nuqta orqali o'tuvchi to'g'ri chiziqlar oilasidan iborat. Integral sirt ana shu to'g'ri chiziqlardan tuziladi. Uning tenglamasi

$$\Phi\left(\frac{x-a}{z-c},\frac{y-b}{z-c}\right)=0$$

ko'rinishda bo'ladi. Bu tenglama uchi berilgan $A(a,b,c)$ nuqtada, joylashgan konik sirtni ifodalaydi. ♣

Endi (VII.4.8) tenglamaning

$$u\big|_{t=t_0}=u_0(x),\ u_0\in C^1, \qquad \text{(VII.4.14)}$$

shartni qanoatlantiruvchi yechimini topish masalasini, ya'ni **Koshi masalasini**, qaraylik. (VII.4.13),(VII.4.14) Koshi masalasining yechimi (VII.4.10). ga ko'ra $u=u_0(\varphi(t_0,t,x))$ ko'rinishda ifodalanadi.

Misol. Ushbu

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x}=0\ (v=\text{const}) \qquad \text{(VII.4.15)}$$

tenglamani yechaylik. Bu tenglama uchun xarakteristik tenglama $\frac{dx}{dt}=v$ osongina yechiladi:

$$x=\varphi(t,t_0,\xi)=\xi+v(t-t_0),\quad \xi=\varphi(t_0,t,x)=x-v(t-t_0)$$

Demak, (VII.4.15) tenglamaning umumiy yechimi

$$u=u(t,x)=u_0(x-v(t-t_0)) \qquad \text{(VII.4.16)}$$

ko'rinishda bo'ladi. Bu yerda $u_0(x)=u\big|_{t=t_0}$ boshlang'ich funksiya. (VII.4.16) formula x o'qi bo'ylab o'zgarmas v tezlik bilan harakat qiluvchi to'lqinni anglatadi.

Endi umumiy ko'rinishdagi chiziqli tenglama (VII.4.8) ni

yechishga qaytalik. (VII.4.8) tenglama (τ,ξ) o'zgaruvchilarga nisbatan quyidagi ko'rinishni oladi:

$$\frac{\partial u}{\partial\tau}=g(\tau,\varphi(\tau,t_0,\xi))u+h(\tau,\varphi(\tau,t_0,\xi))$$

Bu tenglama osongina yechiladi:

$$u=\exp\left[\int_{t_0}^{\tau}g(s,\varphi(\tau,t_0,\xi))ds\right]c(\xi)+$$

$$+\int_{t_0}^{\tau}\exp\left[-\int_{t_0}^{s}g(r,\varphi(r,t_0,\xi))dr\right]\cdot h(s,\varphi(s,t_0,\xi))ds;$$

bu yerda $c(\xi)=c(\xi_1,\ldots,\xi_n)$ - ixtiyoriy $\in C^1$ funksiya (integrallash "o'zgarmasi"). Oxirgi tenglikda (t,x) o'zgaruvchilarga qaytamiz. Bizga ma'lum

$$\varphi(s,t_0,\varphi(t_0,t,x))=\varphi(s,t,x)$$

munosabatdan foydalanib, (VII.4.8) tenglamaning har qanday yechimi, agar u mavjud bo'lsa, ushbu

$$u=\exp\left[\int_{t_0}^{t}g(s,\varphi(s,t,x))ds\right]c(\xi)+$$

$$+\int_{t_0}^{t}\exp\left[-\int_{t_0}^{s}g(r,\varphi(r,t,x))dr\right]\cdot h(s,\varphi(s,t,x))ds \qquad \text{(VII.4.17)}$$

ko'rinishda ifodalanishini topamiz. f,g,h funksiyalari C^1 sinfga tegishli ekanligidan foydalanib, (VII.4.17) formula bilan berilgan u funksiyaning haqiqatdan ham (VII.4.8) tenglama yechimi ekanligini tekshirib ko'rish qiyin emas.

Endi umumiy yechim formulasi (VII.4.17) dan (VII.4.8),(VII.4.14) Koshi masalasining yagona yechimini osongina topamiz:

$$=\exp\left[\int_{t_0}^{t}g(s,\varphi(s,t,x))ds\right]u_0(\xi)+$$

$$+\int_{t_0}^{t}\exp\left[-\int_{t_0}^{s}g(r,\varphi(r,t,x))dr\right]\cdot h(s,\varphi(s,t,x))ds. \quad (VII.4.18)$$

Shunday qilib, biz quyidagi teoremani isbotladik.

Teorema 2. *Faraz qilaylik* $(t_0,x^0)\in\mathbb{R}^{1+n}$ *nuqtaning biror atrofida* $f_1(t,x),...,f_n(t,x),g(t,x),h(t,x)$ *funksiyalai* C^1 *sinfga tegishli bo'lsin. U holda shu nuqtaning yetarlicha kichik atrofida* (VII.4.8) *tenglama yechimga ega va uning har qanday yechimi* (VII.4.17) *formula bilan* ($c\in C^1$) *ifodalanadi;* (VII.4.8),(VII.4.14) *Koshi masalasi yagona yechimga ega va bu yechim* (VII.4.18) *formula bilan aniqlanadi.*

Eslatma. Teoremadagi ushbu

$$\{f_1(t,x),...,f_n(t,x),g(t,x),h(t,x)\}\subset C^1$$

shart ahamiyatlidir. Agar biz funksiyalardan faqat uzluksizlikni talab qilsak, u holda (VII.4.13),(VII.4.14) Koshi masalasi (yoki (VII.4.13) tenglama) birorta ham $u\in C^1$ yechimga ega bo'lmasligi mumkin. Aytaylik, $g(x)$ — sonlar o'qi $\mathbb{R}$ da uzluksiz, lekin birorta nuqtada ham differensiallanuvchi bo'lmasin. U holda ushbu

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=g(x-t)u \quad (VII.4.19)$$

tenglama notrivial yechimga ega bo'lolmaydi. Teskarsini faraz qilaylik. $u\in C^1$, $u(t_0,x_0)\neq 0$, funksiya (VII.4.19) tenglamaning yechimi bo'lsin. Umumiylikni buzmasdan $u(t_0,x_0)>0$ deb hisoblaymiz. Yangi $\tau=t$, $\xi=x-t$ koordinatalarda (VII.4.19) tenglama $u'_\tau=g(\xi)u$ ko'rinishni oladi.

Bu tenglamani yechib,

$$u=\exp[(\tau-t_0)g(\xi)]c(\xi)$$

yoki

$$u=\exp[(t-t_0)g(x-t)]c(x-t) \quad (VII.4.20)$$

bo'lishi kerakligini topamiz: Farazimizga ko'ra $u|_{t_0}=u_0(x)$ funksiya $x_0\in\mathbb{R}$ nuqtaning kichik atrofida musbat va uzluksiz differensiallanuvchi. (VII.4.20) ga ko'ra

$$u(t,x)=\exp[(t-t_0)g(x-t)]u_0(x-t+t_0)$$

funksiya (t_0,x_0) nuqtaning yetarlicha kichik atrofida $u(t,x)>0$ bo'lgani uchun, shu atrofda (VII.4.20) dan

$$(t-t_0)g(x-t)=\ln[u_0(x-t+t_0)/u(t,x)]$$

tenglikni topamiz. Bu tenglikdan $t\neq t_0$ da $g(x-t)$ ning differensiallanuvchi ekanligi kelib chiqadi. Bu esa berilganga zid. Hosil bo'lgan ziddiyat (VII.4.19) tenglamaning noldan farqli yechimga ega bo'la olmasligini isbotlaydi.

Yuqorida aytilgan $g(x)$ funksiyaga ko'ra ushbu

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=g(x-t) \quad (VII.4.21)$$

tenglamani tuzaylik. Bu tenglama tekislikning hech qanday sohasida $u\in C^1$ yechimga ega bo'la olmaydi.Teskarisini faraz qilaylik, u holda (VII.4.36) tenglamaning ixtiyoriy $u\in C^1$ yechimi

$$u=t\cdot g(x-t)+c(x-t)$$

ko'rinishda ifodalanadi. Yuqoridagiga o'xshash fikr yuritib, bu funksiyaning hech qanday sohada birinchi tartibli xususiy hosilalarga ega bo'la olmasligini ko'rsatish qiyin emas.

Demak, (VII.4.21) tenglama birorta ham $u\in C^1$ yechimga ega emas.

Kvazichiziqli tenglama

Ushbu

$$\sum_{i=0}^{n}a_i(x,u)\frac{\partial u}{\partial x_i}=a_{n+1}(x,u) \quad (VII.4.22)$$

kvazichiziqli tenglamani qaraylik; bu yerda $a_i(x,u)$, $i=\overline{0,n+1}$, funksiyalari biror $(x,u)\in G\subset\mathbb{R}^{2+n}$ sohada aniqlangan, C^1 sinfga tegishli va

$$\sum_{i=0}^{n}|a_i(x,u)|>0$$

deb faraz qilinadi.

(VII.4.22) tenglamani yechishni chiziqli tenglamani yechishga keltirish mumkin. (VII.4.22) tenglamaning $u=u(x)$ yechimini

$$v(x,u)=0 \qquad \text{(VII.4.23)}$$

tenglama bilan berilgan oshkormas funksiya sifatida izlaylik. U holda (VII.4.23) dan topilgan

$$\frac{\partial u}{\partial x_i}=-\frac{\partial v}{\partial x_i}\Big/\frac{\partial v}{\partial u};\quad i=\overline{0,n}, \qquad \text{(VII.4.24)}$$

hosilalarni (VII.4.22) ga qo'yib,

$$\sum_{i=0}^{n}a_i(x,u)\frac{\partial v}{\partial x_i}+a_{n+1}(x,u)\frac{\partial v}{\partial u}=0 \qquad \text{(VII.4.25)}$$

tenglikni hosil qilamiz.(VII.4.25) munosabat (VII.4.23) bog'lanish asosida o'rinli bo'lishi kerak, ya'ni bu yerda (x,u) erkli o'zgaruvchi emas. Biz (VII.4.25) ni $x_0,x_1,...,x_{n+1}=u$ erkli o'zgaruvchilarga bog'liq bo'lgan $v=v(x_0,x_1,...,x_n,u)$ funksiyaga nisbatan chiziqli tenglama sifatida qaraymiz. (VII.4.25) tenglama uchun xarakteristik sistema

$$\frac{dx_i}{d\tau}=a_i(x,u),\quad i=\overline{0,n}$$
$$\frac{du}{d\tau}=a_{n+1}(x,u) \qquad \text{(VII.4.26)}$$

(VII.4.26) sistemada τ – parametr; u additiv o'zgarmas aniqligida kiritiladi.(VII.4.26) sistemaning $(x,u)\in G\subset\mathbb{R}^{n+2}$ sohadagi traektoriyalari (VII.4.22) tenglamaning xarakteristikalari (xarakteristik chiziqlari) deb yuritiladi.

Bu yerda shuni ta'kidlash lozimki, xususiy hosilali chiziqli differensial tenglamaning xarakteristikalari yuqorida kiritilishiga ko'ra x lar fazosida joylashgan. Ular hozirgi ma'nodagi (x,u)

lar fazosida yotuvchi xarakteristikalarning x o'zgaruvchilar fazosidagi ortogonal proyeksiyalaridan iborat bo'ladi.

Faraz qilaylik, $\psi_1(x,u),...,\psi_{n+1}(x,u)$ funksiyalar (VII.4.26) xarakteristik sistema uchun birinchi integrallarning to'la sistemasi bo'lsin.

U holda (VII.4.25) chiziqli tenglamaning umumiy yechimi

$$v=c(\psi_1(x,u),...,\psi_{n+1}(x,u))\equiv v(x,u) \qquad \text{(VII.4.27)}$$

ko'rinishda ifodalanadi. Agar

$$c(\psi_1(x,u),...,\psi_{n+1}(x,u))=0\quad (v(x,u)=0)$$

tenglikdan $u=u(x)$ funksiya aniqlansa, hamda

$$\left.\frac{\partial v}{\partial u}\right|_{u=u(x)}\neq 0 \qquad \text{(VII.4.28)}$$

shart bajarilsa, u holda (VII.4.24) formulalarga ko'ra topilgan hosilalarni (VII.4.22) tenglamaga qo'yib, uning (VII.4.25) ga ko'ra qanoatlanishini ko'ramiz.

Shunday qilib, quyidagi teorema isbotlandi.

Teorema 3. *Faraz qilaylik,* (VII.4.25) *chiziqli tenglamaning ixtiyoriy yechimi* $v(x,u)$ *berilgan,* $v(x,u)=0$ *tenglama esa biror* $x\in D\subset\mathbb{R}^{1+n}$ *sohada* $u=u(x)$ *funksiyani aniqlagan hamda* (VII.4.28) *tengsizlik bajarilgan bo'lsin, u holda* $u=u(x)$ *funksiya D sohada kvazichiziqli tenglama* (VII.4.22) *ning yechimi bo'ladi.*

Kvazichiziqli tenglama (VII.4.22) uchun Koshi masalasini yechishda geometrik yondoshish va xarakteristik chiziq tushunchasi juda ham qo'l keladi.

Endi biz ana shu **xarakteristikalar metodi** deb ataluvchi metodda to'xtalaylik.

(VII.4.22) tenglama $G\subset\mathbb{R}^{n+2}$ sohada silliq vektor maydon

$$a(x,u)=\left[(a_0(x,u),a_1(x,u),...,a_{n+1}(x,u)\right]^T$$

ni aniqlaydi. Agar $u=u(x)$ funksiya (VII.4.22) tenglamaning yechimi bo'lsa, bu yechimning grafigiga normal bo'lgan

$$n = n(x) = \left[\frac{\partial u}{\partial x_0}, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, -1\right]^T$$

vektor $a(x,u)$ vektor maydonga mos nuqtada ortogonal bo'lib (bu vektorlarning skalyar ko'paytmasi nolga teng):

$$(a(x,u), n(x)) = 0$$

Bu shart a vektorning integral sirtga urinma ekanligini anglatadi. Oxirgi tenglikdan quyidagi tasdiq kelib chiqadi.

Teorema 4. $u = u(x) \in C^1$ *funksiya* (VII.4.22) *tenglamaning yechimi bo'lishi uchun uning grafigi o'zining har bir nuqtasida shu nuqtadagi* $a(x,u)$ *maydon vektoriga urinishi yetarli va zarurdir.*

Xarakteristik sistema (VII.4.26) xarakteristikalarining a vektor maydonga urinma ekanligi ravshan.

Agar $S \subset \mathbb{R}^{n+2}$ (giper)sirtning har bir nuqtasidan (VII.4.26) sistemaning yagona yechimi o'tsa va S sirtda joylashsa, u holda S sirt (VII.4.22) tenglamaning xarakteristikalaridan tuzilgan deyiladi.

Teorema 5. $u = u(x) \in C^1$ *funksiya* (VII.4.22) *tenglamaning yechimi bo'lishi uchun uning grafigi shu tenglamaning xarakteristikalaridan* ((VII.4.26) *sistema yechimlaridan*) *tuzilgan bo'lishi yetarli va zarurdir.*

⊶ Faraz qilaylik, $u = u(x) \in C^1$ – (VII.4.22) ning yechimi, Γ_u esa uning grafigi bo'lsin. Ixtiyoriy $(x^0, u(x^0)) \in \Gamma_u$ nuqtadan (VII.4.22) ning yagona xarakteristikasi ((VII.4.26) ning yechimi) o'tadi. Bu xarakteristikani χ bilan belgilaymiz. χ ning to'laligicha Γ_u da yotishini ko'rsataylik. $x = x(\tau)$ bilan ushbu

$$\frac{dx_i}{d\tau} = a_i(x, u(x)), \quad i = \overline{0,n} \qquad \text{VII.4.29)}$$

$$x(0) = x^0$$

masalaning yechimini belgilab, parametrik tenglamasi $x = x(\tau)$, $u = u(x(\tau))$ bo'lgan ч* chiziqni qaraylik. $ч^* \subset \Gamma_u$ ekanligi ravshan. $u = u(x)$ (VII.4.22) ning yechimi bo'lgani uchun

tenglama $x = x(\tau)$ da ham qanoatlanadi.

(VII.4.29) va (VII.4.22) ga ko'ra

$$\frac{du(x(\tau))}{d\tau} = a_{n+1}(x(\tau), u(x(\tau))), \; u(x(0)) = x^0 \qquad \text{(VII.4.30)}$$

(VII.4.29) va (VII.4.30) dan χ^* ning $\left(x^0; u(x^0)\right) \in \Gamma_u$ dan o'tuvchi xarakteristika ekanligi kelib chiqadi. Bu xarakteristika yagona bo'lgani uchun $ч = ч^* \subset \Gamma_u$. Endi faraz qilaylik, $u = u(x)$ ning $\Gamma_u \subset \mathbb{R}^{n+2}$ grafigi (VII.4.22) tenglama xarakteristikalaridan tuzilgan bo'lsin. $u = u(x)$ funksiya (VII.4.22) tenglama yechimi ekanligini ko'rsatamiz. Ixtiyoriy $\left(x^0; u(x^0)\right) \in \Gamma_u$ nuqta orqali o'tgan xarakteristika ((VII.4.26) ning yechimi) Γ_u da yotadi va u (VII.4.26) ga ko'ra shu nuqtada $a\left(x^0; u(x^0)\right)$ vektorga urinadi. Demak, ixtiyoriy $\left(x^0; u(x^0)\right) \in \Gamma_u$ nuqtada Γ_u sirt $a\left(x^0; u(x^0)\right)$ vektorga urinadi. Bundan esa $u = u(x)$ ning (VII.4.22) tenglama yechimi ekanligi kelib chiqadi.

Endi (VII.4.22) tenglama uchun Koshi masalasi bilan shug'ullanamiz. Dastlab (VII.4.22) tenglama o'rniga ushbu

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{n} f_i(t,x,u)\frac{\partial u}{\partial x_i} = f_{n+1}(t,x,u) \qquad \text{(VII.4.31)}$$

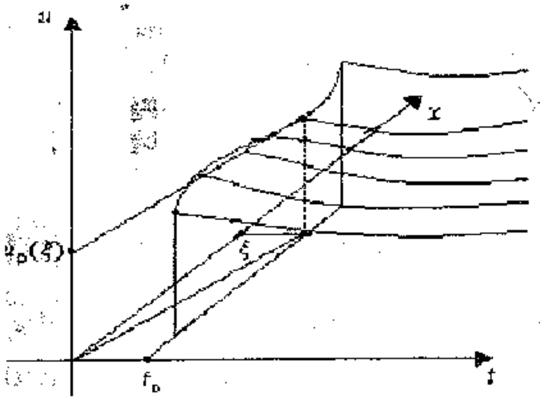
keltirilgan tenglamani qaraymiz; bu yerda $x = (x_1, \dots, x_n)^T \in D \subset \mathbb{R}^{n+2}$. (VII.4.29) tenglamaning xarakteristikalari (t,x,u) nuqtalar fazosida

$$\frac{dx_i}{dt} = f_i(t,x,u), \quad i = \overline{1,n}$$
$$\frac{du}{dt} = f_{n+1}(t,x,u) \qquad \text{(VII.4.32)}$$

sistemadan aniqlanadi (bu holda xarakteristikada τ parametr o'rniga erkli o'zgaruvchi t ni oldik). (VII.4.31) tenglamaning

$$u\Big|_{t=t_0} = u_0(x) \quad (u_0(x) \in C^1(D)) \qquad \text{(VII.4.33)}$$

boshlang'ich shartni qanoatlantiruvchi yechimini topish Koshi masalasi deb ataladi.



VII.6-rasm.

Yuqorida isbotlangan teorema (VII.4.31),(VII.4.33) Koshi masalasining yechimini qurishga imkon beradi. Izlanayotgan yechim grafigi

$$(t_0, \xi, u_0(\xi))$$

nuqtalar orqali o'tkazilgan xarakteristikalardan tuziladi. Bu xarakteristikalar (VII.4.32) sistemaning

$$x\Big|_{t_0} = \xi, \quad u\Big|_{t_0} = u_0(\xi)$$

boshlang'ich shartli yechimlardan iborat (VII.6- rasm). Bu yechimlarni Koshi ko'rinishida yozaylik:

$$x = \Phi(t, t_0, \xi, u_0(\xi)), \; u = \psi(t, t_0, \xi, u_0(\xi)). \qquad \text{(VII.4.34)}$$

(VII.4.34) tengliklar yechim grafigining parametrik tenglamasini beradi (ξ-parametr, t_0-tayinlangan). Yechimning parametrlarga silliq bog'liqligi haqidagi teoremaga ko'ra Φ va ψ funksiyalari t va ξ o'zgaruvchilarning uzluksiz differensiallanuvchi funksiyalaridan iborat. $u = u(t,x) \in C^1$ yechimni topish uchun (VII.4.34) dagi birinchi tenglikdan ξ ni t, x orqali

$$\xi = \Xi(t, x)$$

funksiya sifatida ifodalab, (VII.4.23) tenglikka qo'yish kerak.

$$\det \frac{\partial \Phi(t, t_0, \xi, u_0(\xi))}{\partial \xi}\Bigg|_{t=t_0} = 1 \neq 0$$

bo'lgani uchun teskari funksiya haqidagi teoremaga ko'ra t_0 ga yetarlicha yaqin t larda (VII.4.34) dagi O funksiya mavjud va u C^1 sinfga tegishli, chunki $\text{Ц}, u_0 \in C^1$. Shunday qilib, t_0 ga yetarlicha yaqin t larda (VII.4.31), (VII.4.33) Koshi masalasi

$$u = u(t,x) \equiv \psi\big(t, t_0, \Xi(t,x), u_0(\Xi(t,x))\big)$$

yagona yechimga ega.

Bu yerda shuni ta'kidlash lozimki, (VII.4.31), (VII.4.33) Koshi masalasining yechimi mavjud bo'lgan vaqt oralig'i silliq O (VII.4.34) funksiyaning mavjudlik shartidan aniqlanadi. Bu oraliq umumiy holda yechimning berilgan u_0 qiymatiga bog'liq bo'ladi.

Misol. Ushbu

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \; u\big|_{t=0} = u_0(x) \; (u_0' \in C^1(\mathbb{R}))$$

Koshi masalasini yechaylik.

Xarakteristik sistema (VII.4.32) quyidagi ko'rinishni oladi:

$$\frac{dx}{dt} = u; \quad \frac{du}{dt} = 0.$$

Bu sistemani $x\big|_{t=0} = \xi, \; u\big|_{t=0} = u_0(\xi)$ boshlang'ich shartlarda yechib, izlanayotgan yechim grafigini tashkil etuvchi xarakteriskalarni topamiz:

$$u = u_0(\xi); \; x = tu_0(\xi) + \xi.$$

Umumiy holda oxirgi munosabatlar qaralayotgan masalaning yechimini barcha $t > 0$ paytlarda aniqlamaydi. Haqiqatan ham, agar biror $t = t_*$ da ξ ga nisbatan $x = t_* u_0(\xi) + \xi$ tenglama ikkita $\xi_1 \neq \xi_2$ yechimga ega bo'lsa, ξ o'zgaruvchini $t = t_*$ da (x,t) ning bir qiymatli funksiya sifatida ifodalab bo'lmaydi.

$x=(x_0,x_1,...,x_n)^T$ nuqtalar fazosida regulyar gipersirt S berilgan bo'lsin. S ning regulyarligi u o'zining har bir $x_0\in S$ nuqtasi atrofida

$$F(x)=0 \qquad \text{(VII.4.35)}$$

silliq tenglama bilan berilishi mumkinligini anglatadi; bunda $F(x)$ funksiya $x^0\in\mathbb{R}^{1+n}$ nuqtaning biror atrofida C^1 sinfiga tegishli hamda

$$\operatorname{grad}F\Big|_{x^0}=\left[\frac{\partial F}{\partial x_0},\frac{\partial F}{\partial x_1},...,\frac{\partial F}{\partial x_n}\right]\Bigg|_{x^0}\neq 0 \qquad \text{(VII.4.36)}$$

bo'lishi kerak (S sirt uzluksiz o'zgaruvchi normal vektorga ega).

Endi (VII.4.22) tenglamaning S sirtda berilgan $g\in C^1$ funksiyaga aylanuvchi ya'ni

$$u|_S=g \qquad \text{(VII.4.37)}$$

shartni qanoatlantiruvchi yechimini topish haqidagi Koshi masalasini yechaylik.

Aniqlik uchun

$$\frac{\partial F}{\partial x_0}\Big|_{x^0}\neq 0$$

deyilik. U holda S sirt $x=(x_0^{\ 0},x_1^{\ 0},...,x_n^{\ 0})^T\in S$ nuqtaning biror atrofida

$$x=s(x_1,...,x_n)\quad (s\in C^1) \qquad \text{(VII.4.38)}$$

oshkor ko'rinishda ifodalanadi. $(x_0,x_1,...,x_n)$ o'zgaruvchilar o'rniga yangi $(t,y_1,...,y_n)$ o'zgaruvchilarni ushbu

$$t=F(x_0,x_1,...,x_n)$$
$$y_1=x_1-x_1^{\ 0}$$
$$\cdots\quad\cdots\quad\cdots$$
$$y_n=x_n-x_n^{\ 0}$$

formula bilan kiritaylik. Almashtirish yakobiani $x^0\in\mathbb{R}^{1+n}$ nuqtada

$$\det\frac{\partial(t,y_1,...,y_n)}{\partial(x_0,x_1,...,x_n)}\Big|_{x^0}=\frac{\partial F}{\partial x_0}\Big|_{x^0}\neq 0$$

Demak, shu nuqtaning biror atrofida sistemadan $(x_0,x_1,...,x_n)$ o'zgaruvchilar $(t,y_1,...,y_n)$ o'zgaruvchilari orqali bir qiymatli ifodalanadi:

$$x_0=F^*(t,y_1,...,y_n)$$
$$x_1=y_1+x_1^{\ 0}$$
$$\cdots\quad\cdots\quad\cdots \qquad \text{(VII.4.39)}$$
$$x_n=y_n+x_n^{\ 0}$$

va bunda $F^*\in C^1$ bo'ladi. Ana shu $(t,y_1,...,y_n)$ o'zgaruvchilarda S sirt $t=0$ tenglama bilan beriladi va (VII.4.37) Koshi sharti ushbu

$$u\big|_{t=0}=g \qquad \text{(VII.4.40)}$$

ko'rinishni oladi. (VII.4.39) o'tish formulalariga ko'ra quyidagi hosilalarni hisoblaymiz:

$$\frac{\partial u}{\partial x_0}=\frac{\partial u}{\partial t}\frac{\partial F}{\partial x_0};\quad \frac{\partial u}{\partial x_j}=\frac{\partial u}{\partial t}\frac{\partial F}{\partial x_j}+\frac{\partial u}{\partial y_j};\quad j=\overline{1,n}.$$

Buni (VII.4.22)ga qo'yib,

$$\sum_{i=0}^{n}a_i\frac{\partial F}{\partial x_i}\frac{\partial u}{\partial t}+\sum_{j=0}^{n}a_j\frac{\partial u}{\partial y_j}=a_{n+1} \qquad \text{(VII.4.41)}$$

tenglikni hosil qilamiz. Shunday qilib, (VII.4.22),(VII.4.37) Koshi masalasi (t,y) erkli o'zgaruvchilarda (VII.4.41), (VII.4.40) masalani yechishga keltirildi.

Faraz qilaylik, $x^0\in S$ nuqtada S sirt $(x^0,u(x^0))$ nuqtadan o'tgan xarakteriskaning $\mathbb{R}_x^{n+1}$ fazodagi proyeksiyasiga urinmasin, va'ni

$$\sum_{i=0}^{n} a_i(x^0, u(x^0))\frac{\partial F}{\partial x_i} \neq 0 \qquad \text{(VII.4.42)}$$

bo'lsin. U holda x^0 nuqtaning S sirtdagi biror atrofida ham (VII.4.42) tengsizlik saqlanadi va x^0 ning kichik atrofida (VII.4.41) tenglama (VII.4.31) ko'rinishidagi tenglamaga keladi. Yuqorida isbotlanganga ko'ra (VII.4.41), (VII.4.40) masala yagona yechimga ega. Hosil bo'lgan natijani teorema sifatida ifodalaylik.

Teorema 6. *S (VII.4.35) regulyar sirtning (VII.4.42) shartni qanoatlantiruvchi har qanday nuqtasining yetarlicha kichik atrofida (VII.4.22), (VII.4.37) Koshi masalasi yagona yechimga ega. Yechimning grafigi $\{(x,u)|\ x \in S,\ u = u(x)\}$ sirt nuqtalaridan o'tkazilgan xarakteristikalaridan tuzilgan.*

Agar $x = x^0$ nuqtada

$$\sum_{i=0}^{n} a_i(x, u(x))\frac{\partial F}{\partial x_i} = 0 \qquad \text{(VII.4.43)}$$

bo'lsa, u holda (VII.4.41) tenglik

$$\sum_{j=1}^{n} a_j \frac{\partial u}{\partial y_j} = a_{n+1} \qquad \text{(VII.4.44)}$$

ko'rinishga keladi va u berilgan g funksiyaning hosilalari orasidagi bog'lanishni ifodalaydi. Tabiiyki, umumiy holda bu shart hech qanday atrofda o'rinli bo'la olmaydi.

Agar (VII.4.43) va (VII.4.44) tengliklar x^0 nuqtaning S sirtdagi biror atrofida ham qanoatlansa, u holda (VII.4.22), (VII.4.40) Koshi masalasi cheksiz ko'p yechimga ega bo'ladi.

Eslatma. Umumiy ko'rinishdagi ushbu

$$F\left(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots \frac{\partial u}{\partial x_n}\right) = 0$$

tenglama ham xarakteristikalar (aniqrog'i xarakteristik polosa) usuli yordamida tekshirilishi mumkin.

Masalalar

1. $g(x)$ — $\mathbb{R}$ da uzluksiz, lekin birorta nuqtada ham differensiallanuv... bo'lmasin. Ushbu

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = g(x - t)$$

tenglamani qaraylik. Tenglama tekislikning hech qanday sohasida $u \in C$ yechimga ega bo'la olmasligini isbotlang.

2. Quyidagi Koshi masalalari $(0,0)$ nuqta atrofida analitik yechimga egami?

a) $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x},\ u|_{t=0} = \dfrac{1}{1+x^2}$;

b) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2},\ u|_{t=0} = \dfrac{1}{1+x^2}$.

JAVOBLAR, KO`RSATMALAR, YECHIMLAR

III.1.

1. $\|A\| = \sqrt{30}$, $\|A\|^2 = \sqrt{15+\sqrt{221}}$.

3. Ixtiyoriy tayinlangan $z \in \mathbb{R}^n$ uchun $\varphi(s) = (z, f(x + s(y - x)))$, $s \in [0;1]$, bir o`zgaruvchining funksiyasini qarang. $\varphi$ funksiyaning $[0;1]$ da differensiallanuvchiligini asoslang va bir o`zgaruvchining haqiqiy funksiyasi uchun Lagranj teoremasiga ko'ra biror $u \in (0;1)$ uchun $|\varphi(1) - \varphi(0)| = |\varphi'(u)| \le \sup_{0<\tau<1} |\varphi'(\tau)|$ ekanligidan foydalanib, fikrlashni davom ettiring.

8. Funksiya $|t| < 1$, $|x_1| < 1$, $|x_2| < 1$ to`plamda $x_1, x_2$ bo'yicha Lipshits shartini qanoatlantirmaydi, $|t| < 1$, $\varepsilon < |x_1| < 1$, $\varepsilon < |x_2| < 1$ $(0 < \varepsilon < 1)$ to`plamda esa – qanoatlantiradi.

9. Agar $x \in E$ bo'lsa, ixtiyoriy $y \in E$ uchun berilganga ko'ra
$f(x) - f(y) \le |f(x) - f(y)| \le L|x - y| \;\Rightarrow\; f(x) \le f(y) + L|x - y|$
va aniq quyi chegara (inf) ta'rifiga ko'ra

$$f(x) \le \inf_{y\in E}\{f(y) + L|x - y|\} = \tilde{f}(x).$$

Ravshanki, $x \in E$ bo'lgani uchun $\tilde{f}(x) = \inf_{y\in E}\{f(y) + L|x - y|\} \le f(x)$ ($y = x$ olish mumkin). Demak, $x \in E$ uchun $\tilde{f}(x) = f(x)$. Endi $\tilde{f}$ nin Lipshits shartini qanoatlantirishini ko'rsatamiz. $x \in \mathbb{R}^n$ va $z \in \mathbb{R}$ bo'lsin. U holda $L|x - y| \le L|z - y| + L|x - z|$ uchburcha tengsizligiga ko'ra

$$\tilde{f}(x) = \inf_{y\in E}\{f(y) + L|x - y|\} \le \inf_{y\in E}\{f(y) + L|z - y| + L|x - z|\} =$$

$$= \tilde{f}(z) + L|x - z|$$

ya`ni $\tilde{f}(x) \le \tilde{f}(z) + L|x - z|$. Bu yerda x va z ning o'rinlarini almashtirib, $\tilde{f}(z) \le \tilde{f}(x) + L|z - x|$ tengsizlikni ham topamiz. Demak, $|\tilde{f}(x) - \tilde{f}(z)| \le L|x - z|$.

III.2.

1. $y''^2 - 4y(y^2+1)y'' - 16y^2y' + 4(y^4 - 3)y^2 = 0$.

2. Sistemadagi (1) tenglamani differensiallash natijasida hosil bo'lgan tenglamadan y' hosilani (2) tenglamadan foydalanib yo'qoting:

$$x'' + 3x^2x' - x^3 - (x' + 3x^2)y - xy^2 + 2y^3 = 0 \qquad (3)$$

Endi (1) va (3) tenglamalardan y noma'lumni yo'qotish uchun (3)ni y ga, 1 ni y va y^2 ko'paytirib, quyidagi tengliklar sistemasini hosil qiling:

$$x'' + 3x^2x' - x^3 - (x' + 3x^2)y - xy^2 + 2y^3 = 0 \qquad (3)$$
$$(x'' + 3x^2x' - x^3) - (x' + 3x^2)y^2 - xy^3 + 2y^4 = 0 \qquad (4)$$
$$x' + x^3 - xy - y^2 = 0 \qquad (1)$$
$$(x' + x^3)y - xy^2 - y^3 = 0 \qquad (5)$$
$$(x' + x^3)y^2 - xy^3 - y^4 = 0 \qquad (6)$$

Bu sistemani $1, y, y^2, y^3, y^4$ "noma'lumlar"ga nisbatan chiziqli bir jinsli algebraik tenglamalar sistemasi deb qarab, u notrivial yechimga ega bo'lgani sababli uning determinantini nolga tengligi shartini yozib, izlangan tenglamani toping:

$$x''^2 + (6xx' - 7x' - 8x^3 - 2x^2)xx' - x'^3 + (9x^2 - 26x + 12)x^2x' -$$
$$-(32x^2 - 21x - 7)x^4x' - 4x^9 + 15x^8 + 8x^7 + x^6 = 0.$$

III.3.

1. $u(t)$ va $\varphi(t)$ funksiyalarning $t = a$ nuqtada uzluksizligidan $u(t) > \varphi(t)$ tengsizlikning biror $t \in [a, a+\delta]$ $(\delta > 0)$ oraliqda bajarilishi kelib chiqadi. Agar bu tengsizlik $[a,b]$ segmentda bajarilmasa, ya'ni $\delta < b - a$ bo'lsa, u holda shunday eng kichik $c > a + \delta$ son va $j \in [1,n]$ indeks topiladiki, ular uchun

$$u(t) > \varphi(t),\ a \le t < c,\ u(c) \ge \varphi(c),\ u^j(c) = \varphi^j(c)$$

bo'ladi. Bu holda shartga ko'ra

$$D^- u^j(c) > f^j(t, u(c)) \ge f^j(t, \varphi(c)) = \varphi^{j\prime}(c).$$

Lekin

$$D^- u^j(c) = \overline{\lim_{h\to 0-}} \frac{u^j(c+h) - u^j(c)}{h} \le \overline{\lim_{h\to 0-}} \frac{\varphi^j(c+h) - \varphi^j(c)}{h} = \varphi^{j\prime}(c).$$

2. $x'(t)=f(x(t))$ ayniyatni $x'(t)$ ga ko'paytirib, uni hadma-had $t\in[a,b]$ segment bo'yicha integrallaymiz va $x(a)=x(b)$ ekanligini hisobga olib topamiz:

$$\int_a^b x'^2(t)\,dt=\int_a^b f(x(t))x'(t)\,dt=\int_a^b \frac{dF(x(t))}{dt}\,dt=F(x(b))-F(x(a))=0,$$

bu yerda $F(x)=\int_0^x f(s)ds$. $x'^2(t)$ uzluksiz va nomanfiy bo'lgani uchun

$\int_a^b x'^2(t)dt=0$ tenglikdan $x'(t)\equiv 0$, $t\in[a,b]$, ya'ni $x(t)=\text{const}$ ekanligi kelib chiqadi.

Tasdiqning $f\in C(\mathbb{R}^n,\mathbb{R}^n)$, $n>1$, holida o'rinli emasligi quyidagi misoldan kelib chiqadi:

$$\begin{cases} x_1'=-x_2 \\ x_2'=x_1 \end{cases}$$

sistema $x_1=\cos t$, $x_2=\sin t$ o'zgarmasdan farqli yechimga ega va bu yechim uchun $x_1(0)=x_1(2\pi)$, $x_2(0)=x_2(2\pi)$.

Endi faraz qilaylik, $f=\text{grad}\,\varphi$, $\varphi\in C^1(\mathbb{R}^n,\mathbb{R})$, bo'lsin. Bu holda $x'=f(x)$ sistemaning $x=x(t), t\in[a,b]$, $x(a)=x(b)$, yechimi uchun quyidagilarga egamiz:

$$\int_a^b \|x'(t)\|^2dt=\int_a^b (x'(t),x'(t))dt=\int_a^b (f(x(t)),x'(t))dt=$$

$$=\int_a^b\sum_{j=1}^n \frac{\partial\varphi(x(t))}{x_j}\cdot x_j'(t)dt=\int_a^b\frac{d\varphi(x(t))}{dt}dt=$$

$$=\varphi(x(b))-\varphi(x(a))=0.$$

Demak, $\|x'(t)\|\equiv 0$, $t\in[a,b]$, ya'ni $x(t)=\text{const}$.

4. (K) masalaning $t\ge t_0$ da $x=x(t)$ va $y=y(t)$ yechimlari berilgan bo'lsin:

$$x'(t)=f(t,x(t)),\quad y'(t)=f(t,y(t)),\quad x(t_0)=y(t_0)=x^0.$$

Ushbu $z(t)=x(t)-y(t)$, $u(t)=|z(t)|$ belgilashlarni kiritaylik.

Ravshanki, $z(t)\cdot z(t)=|z(t)|^2=u^2(t)$, $u(t_0)=0$. Quyidagilarga egamiz:

$$2u(t)\frac{du(t)}{dt}=2z(t)\cdot\frac{dz(t)}{dt}=2(x(t)-y(t))\cdot(f(t,x(t))-f(t,y(t)))\le$$

$$\le 2\cdot|x(t)-y(t)|\varphi(|x(t)-y(t)|)=2u(t)\cdot\varphi(u(t)).$$

Demak,

$$u(t)\frac{du(t)}{dt}\le u(t)\varphi(u(t)),\ u(t_0)=0 \qquad (*)$$

Biz $t\ge t_0$ bo'lganda $u(t)=0$ bo'lishini ko'rsatishimiz kerak. Teskarisini faraz qilaylik, ya'ni biror $t_*>t_0$ nuqtada $u(t_*)>0$ bo'lsin. $u(t)$ funksiyaning $[t_0;t_*]$ segmentdagi nollari to'plamini qaraylik:

$$F=\{t\in[t_0;t_*]\,|\,u(t)=0\}.$$

$F\ne\varnothing$, chunki $t_0\in F$. $u(t)$ ning uzluksizligidan F ning yopiqligi ravshan. F yuqoridan t_* bilan chegaralangan. Demak, uning aniq yuqori chegarasi mavjud

$$\bar t \overset{def}{=} \sup F,\ \ \bar t\le t_*.$$

F yopiq bo'lgani uchun $\bar t\in F$, ya'ni $u(\bar t)=0$. Bundan $\bar t<t_*$ ekanligi kelib chiqadi. Endi ravshanki, $(\bar t,t_*]$ oraliqda $u(t)>0$ va (*) dan shu oraliqda

$$\frac{u'(t)}{\varphi(u(t))}\le 1,\ \ t\in(\bar t,t_*].$$

Oxirgi tengsizlikni $[\tau;t_*]\subset(\bar t,t_*]$ segmentda integrallaymiz:

$$\int_\tau^{t_*}\frac{du(t)}{\varphi(u(t))}\le t_*-\tau$$

ya'ni

$$\int_{u(\tau)}^{u(t_*)}\frac{ds}{\varphi(s)}\le t_*-\tau,\ \ \bar t<\tau<t_*.$$

Oxirgi tengsizlikda $\phi\to\bar t+0$ deb limitga o'tib,

$$\int_0^{u(t_*)} \frac{ds}{\varphi(s)} \le t_* - \tau < +\infty$$

munosabatni hosil qilamiz. Bu esa berilganga zid. Shunday qilib $u(t) \equiv 0$, ya'ni $x(t) \equiv y(t)$.

5. Yuqoridagi masaladan Koshi-Bunyakovskiy tengsizligiga ko`ra kelib chiqadi.

III.4.

1. Yangi $\tau = \ln t$ erkli o'zgaruvchiga o`ting. Yechim ko'rinishidan ravshanki, u $t = 0$ nuqtaga davom etmaydi.

3. Teskarisini faraz qilamiz.

$(y' > 0,\ y(0) \ge 0 \Rightarrow y$ – o'suvchi $\Rightarrow y(x) \underset{x \to 2,6-}{\longrightarrow} +\infty$

$y'(x) = y^2(x) + x^2 \ge y^2 + \varepsilon^2,\ \varepsilon \le x < 2,6\ (0 < \varepsilon < 2,6).$

$$\frac{dy}{y^2 + \varepsilon^2} \ge dx \quad \int_\varepsilon^x \Rightarrow \frac{1}{\varepsilon}\left(\mathrm{arctg}\frac{y(x)}{\varepsilon} - \mathrm{arctg}\frac{y(\varepsilon)}{\varepsilon}\right) \ge x - \varepsilon$$

$\mathrm{arctg}\dfrac{y(x)}{\varepsilon} - \mathrm{arctg}\dfrac{y(\varepsilon)}{\varepsilon} \ge \varepsilon(x - \varepsilon)$, bu yerda $x \to 2,6-$ deymiz va

$\mathrm{arctg}\dfrac{y(\varepsilon)}{\varepsilon} \le \dfrac{\pi}{2} - \varepsilon(2,6 - \varepsilon)$ ni hosil qilamiz; oxirgi tengsizlikda $\varepsilon = 1,3$

deb ziddiyatga kelamiz: $\mathrm{arctg}\dfrac{y(1,3)}{1,3} \le \dfrac{\pi}{2} - (1,3)^2 < 0$.

4. Faraz qilaylik, berilgan Koshi masalasining $x = x(t)$ yechimi $[0, b)\ (b < +\infty)$ oraliqda aniqlangan bo'lsin. Odatdagidek normal sistemaga o'tamiz:

$$\begin{cases} x' = y \\ y' = -g(x) - f(y) \\ x(0) = x_0,\ x'(0) = v_0. \end{cases}$$

$x'' + f(x') + g(x) = 0$ tenglikni x' ga ko'paytirib, uni 0 dan t gacha integrallaymiz. Berilgan $G(x) \ge mx^2$, $yf(y) \ge 0$ shartlarga ko'ra ushbu

$$\frac{1}{2}|y(t)|^2 + m|x(t)|^2 \le \frac{1}{2}|v_0|^2 + m|x_0|^2,\ t \in [0, b),$$

tengsizlikni hosil qilamiz. Demak, $x(t)$, $y(t)$ yechim chegaralangan, va shuning uchun u $[0, +\infty)$ gacha davom etadi.

III.5.

1. Yechim $[0, b]$ da aniqlangan aniqlangan bo'lsin. Bu oraliqda $y' > 0,\ y(0) = 0 \Rightarrow y(x) > 0$. Ravshanki,

$$y(x) = \int_0^x (s^2 + y^2(s))ds = \frac{x^3}{3} + \int_0^x y^2(s)ds \le \frac{b^3}{3} + \int_0^x y^2(s)ds.$$

$u(x) = \dfrac{b^3}{3} + \displaystyle\int_0^x y^2(s)ds$ deylik. U holda $y(x) \le u(x)$ va

$u'(x) = y^2(x) \le u^2(x)$. Bundan $\dfrac{u'}{u^2} \le 1,\ -\left(\dfrac{1}{u}\right)' \le 1,\ \dfrac{1}{u(0)} - \dfrac{1}{u(x)} = x$,

$\dfrac{3}{b^3} - \dfrac{1}{u(x)} = x, \quad \dfrac{3}{b^3} - x = \dfrac{1}{u(x)}, \quad u(x) = \dfrac{b^3}{3 - b^3 x}.$ Demak,

$$y(x) \le u(x) = \frac{b^3}{3 - b^3 x},\ x \in [0; 3/b^3].\ b = 3/b^3 \Rightarrow b^4 = 3 \Rightarrow b \approx 1,3161.$$

Yechim kamida $[0; b] \approx [0; 1,3161]$ oraliqda aniqlangan.

x ni $-x$, y ni $-y$ bilan almashtirib, $y(x)$ yechim kamida $[-b, b] \approx [-1,3161; 1,3161]$ oraliqda aniqlanganligini topamiz.

2. 1 masalaning yechilishidan foydalanib, yechim o`ngga kamida $[1; b] \approx [1; 1,1549]$ gacha davom etishini asoslang; aniq hisoblashlar yechim $[1; 1,25609]$ gacha davom etishini ko`rsatadi. Yechimni chapga davom ettirish uchun $x = 1 - t$ almashtirish bajaring.

III.6.

1. Teoremani to'g'ridan-to'g'ri ushbu

$$\varphi(t;\xi') - \varphi(t;\xi'') = \xi' - \xi'' + \int_{t_0}^t \big(f(s, \varphi(s;\xi')) - f(s, \varphi(s;\xi''))\big)ds$$

tenglik va Gronuoll tengsizligidan foydalanib isbotlang.

IV.4.

1. $\Phi'(t) = A(t)\Phi(t)$ bo`lganligi uchun berilganga ko`ra

$$\frac{d\Phi^T(t)}{dt} = \big(A(t)\Phi(t)\big)^T = \Phi^T(t)A^T(t) = -\Phi^T(t)A(t).$$

Demak, $\Psi = \Phi^T(t)$ quyidagi Koshi masalasining yechimi:

$$\begin{cases} \Psi' = -\Psi A(t) \\ \Psi|_{t_0} = \Phi^T(t_0) \end{cases}$$

$\Psi = \Phi^{-1}(t)$ ham shu masalaning yechimi, chunki

$$\frac{d\Phi^{-1}(t)}{dt} = -\Phi^{-1}(t)\frac{d\Phi(t)}{dt}\Phi^{-1}(t) = -\Phi^{-1}(t)A(t)\Phi(t)\Phi^{-1}(t) = -\Phi^{-1}(t)A(t),$$

va $\Phi(t_0)$ ortogonal matritsa bo'lganligi uchun $\Phi^{-1}(t_0) = \Phi^T(t_0)$. Chiziqli normal sisitema uchun yagonalik xossasiga ko'ra $\Phi^T(t) = \Phi^{-1}(t)$, ya'ni $\Phi(t)$ – ortogonal matritsa.

2. $\Phi(t)$ fundamental matritsa

$$\Phi(t) = \Phi(t_0) + \int_{t_0}^{t} A(s)\Phi(s)ds$$

integral tenglamalar sistemasinining yechimidir. Uni ushbu

$$\Phi_0(t) = \Phi(t_0),\ \Phi_k(t) = \Phi(t_0) + \int_{t_0}^{t} A(s)\Phi_{k-1}(s)ds,\ k = 1,2,3,\ldots,$$

ketma-ket yaqinlashishlarning limiti sifatida topish mumkin. Ravshanki, simmetrik matritsalar ko'paytmasi va yig'indisi yana simmetrik matritsa bo'ladi. Shuning uchun ketma-ket yaqinlashishlarning barchasi simmetrik matritsalardan iborat. Demak, ularning limiti bo'lmish $\Phi(t)$ ham simmetrik matritsadir.

IV.7.

2. Berilganga ko'ra

$$\left.\begin{array}{l} e^{tA+tB} = e^{tA}e^{tB} \\ e^{tB+tA} = e^{tB}e^{tA} \end{array}\right\} \Rightarrow e^{tA}e^{tB} = e^{tB}e^{tA},\ \text{demak,}$$

$$\left(E + tA + \frac{t^2}{2}A^2 + \ldots\right)\left(E + tB + \frac{t^2}{2}B^2 + \ldots\right) =$$

$$= \left(E + tB + \frac{t^2}{2}B^2 + \ldots\right)\left(E + tA + \frac{t^2}{2}A^2 + \ldots\right).$$

t^2 oldidagi koeffitsientlarni tenglashtiramiz:

$$AB + \frac{1}{2}A^2 + \frac{1}{2}B^2 = BA + \frac{1}{2}B^2 + \frac{1}{2}A^2 \Rightarrow AB = BA.$$

6. 2^0. $\frac{d}{dt}(\cos tA) = -A\sin tA$, $\frac{d}{dt}(\sin tA) = A\cos tA$.

IV.8.

1. A matritsaning Jordan kanonik ko'rinishidan foydalaning, yoki

$$\left\| e^A - \left(E + \frac{1}{k}A\right)^k \right\| = \left\| \sum_{j=0}^{\infty}\left(\frac{1}{j!} - \frac{1}{k^j}C_k^j\right)A^j \right\| \le e^{\|A\|} - \left(1 + \frac{1}{k}\|A\|\right)^k$$

baholashni asoslang.

3. Berilgan ayniyatni differensiallang va $X'(t) = X(0)X(t)$ tenglamani hosil qiling.

V.

1. Berilganga ko'ra $f(x)$ funksiya $(-\infty;a),(a;b),(b;+\infty)$ oraliqlarining har birida o'z ishorasini saqlaydi. Ixtiyoriy $x = x(t)$ yechimni qaraylik. Aytaylik, biror t_0 uchun $x_0 = x(t_0) \in (a;b)$ bo'lsin. U holda t ning o'zgarish jarayonida bu yechim chekli paytda a ga ham b ga ham yetib bora olmaydi (yechimning yagomalik xossasiga ko'ra). Demak, u barcha $t \in (-\infty;+\infty)$ larda aniqlangan va $x(t) \in (a;b)$. $x = x(t)$ yechim monoton bo'lganligi uchun ($f(x)$ funksiya $(a;b)$ da o'z ishorasini saqlaydi) bu funksiyaning qiymatlar to'plami, ya'ni fazaviy traektoriya $(a;b)$ intervaldan iborat bo'ladi.

Endi faraz qilaylik, $x = x(t)$ yechim uchun biror $t = t_0$ da $x_0 = x(t_0) \in (-\infty;a)$ bo'lsin. Aniqlik uchun $f(x_0) < 0$ deylik. Demak, $x = x(t)$ kamayuvchi. t kamayishi bilan $x(t)$ ortadi va $\lim_{t\to-\infty} x(t) = a$ bo'ladi. t ortishi bilan esa $x(t)$ kamayadi va u yo chekli paytda $-\infty$ ga ketib qoladi, yoki $[t_0;+\infty)$ oraliqqacha davom etadi. Oxirgi holda ham $\lim_{t\to-\infty} x(t) = -\infty$ bo'ladi, chunki aks holda $x(t)$ quyidan biror α son bilan chegaralangan, ya'ni $x(t) \ge \alpha$, $t \in [t_0;+\infty)$, va

$$x'(t) = f(x(t)) \le \sup_{[\alpha;x_0]} f(x) = \beta < 0,\ t \in [t_0;+\infty),$$

baholashlardan hosil bo'luvchi $x(t) \le \beta(t - t_0) + x_0$, $t \in [t_0;+\infty)$, tengsizlikdan yetarlicha katta t lar uchun $x(t) < \alpha$ ziddiyat hosil bo'lar edi.

4. $F(x) = \int_0^x \frac{1}{f(s)}ds$ deylik. $F(x+\tau) - F(x)$ funsiyaning

o'zgarmasligini va $F(x(b))-F(x(a))=b-a$ ekanligini ko'rsating.

5. Bu sistema $-p(x')x'$ qarshilik (ishqalanish) kuchi va $-x$ elastik kuch ta'siri ostida moddiy nuqtaning ($m=1$) harakati tenglamasini ifodalaydi. $x''=-p(x')x'-x$. Moddiy nuqtaning $v=v(x,y)=(x^2+y^2)/2$ to'la mexanik energiyasini qarang. $x=x(t), y=y(t)$ harakat mobaynida bu energiya kamayadi:

$$\frac{dv}{dt}=-p(y)y^2<0, y\neq 0.$$

Sistemaning biror

$$\begin{cases} x=x(t) \\ y=y(t) \end{cases}, 0\leq t\leq T,$$

T-davriy yecimi mavjud deb hisoblab, bu yechim bo'ylab

$$\int_0^T \frac{dv}{dt}dt$$

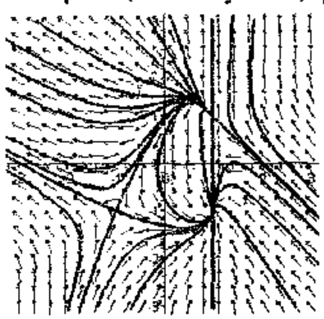
integralni ikki usul bilan hisoblab, ziddiyat hosil qiling.

6. Oshkormas funksiyalar haqidagi teoremani

$$\begin{cases} \varphi(t,\xi,\eta)=\alpha_1+\beta_1 u \\ \psi(t,\xi,\eta)=\alpha_2+\beta_2 u \end{cases}$$

sistemaga qo'llab, uni t va u noma'lumlarga nisbatan yeching [12].

7. Muvozanat nuqtalari va ularning tabiatini aniqlang. $\{(1;y)| \, |y|<1\}$ kesmaning hamda $\{(1;y)| -\infty<y\}$, $\{(1;y)| \, y>1\}$, $\{(x;y)| \, y=2-x, x<1\}$ va $\{(x;y)| \, y=2-x, x>1\}$ nurlarning traektoriya ekanligini asoslang. Tezliklar maydonini va maydon vektorlariga urinuvchi chiziqlarni (traektoriyalarni) quring.



8. Muvozanat nuqtalar to'rtta:

$(1;1)$ – egar, chunki chiziqlashtirilgan sistema matitsasinig xos sonlari

$$\lambda_1=\frac{-1+\sqrt{17}}{2}>0, \ \lambda_2=\frac{-1-\sqrt{17}}{2}<0;$$

$(3;3)$ – turg'un tugun, chunki mos matritsaning xos sonlari

$$\lambda_1=-3<0, \ \lambda_2=-4<0;$$

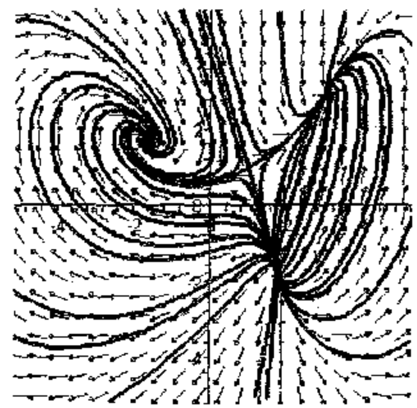
$(\sqrt{3};-\sqrt{3})$ – noturg'un tugun, chunki mos matritsaning xos sonlari

$$\lambda_1=\frac{2+3\sqrt{3}+\sqrt{12\sqrt{3}-\sqrt{17}}}{2}\approx 4{,}6>0, \ \lambda_2=\frac{2+3\sqrt{3}-\sqrt{12\sqrt{3}-\sqrt{17}}}{2}\approx 2{,}6>0;$$

$(-\sqrt{3};\sqrt{3})$ – turg'un fokus, chunki mos matritsaning xos sonlari

$$\lambda_1=\frac{2-3\sqrt{3}}{2}+i\frac{\sqrt{12\sqrt{3}+\sqrt{17}}}{2}\approx -1{,}6+i3{,}1; \ \lambda_2=\frac{2-3\sqrt{3}}{2}-i\frac{\sqrt{12\sqrt{3}+\sqrt{17}}}{2}\approx -1{,}6-i3{,}1;$$

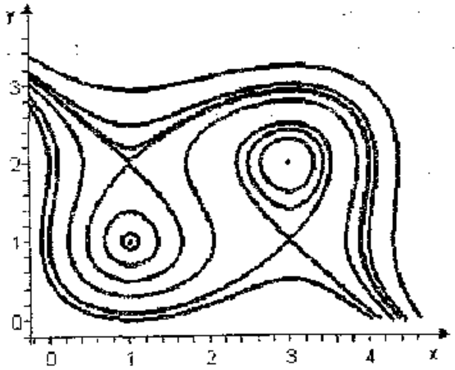
$\mathrm{Re}(\lambda_1)=\mathrm{Re}(\lambda_2)<0$



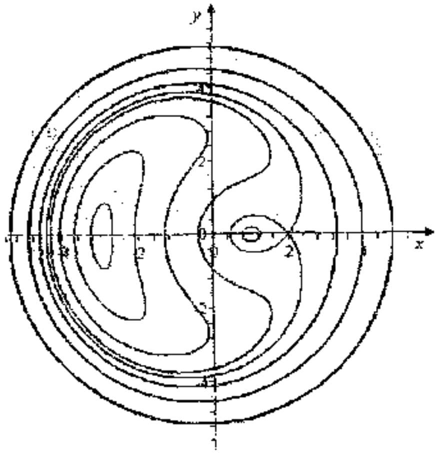
9. To'rtta muvozanat nuqtasi bor. Ular:

$(1;1)$ va $(3;2)$ - markazlar;

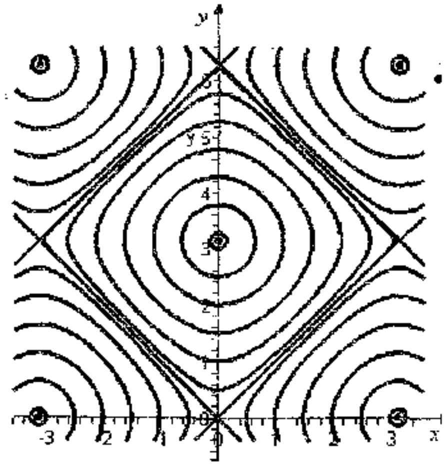
$(1;2)$ va $(3;1)$ - egarlar.



10. Muvozanat nuqtalari uchta: $(-3;0),(1;0)$ – markazlar; $(2;0)$ – egar.



11. Davriy manzarning bir qismi quyidagi rasmda tasvirlangan:



12. 1). Bendikson-Dyulak teoremasidan foydalaning ($h = be^{-\beta x}$).
2). Bendikson-Dyulak teoremasidan foydalaning ($h = x^k y^l$; k,l larni tanlang).
3). $(x^2 + y^2)' \ge 0$ munosabatdan foydalaning.

13. $\dfrac{dx}{d\tau}$ va $\dfrac{dy}{d\tau}$ hosilalarning ishoralarini o`rganing va $A_7A_6$ kesmaning ixtiyoriy $(x;0)$ nutasidan chiqqan yechim $\tau$ ning ortishi bilan $(0;0)$ nuqta atrofida aylanib, $A_7A_6$ ning $(\varphi(x);0)$ nuqtasiga qaytishini ko`rsating. $\varphi(x)$ ning o`suvchi va uzluksiz funksiya ekanligini asoslang. $x = x_{A_7}$ da $\varphi(x) - x > 0$, $x = x_{A_6}$ da esa $\varphi(x) - x < 0$ bo`lgani uchun $\exists \tilde{x} \in [x_{A_7}; x_{A_6}]$ $\varphi(\tilde{x}) = \tilde{x}$. Sistemaning $\tilde{x}$ nuqtadan chiqqan traektoriyasi yopiq chiziqdan iborat.

VI.

4. Nol-yechim noturg'un, $v = x^2 + y^2$.

5. Nol-yechim turg`un,

$$v = x + y - \ln(1+x) - \ln(1+y), (x,y) \in U, U = (-1;1)\times(-1;1).$$

6. $v = x^4 + 2y^2$.

8. 1). $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{xy}$ tenglamada $y = xu$ deb traektoriyalarni (birinchi integralni) toping.

2). $x^4 + y^2 = cx^2$ egri chiziqlar traektoriyalarni ifodalaydi.

3). $\dfrac{dy}{dx} = \dfrac{y^2 - 6x^2 y + x^4}{xy}$ tenglamada $y = x^2 u$ almashtirish bajaring.

4). Birinchi integralni toping.

5). Muvozanat nuqtalari atrofida traektoriyalar tabiatini tekshiring.

7). Tekshirishda $v(x,y) = (x^2 - 1)e^{-y^2}$ ning Lyapunov funksiyasi ekanligidan foydalaning.

10). Qutb koordinatalariga o'ting.

10. Yetarlicha katta x, y, z lar uchun $(rx^2 + \sigma y^2 + \sigma(z - 2r)^2)' < 0$ va sistema aniqlagan mos oqimning divergensiyasi $-(\sigma + b + 1)$ ga teng ekanligini ko'rsating.

VII.1.

1. 1) Qisqalik uchun $\psi(t) = \psi(t; t_0, x^0)$ deylik. Ma'lum (VII.1.26)

$$\frac{\partial \varphi(t; t_0, x^0)}{\partial t_0} = -\frac{\partial \varphi(t; t_0, x^0))}{\partial x^0} f(t_0, x^0)$$

formulaga ko'ra

$$\frac{\partial \varphi(t; s, \psi(s))}{\partial s} = \frac{\partial \varphi(t; s, \psi(s))}{\partial t_0} + \frac{\partial \varphi(t; s, \psi(s))}{\partial x^0} \psi'(s) =$$
$$= \Phi(t; s, \psi(s))(\psi'(s) - f(s, \psi(s)) =$$
$$= \Phi(t; s, \psi(s)) r(s, \psi(s)).$$

Bu tenglikni $s = t_0$ dan $s = t$ gacha integrallab, va $\varphi(t; t, \psi(t)) = \psi(t)$ ekanligidan foydalanib, V. A. Alekseev formulasini hosil qiling.

2). Ushbu

$$\frac{\partial \varphi(t; t_0, x^0 + s(y^0 - x^0))}{\partial s} = \Phi(t; t_0, x^0 + s(y^0 - x^0))(y^0 - x^0), \ s \in [0;1]$$

ayniyatni integrallang.

3). Bu formulani 1) va 2) formulalardan keltirib chiqaring.

VII.2.

1. $x = \dfrac{e^t}{1 + \mu(e^t - 1)}$.

VII.3.

1. Tenglamalarni hadma-had qo'shib va ayirib, toping:

$$\begin{cases} (x+y)' = (x+y)^2 \\ (x-y)' = (x-y)^2 \end{cases}$$

Tenglamalarni alohida-alohida yeching ($x + y \neq 0$, $x - y \neq 0$ deb faraz qiling):

$$\begin{cases} x + y = \dfrac{1}{c_1 - t} \\ x - y = \dfrac{1}{c_2 - t} \end{cases} \Rightarrow \begin{cases} x = \dfrac{1}{2}\left(\dfrac{1}{c_1 - t} + \dfrac{1}{c_2 - t}\right) \\ y = \dfrac{1}{2}\left(\dfrac{1}{c_1 - t} - \dfrac{1}{c_2 - t}\right) \end{cases}$$

Yo'qolgan yechimlarni toping.

2. 1). Teskarisini faraz qilaylik: biror B_δ doirada aniqlangan $u(x,y)$ birinchi integral mavjud bo'lsin. Demak, $u(x,y) \in C^1(B_\delta)$, $u(x,y) \neq \text{const}$ va berilgan sistemaning B_δ da joylashgan har qanday $x = x(t)$, $y = y(t)$ yechimi bo'ylab $u(x(t), y(t)) = \text{const}$. Ixtiyoriy $(x_0, y_0) \in B_\delta$ nuqtani olib, berilgan sistemaning $x(t) = x_0 e^{-t}$, $y(t) = y_0 e^{-t}$ yechimini qaraylik. Ravshanki, ixtiyoriy $t \geq 0$ uchun $(x(t), y(t)) = (x_0 e^{-t}, y_0 e^{-t}) \in B_\delta$. Demak, $u(x_0 e^{-t}, y_0 e^{-t}) = u(x_0, y_0)$, $t \geq 0$. Demak, $u(x_0 e^t, y_0 e^t) = u(x_0, y_0)$. Bu ayniyatda $t \to +\infty$ da limitga o'tamiz. Natijada ixtiyoriy $(x_0, y_0) \in B_\delta$ uchun $u(0,0) = u(x_0, y_0)$ ekanligini hosil qilamiz. Bu esa $u(x,y) \neq \text{const}$ ekanligiga zid.

2). *$x > 0$ yarim tekislikda aniqlangan birinchi integral osongina topiladi:*

$$xy' - yx' = 0 \Rightarrow \left(\frac{y}{x}\right)' = 0 \Rightarrow \frac{y}{x} = c.$$

3. Sistemaning ikkinchi va uchinchi tenglamalaridan bitta birinchi integralni topamiz

$$zy' + yz' = 0 \Rightarrow zy = c_1 \ (c_1 - \text{ixtiyoriy o'zgarmas}).$$

Demak, har qanday $x = x(t)$, $y = y(t)$, $z = z(t)$ yechim bo'ylab zy ko'paytma o'zgarmas. Ikkinchi tenglamani birinchisiga hadma-had bo'lib, va yechim bo'ylab $c_1 = yz$ o'zgarmas ekanligini hisobga olib, ushbu

$$\frac{dy}{dx}=\frac{c_1 x}{1+3y^2}$$

o'zgaruvchilari ajraladigan tenglamani hosil qilamiz. Bundan

$$c_1\frac{x^2}{2}=y+y^3+\frac{c_2}{2}, \text{ ya'ni } x^2c_1-2y-2y^3=c_2$$

ekanligi kelib chiqadi. Oxirgi tenglikdan yechim bo'ylab $c_1=yz$ bo'lganligi uchun yana bir birinchi integralni hosil qilamiz:

$$x^2yz-2y-2y^3=c_2.$$

Topilgan $u_1=yz$ va $u_2=x^2yz-2y-2y^3$ birinchi integrallarni erklilikka tekshiramiz. Buning uchun ushbu

$$\begin{pmatrix}\frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z}\\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z}\end{pmatrix}=\begin{pmatrix}0 & z & y\\ 2xyz & x^2z-2-6y^2 & x^2y\end{pmatrix}$$

Yakobi matritsasini tuzib, uning rangini hisoblaymiz. Agar $y\neq 0$ bo'lsa, quyidagi ikkinchi tartibli minorning qiymati noldan farqli:

$$\begin{vmatrix} z & y\\ x^2z-2-6y^2 & x^2y\end{vmatrix}=y(2+6y^2)\neq 0;$$

demak, tuzilgan matritsaning rangi ikkiga teng va $y>0$ (yoki $y<0$) sohada topilgan birinchi integrallar erkli.

4. Volterra-Lotka sistemasining $x>0, y>0$ sohadagi birinchi integrali topilgan edi:

$$u(x,y)=\frac{x^a y}{e^{ax}e^y}.$$

Har bir traektoriya to'laligicha bitta $u(x,y)=\text{const}$ sath chizig'ida yotadi. $u(x,y)$ funksiyani va uning sath chiziqlarini tekshiring.

5. Sistemaning birinchi integrallari:

$$x_1'+x_2'+x_3'=0 \Rightarrow x_1+x_2+x_3=c_1 \text{ (tekislik)}$$

$$x_1x_1'+x_2x_2'+x_3x_3'=0 \Rightarrow x_1^2+x_2^2+x_3^2=c_2 \text{ (sfera)}$$

Demak, traektoriyalar aylanalardan iborat.

VII. 4.

1. $\xi=x-t$, $\tau=t$ almashtirish bajaring.
2. 1) Ha; 2) Yo'q.

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O'quv-uslubiy nashr

N. Dilmurodov

DIFFERENSIAL TENGLAMALAR KURSI

II jild

Texnik muharrir: **M. Raxmatov**

Muharrir: **E. Jabborov**

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