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NIZOMIY NOMIDAGI TOSHKENT DAVLAT
PEDAGOGIKA UNIVERSITETI

MATEMATIK TAHLILDAN
MISOL VA MASALALAR YECHISH

(Kompleks o`zgaruvchining funksiyalari nazariyasi,
Furye qatori va integrali)
uslubiy qo`llanma

Tuzuvchilar: E.A.Yo`ldoshev
R.M.Turgunbayev

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ANNOTATSIYA

Ushbu o`quv qo`llanma pedagogika oliy o`quv yurtlari «Fizika va astronomiya» mutaxassisligining «Matematik tahlil» fani dasturiga mos yozilgan bo`lib, bunda matematik tahlilning kompleks o`zgaruvchining funksiyalari nazariyasi, Furye qatori va integrali bo`limlaridan misol va masalalar yechish bo`yicha namunalar va mustaqil yechish uchun misol va masalalar berilgan. Bu uslubiy qo`llanmadan «Matematika va informatika» mutaxassisligi talabalari ham foydalanishi mumkin.

Tuzuvchilar: f.-m.f.n., dotsent E.Yo`ldoshev
f.-m.f.n. R.M.Turgunbayev

Taqrizchilar: f.-m.f.n., dotsent M.Madirimov
f.-m.f.n., dotsent M.Berdiqulov

Muharrir: prof. O`.Toshmetov

Nizomiy nomidagi Toshkent davlat pedagogika universiteti Ilmiy Kengashining 200 yil -raqamli qaroriga binoan nashrga tavsiya etilgan.

MUNDARIJA

Kirish

Ushbu o`quv qo`llanma pedagogika institutlari va universitetlarning fizika fakulьtetlarida tahsil olayotgan talabalarga mo`ljallangan T.Sharifova, E.Yo`ldoshevlarning “Matematik analizdan misol va masalalar yechish” o`quv qo`llanmasining tadrijiy davomi bo`lib, bu qo`llanma matematik analiz kursining kompleks o`zgaruvchining funksiyalari nazariyasi, furьe qatori va furьe integrali bo`limlariga bag`ishlangan.

Qo`llanma pedagogika oliy o`quv yurtlarining fizika-astronomiya bakalavriat yo`nalishining “Matematik tahlil” fani dasturiga mos holda yozilgan bo`lib, bunda kompleks o`zgaruvchining funksiyalari nazariyasi, Furye qatori va integrali bo`limlaridan asosiy nazariy tushunchalar, misol, masalalar yechish bo`yicha namunalar, mustaqil yechish uchun misol va masalalar keltirilgan.

Q`llanmaning I bobini R.M.Turgunbayev, II bobini E.Yo`ldoshev yozgan.

Bu o`quv qo`llanmadan matematika va informatika bakalavriat yo`nalishida tahsil olayotgan talabalar ham foydalanishi mumkin.

I bob. KOMPLEKS O`ZGARUVCHINING FUNKSIYALARI NAZARIYASI

1-§. Kompleks sonlar va ular ustida amallar. Kompleks sonning geometrik ma'nolari. Kompleks sonning moduli va argumenti. Kompleks sonning trigonometrik shakli. Muavr formulasi. Kompleks sonning ildizi.

Kompleks son deb $x+iy$ ko`rinishdagi ifodaga aytiladi, bu erda $x$, $y$ lar ixtiyoriy haqiqiy sonlar, $i$ mavqum birlik bo`lib, u $i^2=-1$ shartni qanoatlantiradi. Ko`pincha $x+iy$ kompleks sonni bitta z harfi bilan belgilanadi va kompleks sonning algebraik shakli deb yuritiladi: $z=x+iy$. Ushbu x va y sonlar z kompleks sonning mos ravishda haqiqiy va mavqum qismi deyiladi va quyidagicha belgilanadi: $x=Rez$, $y=Imz$.

$\bar{z}=x-iy$ kompleks son $z=x+iy$ kompleks sonning qo`shmasi deyiladi.

$z=x+iy$ kompleks son tekislikdagi dekart koordinatalari sistemasida absissasi x, ordinatasi y bo`lgan $M(x;\ y)$ nuqta yoki boshi $O(0,0)$ nuqtada uchi $M(x;\ y)$ nuqtada bo`lgan vektor bilan tasvirlanadi. Ox o`qi haqiqiy o`q, Oy o`qi esa mavqum o`q deyiladi.

$\overrightarrow{OM}$ vektorning r uzunligi z kompleks sonning moduli deyiladi va $|z|$ kabi belgilanadi. Demak, $|z|=\sqrt{x^2+y^2}$.

Kompleks sonning moduli quyidagi xossalarga ega:

1. $|\bar{z}|=|z|$; 2. $z\cdot\bar{z}=|z|^2$; 3. $|z_1 z_2|=|z_1|\,|z_2|$;

4. $|z^n|=|z|^n$; 5. $\left|\dfrac{z_1}{z_2}\right|=\dfrac{|z_1|}{|z_2|}$; 6. $|Rez|\le|z|$; $|Imz|\le|z|$;

7. $|z_1+z_2|\le|z_1|+|z_2|$; 8. $||z_1|-|z_2||\le|z_1-z_2|$.

$\overrightarrow{OM}$ vektor va Ox o`q orasidagi $\varphi$ burchak $z$ kompleks sonning argumenti deyiladi va $\varphi=Argz$ kabi belgilanadi. Kompleks sonning argumenti $2\pi$ ga karrali qo`shiluvchi aniqligida bir qiymatli aniqlanadi: $Argz=argz+2k\pi\ (k\in Z)$, bu erda $argz$ quyidagi $0\le argz<2\pi$ shart bilan aniqlanib, argumentning bosh qiymati deyiladi va ushbu formuladan topiladi:

$$\arg z=\begin{cases} arctg\dfrac{y}{x}, & agar \quad x>0, \quad y\ge 0,\\ \pi+arctg\dfrac{y}{x}, & agar \quad x<0,\\ 2\pi+arctg\dfrac{y}{x}, & agar \quad x>0, \quad y<0,\\ \dfrac{\pi}{2}, & agar \quad x=0, \quad y>0\\ \dfrac{3\pi}{2}, & agar \quad x=0, \quad y<0. \end{cases} \qquad (1)$$

$z=0$ ning argumenti aniqlanmagan.

Ikkita $z_1=x_1+iy_1$ va $z_2=x_2+iy_2$ kompleks sonlar berilgan bo`lsin.

Agar $z_1=x_1+iy_1$ va $z_2=x_2+iy_2$ kompleks sonlar uchun $x_1=x_2$ va $y_1=y_2$ bo`lsa, u holda bu sonlar teng deyiladi.

Ikkita $z_1=x_1+iy_1$ va $z_2=x_2+iy_2$ kompleks sonlar teng bo`lishi uchun ularning modullari va argumentlarining bosh qiymatlari teng bo`lishi zarur va etarli: $|z_1|=|z_2|$, $argz_1=argz_2$.

$z_1=x_1+iy_1$ va $z_2=x_2+iy_2$ kompleks sonlarning yig`indisi deb ushbu $z_1+z_2=(x_1+x_2)+i(y_1+y_2)$ songa aytiladi.

$z_1=x_1+iy_1$ *va* $z_2=x_2+iy_2$ kompleks sonlarning ayirmasi deb ushbu $z_1-z_2=(x_1+x_2)-i(y_1+y_2)$ songa aytiladi.

$z_1=x_1+iy_1$ va $z_2=x_2+iy_2$ kompleks sonlarning kщpaytmasi deb ushbu $z_1\cdot z_2=(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$ songa aytiladi. Xususan, $z\bar{z}=x^2+y^2=|z|^2$ bo`ladi.

$z_1=x_1+iy_1$ kompleks sonning $z_2=x_2+iy_2$ ($z_2\neq0$) kompleks songa bo`linmasi deb $z_1=z\cdot z_2$ tenglikni qanoatlantiradigan z kompleks songa aytiladi. U ushbu formuladan topiladi:

$$z=\frac{z_1}{z_2}=\frac{z_1\cdot\bar{z}_2}{|z_2|^2} \quad \text{yoki} \quad z=\frac{z_1}{z_2}=\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+i\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}.$$

Har qanday $z=x+iy$ ($z\neq0$) kompleks sonni trigonometrik shaklda yozish mumkin:

$z=r(\cos\varphi+i\sin\varphi)$, bu erda $r=|z|$, $\varphi=Argz$.

Faraz qilaylik z_1 va z_2 kompleks sonlar trigonometrik shaklda berilgan bo`lsin: $z_1=r_1(\cos\varphi_1+i\sin\varphi_1)$, $z_2=r_2(\cos\varphi_2+i\sin\varphi_2)$. U holda

a) ularning ko`paytmasi

$z_1\cdot z_2= r_1\cdot r_2(\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2))$

formula yordamida topiladi, ya’ni kompleks sonlarni ko`paytirganda ularning modullari ko`paytiriladi, argumentlari qo`shiladi:

$|z_1\cdot z_2|=|z_1|\cdot|z_2|$, $Arg(z_1\cdot z_2)=Argz_1+Argz_2$.

b) ikkita z_1 va $z_2\neq0$ kompleks sonlarning nisbati

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}(\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)), \qquad (2)$$

formuladan topiladi, bundan

$$\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}; \quad Arg\frac{z_1}{z_2}=Argz_1-Argz_2.$$

$z=r(\cos\varphi+i\sin\varphi)$ kompleks sonning natural ko`rsatkichli n-darajasi ushbu formuladan topiladi:

$z^n=r^n(\cos n\varphi+i\sin n\varphi)$,

bundan $|z^n|=|z|^n$, $Argz^n=nArgz+2k\pi$, $k\in Z$ ekanligi, xususan

$$(\cos\varphi+i\sin\varphi)^n=(\cos n\varphi+i\sin n\varphi) \qquad (3)$$

Muavr formulasi kelib chiqadi.

$z=r(\cos\varphi+i\sin\varphi)$ kompleks sonning n-darajali ildizi n-ta turli qiymatlarga ega va quyidagi formuladan topiladi:

$$\sqrt[n]{z}=\sqrt[n]{|z|}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right), \qquad (4)$$

bu erda k=0, 1, 2,…, n-1, $\varphi=argz$.

$\sqrt[n]{z}$ qiymatlariga mos keluvchi nuqtalar radiusi $R=\sqrt[n]{|z|}$ ga teng bo`lgan aylanaga ichki chizilgan muntazam $n$ burchakning uchlaridan iborat bo`ladi.

№1. Quyidagi amallarni bajaring:

a) $(3-4i)+(5+i)$; b) $(4+3i)-(1-2i)$; c) $(1-i)\cdot(2+3i)$; d) $5i\cdot(-3+2i)$;

e) $5i:(1-i)$; f) $(4-5i):3$; g) $(6+5i):2i$; i) $(2-i):(5+12i)$.

Yechish. a) $(3-4i)+(5+i)=(3+5)+i(-4+1)=8-3i$;

b) $(4+3i)-(1-2i)=(4-1)+i(3-(-2))=3+5i$;

c) $(1-i)\cdot(2+3i)=(1\cdot2-(-1)\cdot3)+i(1\cdot3+(-1)\cdot2)=5+i$;

d) $5i\cdot(-3+2i)=-15i+10i^2=-10-15i$;

e) $5i:(1-i)=\frac{5i\cdot(1+i)}{|1-i|}=\frac{-5+5i}{\sqrt{1^2+(-1)^2}}=-\frac{5}{\sqrt{2}}+\frac{5}{\sqrt{2}}i$;

f) $(4-5i):3=\frac{4-5i}{3}=\frac{4}{3}-\frac{5}{3}i$;

g) $(6+5i):2i=\frac{6+5i}{2i}=\frac{(6+5i)\cdot i}{2i\cdot i}=\frac{-5+6i}{-2}=2{,}5-3i$;

i) $(2-i):(5+12i)=\frac{2-i}{5+12i}=\frac{(2-i)\cdot(5-12i)}{(5+12i)\cdot(5-12i)}=\frac{-2-29i}{13}=-\frac{2}{13}-\frac{29}{13}i$.

№2. Quyidagi sonlarning moduli va argumentini toping:

a) $z=-2+2\sqrt{3}\,i$; b) $-3i$; c) $-\sin\frac{\pi}{8}-i\cos\frac{\pi}{8}$.

Yechish. a) $|z|=|-2+2\sqrt{3}\,i|=\sqrt{(-2)^2+(2\sqrt{3})^2}=\sqrt{4+12}=\sqrt{16}=4$,

(1) formulaga asosan $argz=arg(-2+2\sqrt{3}\,i)=\pi+arctg\frac{2\sqrt{3}}{-2}=\pi-arctg\sqrt{3}=$ $=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$.

Demak, $Argz=\frac{2\pi}{3}+2k\pi,\ k\in Z,\ |z|=4$.

b) $|-3i|=|0-3i|=\sqrt{0^2+(-3)^2}=\sqrt{9}=3$, $\arg(-3i)=\frac{3\pi}{2}$ ((1) ga asosan).

Demak, $|z|=3$, $Argz=\frac{3\pi}{2}+2k\pi,\ k\in Z$.

c) $|z|=\sqrt{(-\sin\frac{\pi}{8})^2+(-\cos\frac{\pi}{8})^2}=\sqrt{\sin^2\frac{\pi}{8}+\cos^2\frac{\pi}{8}}=1$,

Argumentning bosh qismi ni topish uchun (1) formuladan foydalanamiz, bunda $x=-sin\frac{\pi}{8}<0$, $y=-\cos\frac{\pi}{8}<0$. Demak,

$$argz=\pi+arctg\frac{y}{x}=\pi+arctg\frac{-\cos\frac{\pi}{8}}{-\sin\frac{\pi}{8}}=\pi+arctg(\text{ctg}\frac{\pi}{8})=\pi+\quad arctg(tg(\frac{\pi}{2}-\frac{\pi}{8})=$$

$=\pi+\frac{3\pi}{8}=\frac{11\pi}{8}$, bundan $Argz=\frac{11\pi}{8}+2k\pi,\ k\in Z.$

№3. Kompleks sonni trigonometrik shaklda yozing:

a) 3; b) $-2+2\sqrt{2}\,i$.

Yechish. a) $|3|=3$, $arg3=0$, demak $3=3(cos2k\pi+isin2k\pi)$;

b) $|-2+2\sqrt{2}\,i|=\sqrt{4+8}=2\sqrt{3}$; $arg(-2+2\sqrt{2}\,i)=\pi+arctg\frac{2\sqrt{2}}{-2}=\pi-arctg\sqrt{2}$,

demak, $-2+2\sqrt{2}\,i=2\sqrt{3}\,(cos(\pi-arctg\sqrt{2}+2k\pi)+isin(\pi-arctg\sqrt{2}+2k\pi)),\ k\in Z.$

№4. Hisoblang:

a) $(1+\sqrt{3}\,i)^3$; b) $(1+i)^{20}$; c) $\frac{\left(1+\sqrt{3}i\right)^{16}}{\left(1+i\right)^{20}}$.

Yechish. a) ikki usulda hisoblaymiz. 1-usul. $(1+\sqrt{3}\,i)^3=1^3+3\cdot1^2\cdot\sqrt{3}\,i+$

$+3\cdot1\cdot(\sqrt{3}\,i)^2+(\sqrt{3}\,i)^3=1+3\sqrt{3}\,i-9-3\sqrt{3}\,i=-8$, bu erda $i^2=-1$, $i^3=-i$ ekanligidan foydalandik.

2-usul. Avval $1+\sqrt{3}\,i$ sonni trigonometrik shaklda yozib olamiz. Bunda $|1+\sqrt{3}\,i|=2$, $\arg(1+\sqrt{3}\,i)=\text{arctg}\sqrt{3}=\frac{\pi}{3}$.

Demak, $1+\sqrt{3}\,i=2(\cos(\frac{\pi}{3}+2k\pi)+i\sin(\frac{\pi}{3}+2k\pi))$, $k\in Z.$ Hisoblashlar bajarganda $k=0$ deb olish etarli. Shu sababli $1+\sqrt{3}\,i=2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$ tenglikdan foydalanamiz. Bu tenglikni ikkala tomonini uchinchi darajaga ko`taramiz va (3) Muavr formulasidan foydalanamiz:

$(1+\sqrt{3}\,i)^3=(2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}))^3=8(\cos\pi+i\sin\pi)=-8.$

b) $(1+i)^{20}$ ni hisoblash uchun $1+i$ ni trigonometrik shaklda yozamiz: $1+i=|1+i|\cdot(cosarg(1+i)+isinarg(1+i))=\sqrt{2}\,(cos\frac{\pi}{4}+isin\frac{\pi}{4})$. Endi (3) formuladan foydalanamiz. $(1+i)^{20}=(\sqrt{2}\ (cos\frac{\pi}{4}+isin\frac{\pi}{4}))^{20}=(\sqrt{2})^{20}(cos5\pi+isin5\pi)=$

$=2^{10}(-1+0i)=-1024.$

c) $\frac{\left(1+\sqrt{3}i\right)^{16}}{\left(1+i\right)^{20}}$ ni hisoblashda yuqorida olingan $1+\sqrt{3}\,i=2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$, $1+i=\sqrt{2}\,(cos\frac{\pi}{4}+isin\frac{\pi}{4})$ natijalar va (2) formuladan foydalanamiz.

$$\frac{\left(1+\sqrt{3}i\right)^{16}}{\left(1+i\right)^{20}}=\frac{(2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}))^{16}}{(\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})^{20}}=\frac{2^{16}(\cos(5\pi+\frac{\pi}{3})+i\sin(5\pi+\frac{\pi}{3}))}{2^{10}(\cos 5\pi+i\sin 5\pi)}=$$

$=-2^6(-\cos\frac{\pi}{3}-i\sin\frac{\pi}{3})=64(\frac{1}{2}+\frac{\sqrt{3}}{2}i)=32+32\sqrt{3}\,i.$

№5. $\sqrt[4]{1-i}$ ning barcha qiymatlarini toping.

Yechish. 1-i ni trigonometrik shaklda yozib olamiz: $1-i=|1-i|\cdot(\cos\arg(1-i)+i\sin\arg(1-i))=\sqrt{2}\,(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})$. Demak, (4) formulaga asosan

$\sqrt[4]{1-i}=\sqrt[8]{2}(\cos\frac{\frac{3\pi}{4}+2k\pi}{4}+i\sin\frac{\frac{3\pi}{4}+2k\pi}{4})$ bo`lib, bu erda k=0, 1, 2, 3 qiymatlar qabul qiladi. Bundan

k=0 da $\sqrt[4]{1-i}=\sqrt[8]{2}(\cos\frac{\pi}{16}+i\sin\frac{\pi}{16})$,

k=1 da $\sqrt[4]{1-i}=\sqrt[8]{2}(\cos\frac{11\pi}{16}+i\sin\frac{11\pi}{16})$,

k=2 da $\sqrt[4]{1-i}=\sqrt[8]{2}(\cos\frac{19\pi}{16}+i\sin\frac{19\pi}{16})$,

k=3 da $\sqrt[4]{1-i}=\sqrt[8]{2}(\cos\frac{27\pi}{16}+i\sin\frac{27\pi}{16})$.

№6. Quyidagi amallarni bajaring:

a) $(7+3i)+(4-2i)$; b) $(2+3i)-(4-5i)$; c) $(1-2i)\cdot(1+2i)$; d) $-2i\cdot(4+3i)$;
e) $5:(1+i)$; f) $(6+5i):(3i)$; g) $(6+5i):(1-2i)$; i) $(2-i)^2$.

№7. Quyidagi sonlarning moduli va argumentini toping.

a) $z=4-3i$; b) $z=2+2\sqrt{3}\,i$; c) $z=3$; d) $z=-\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}$.

№8. Kompleks sonni trigonometrik shaklda yozing:

a) -3; b) $4i$; c) $-\sqrt{2}+\sqrt{2}\,i$; d) $-1-\sqrt{3}\,i$.

№9. Hisoblang:

a) $(\sqrt{3}-3i)^{12}$; b) $(2+3i)^3$; c) $(-2-2i)^8$; d) $\left(\frac{1-i}{1+\sqrt{3}i}\right)^{30}$; e) $\left(\frac{1+i}{1-i}\right)^{9}$.

№10. Quyidagi ildizlarning barcha qiymatlarini toping:

a) $\sqrt[4]{-i}$; b) $\sqrt[3]{1}$; c) $\sqrt[3]{2-2\sqrt{3}i}$; d) $\sqrt{-1+i}$; e) $\sqrt[4]{-16}$.

2-§. Kompleks tekislikdagi chiziqlar va sohalar

Matematik analiz kursida tekislikdagi chiziqlar odatda $F(x,y)=0$ (x va y dekart koordinatalari), yoki $(r,\varphi)=0$ (r va φ qutb koordinatalari) yoki parametrik tenglamalari

$$x=x(t),\ y=y(t)$$

bilan berilar edi.

Kompleks analizda chiziqlarning berilishi biroz boshqacha ko`rinishga ega bo`ladi:

Agar *xOy* tekislikda egri chiziq $x=x(t),\ y=y(t),\ \alpha\le t\le\beta$ parametrik tenglamalar bilan berilgan bo`lsa, uning kompleks tekislikdagi tenglamasi

$$z(t)=x(t)+iy(t), \quad \alpha\le t\le\beta \qquad (5)$$

bo`ladi.

№11. $x=a\cos t,\ y=b\sin t,\ 0\le t\le 2\pi$ ellips tenglamasini kompleks shaklda yozing.

Yechish. (5) formuladan foydalansak, ellips tenglamasi kuyidagicha bo`ladi: $z=a\cos t+ib\sin t,\ 0\le t\le 2\pi$.

Agar chiziq $F(x,y)=0$ ko`rinishdagi tenglama bilan berilgan bo`lsa, u holda x va y o`zgaruvchilarning o`rniga ularning z va $\bar z$ o`zgaruvchilar bilan ifodalarini

$$x=\frac{z+\bar z}{2},\ y=\frac{z-\bar z}{2i} \qquad (6)$$

almashtirib $F(\frac{z+\bar z}{2},\frac{z-\bar z}{2i})=0$ tenglamaga ega bo`lamiz.

№12. Aylananing

$$x^2+y^2+2ax+2by+c=0$$

umumiy tenglamasini kompleks ko`rinishda yozing.

Yechish. Yuqoridagi (6) formulalardan foydalanamiz. U holda $z\bar z+2a\frac{z+\bar z}{2}+2b\frac{z-\bar z}{2i}+c=0 \Rightarrow z\bar z+(a-ib)z+(a+ib)\bar z+c=0$.

$a+ib=A$, $a-ib=\bar A$ belgilashlar kiritsak, aylananing umumiy tenglamasi

$$z\bar z+\bar A z+A\bar z+c=0, \qquad \operatorname{Im}c=0$$

hosil bo`ladi.

№13. $\operatorname{Im}z^2=4$ tenglama qanday chiziqni aniqlaydi?

Yechish: $z^2=(x+iy)^2=x^2-y^2+2xyi$ bo`lganligidan, $\operatorname{Im}z^2=2xy$ bo`ladi. Bundan $2xy=4$, $xy=2$ yoki $y=2/x$. Bu esa giperbola tenglamasidir, demak $\operatorname{Im}z^2=4$ tenglama tekislikda giperbolani ifodalaydi.

L chiziq (5) tenglama bilan berilgan bo`lsin. Agar $x(t)$ va $y(t)$ funksiyalar $[\alpha,\beta]$ kesmada uzluksiz bo`lsa, u holda bu tenglama shu kesmada uzluksiz egri chiziqni aniqlaydi deyiladi. Agar $[\alpha,\beta]$ kesmadan olingan turli t_1 va t_2 uchun $z(t_1)\ne z(t_2)$ bo`lsa, uzluksiz $L$ egri chiziq sodda chiziq deyiladi. Agar $z(\alpha)=z(\beta)$ bo`lsa, L yopiq chiziq deyiladi.

Agar uzluksiz L egri chiziqni

$$z=x(t)+iy(t), \quad \alpha\le t\le\beta,$$

bu erda $x(t),\ y(t)$ funksiyalar $[\alpha,\beta]$ kesmada uzluksiz differensiallanuvchi va $z'(t)=x'(t)+iy'(t)\ne 0,\ \forall t\in[\alpha,\beta]$, shartlarni qanoatlantiruvchi parametrik tenglama bilan ifodalash mumkin bo`lsa, u holda bu chiziq silliq chiziq deyiladi.

Agar uzluksiz egri chiziqning silliqligi faqat chekli sondagi nuqtalardagina bajarilmasa, u holda bu chiziqni bo`lakli silliq chiziq deyiladi.

№14. Quyidagi

a) $z=1+it$ $(-1\le t\le 1)$; b) $z=1+i\sin t$ $(0\le t\le 2\pi)$

tenglamalar bilan qanday chiziqlar berilgan? Ulardan qaysilari sodda, yopiq, silliq bo`ladi?

Yechish: a) z ni $x+iy$ bilan almashtirib, $z=1+it$ $(-1\le t\le 1)$ tenglamadan izlanayotgan chiziqning $x=1$, $y=t$ $(-1\le t\le 1)$ parametrik tenglamasiga ega bo`lamiz. Bundan $z=1+it$ $(-1\le t\le 1)$ chiziq uchlari $z_1=1-i$ va $z_2=1+i$ nuqtalarda bo`lgan kesmadan iborat ekanligi kelib chiqadi. t parametrning turli qiymatlariga turli nuqtalar mos keladi, demak bu chiziq sodda siziq bo`ladi. Shuningdek, $z(-1)\ne z(1)$ bo`lganligi sababli, u yopiq emas. $x=1$, $y=t$ funksiyalar [-1,1] kesmada uzluksiz differensiallanuvchi va $z'(t)\ne 0$ bo`lganligidan bu chiziq silliq chiziq bo`ladi.

b) Yuqoridagi kabi $z=1+i\sin t$ $(0\le t\le 2\pi)$ tenglamadan izlanayotgan chiziqning $x=1$, $y=\sin t$ $(0\le t\le 2\pi)$ parametrik tenglamasini hosil qilamiz. t parametr 0 dan 2π gacha o`zgarganda $\sin t$ funksiya –1 dan 1 gacha bo`lgan barcha qiymatlarni qabul qiladi. Shu sababli $z=1+i\sin t$ $(0\le t\le 2\pi)$ chiziq xam $z=1+it$ $(-1\le t\le 1)$ chiziq kabi kompleks tekislikda uchlari $z_1=1-i$ va $z_2=1+i$ nuqtalarda bo`lgan kesmadan iborat bo`ladi. Lekin bu chiziqlar bir-biridan farq qiladi.

Haqiqatan ham $z=1+i\sin t$ $(0\le t\le 2\pi)$ chiziq birinchidan sodda emas (masalan, $t_1=\pi/3$, $t_2=2\pi/3$ lar uchun $z(t_1)=z(t_2)$ bo`ladi), ikkinchidan bu chiziq yopiq (chunki $z(0)=z(2\pi)$), uchinchidan bu chiziq bo`lakli silliq chiziqdir (chunki $t=\pi/2$, $t=3\pi/2$ nuqtalarda $z'(t)=0$).

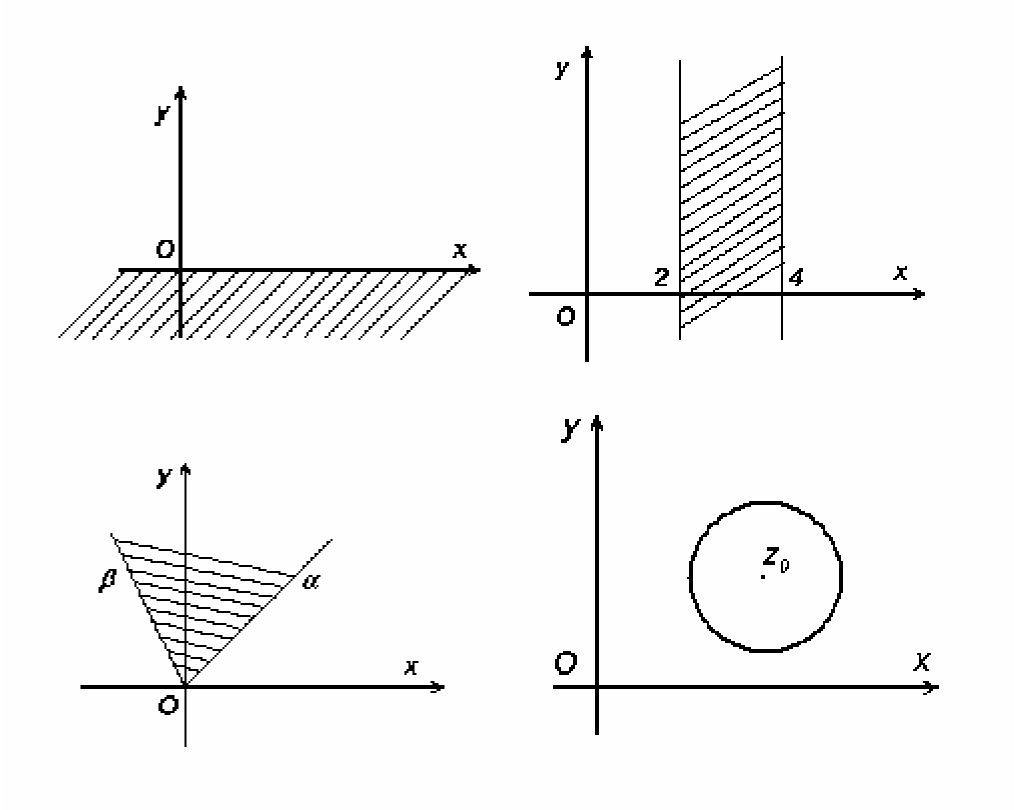
№15. Quyidagi

a) quyi yarimtekislikda yotgan nuqtalar to`plamini;

b) $x=2$ va $x=4$ to`g`ri chiziqlar orasida yotgan nuqtalar to`plamini;

c) uchlari koordinata boshida bo`lgan $\varphi=\alpha$ va $\varphi=\beta$ nurlar $(-\pi<\alpha<\beta<\pi)$ orasida yotgan nuqtalar to`plamini;

d) radiusi r va markazi (x_0,y_0) nuqtada bo`lgan doira nuqtalari to`plamini (1-rasm) aniqlovchi shartlarni kompleks shaklda yozing:



1-rasm

Yechish: a) Quyi yarimtekislikda yotgan nuqtalarning ordinatasi manfiy bo`ladi, ya'ni $y<0$. Demak, $Imz<0$ bo`ladi.

b) $x=Rez$ ekanligini e'tiborga olsak, $2<Rez<4$ bo`ladi.

c) $\alpha<argz<\beta$

d) $|z-z_0|<r$, $(|z-z_0|\leq r)$ $z_0=x_0+iy_0$

Odatta, $|z-a|<\varepsilon$ ochiq doira a nuqtaning ε atrofi deyiladi. R^2 tekislikdagi kabi, kompleks tekislikda ham to`plamning ichki nuqtasi, ochiq to`plam, limit nuqta, yopiq to`plam, bog`lamli to`plam tushunchalari aniqlanadi. Kompleks tekislikda soha tushunchasi muhim ahamiyatga ega. Soha deb ochiq va bog`lamli to`plamga aytiladi. Sohaning chegaraviy nuqtalari (sohaga tegishli bo`lmagan limit nuqtalari) to`plami bog`lamli to`plam bo`lsa, u holda bu soha bir bog`lamli to`plam deyiladi, aks holda ko`p bog`lamli soha deyiladi.

Tekislikda har xil figuralarni (xususan sohalarni) berish uchun ko`pincha tengsizliklardan foydalaniladi. Bunda figura kompleks koordinatalari biror

$$F(z)<0 \text{ yoki } F(z)\leq 0 \qquad (7)$$

bu erda $F(z)$ haqiqiy qiymatli kompleks o`zgaruvchining funksiyasi, tengsizlikni qanoatlantiruvchi barcha nuqtalar to`plami kabi beriladi. Ravshanki, bunday funksiyani ikkita haqiqiy o`zgaruvchining haqiqiy qiymatli funksiyasi ko`rinishda ifodalash, demak (7) tengsizlik bilan bog`liq masalani

$$F(x,y)<0 \text{ yoki } F(x,y)\leq 0$$

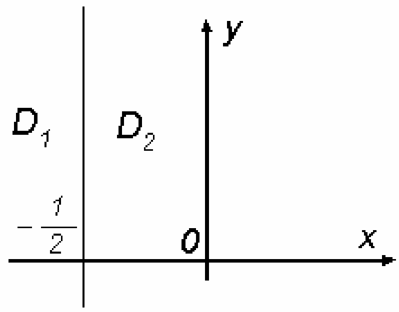
tengsizlikliklarni yechishga keltirish mumkin.

Quyida keltirilgan masalalarda odatta $F(x,y)$ funksiya butun tekislikda uzluksiz (chekli sondagi nuqtalardan boshqa) bo`ladi. Yuqoridagi tengsizliklarni yechishda avval $F(z)=0$ ($F(x,y)=0$) tenglamani qanoatlantiradigan $L$ nuqtalar to`plamini aniqlash maqsadga muvofiq. L nuqtalar to`plami butun tekislikni birnechta $D_1$, $D_2$, $D_3$, ..., $D_m$ sohalarga ajratadi. Har bir sohada $F(z)$ (ya'ni $F(x,y)$) funksiya ishorasini saqlaydi. (Bu funksiyaning sohada uzluksizligidan kelib chiqadi, u o`z ishorasini nolga aylanadigan nuqtadan o`tganda o`zgartiradi). Shu sababli har bir sohada tengsizlik ishorasini tekshirish uchun quyidagi tasdiqdan foydalanish mumkin: agar D_k sohaga tegishli bo`lgan aniq $(x_k,y_k)$ nuqtada $F(xk,yk)<0$ bo`lsa, u holda shu sohaning barcha nuqtalarida $F(x,y)<0$ bo`ladi.

Ba'zi hollarda geometrik tasavvurlardan foydalanish tengsizliklarni yechishni osonlashtiradi.

№16. $\left|\frac{z}{z+1}\right|<1$ tengsizlikni qanoatlantiruvchi nuqtalar to`plamini aniqlang.

Yechish. Avval $\left|\frac{z}{z+1}\right|=1$ tenglamani qanoatlantiruvchi L nuqtalar to`plamini aniqlaymiz. $\left|\frac{z}{z+1}\right|=\left|\frac{x+iy}{x+1+iy}\right|=\frac{|x+iy|}{|x+1+iy|}=\sqrt{\frac{x^2+y^2}{(x+1)^2+y^2}}=1$, bundan $x^2+y^2=x^2+2x+1+y^2 \Rightarrow 2x+1=0$. Demak, $L$ nuqtalar to`plami $x=-1/2$ to`g`ri chiziqdan iborat va bu to`g`ri chiziq tekislikni ikki D_1 va D_2 sohaga ajratadi (2-rasm). Tengsizlikning ishorasini aniqlash uchun D_2 sohaga tegishli bo`lgan, masalan $z=0$ nuqtani olamiz. $\left|\frac{0}{0+1}\right|=0>-1/2$ bo`lganligi sababli, tengsizlikning yechimi $x=-1/2$ to`g`ri chiziqdan chapda yotgan nuqtalar to`plami- $D_2$ sohadan iborat bo`ladi.



2-rasm

№17. $Ax+Bu+C=0$ to`g`ri chiziqning umumiy tenglamasini kompleks shaklda yozing.

№18. a) Koordinata o`qlari, b) $y=2x+1$ to`g`ri chiziqning tenglamalarini kompleks shaklda yozing.

№19. a) $x^2+y^2+4x+2y=0$ aylana, b) $x^2-y^2=a^2$ giperbola, c) $y=x^2$ parabola tenglamalarini kompleks shaklda yozing.

№20. Quyidagi tenglamalar bilan aniqlangan chiziqlarni ko`rsating:

a) $Re\left(\frac{1}{z}\right)=0{,}25$;   b) $Rez^2=4$; c) $z^2+\bar{z}^2=1$;

d) $2z\bar{z}+(2+i)z+(2-i)\bar{z}=2$; e) $|z-i|+|z+i|=4$

f) $\mathrm{Re}(1+z)=|z|$;   g) $\mathrm{Im}((1+i)z)=0$.

№21. Quyidagi tenglamalar bilan qanday chiziqlar berilganligini aniqlang va ularni kompleks tekislikda chizing:

a) $z=2i+e^{it}$, $(\pi\le t\le 3\pi)$;   b) $z=3t+it$, $(0\le t<\infty)$;

c) $z=t+it^2$, $(-\infty<t<+\infty)$;   d) $z=\frac{1}{t}+it$, $(0<t<\infty)$

№22. a) Birinchi chorakda yotgan nuqtalar to`plamini;

b) $u=1$ va $u=2$ tug`ri chiziqlar orasida yotgan nuqtalar to`plamini;

c) $u=x$ va $u=\sqrt{3}x$ to`g`ri chiziqlar orasida va uchinchi chorakka tegishli nuqtalar to`plamini; d) markazi (-2;3) nuqtada va radiusi 3 ga teng doiraga tegishli nuqtalar to`plamini aniqlovchi shartlarni kompleks shaklda yozing.

№23. Quyidagi tengsizliklar bilan berilgan nuqtalar to`plamini aniqlang.

a) $Imz<1$; b) $Rez>1$; c) $|z-i|\leq 1$; d) $1<|z-2|<3$;

e) $0<|z-1+i|<1$; f) $0<\arg z<\pi/3$; g) $\left|\frac{z-1}{z+1}\right|<2$.

3-§. Kompleks o`zgaruvchili funksiyalar, ularning limiti va uzluksizligi

Agar E to`plamdan olingan har bir $z=x+yi$ songa biror qonun yoki qoida bo`yicha aniq bir $w=u+vi$ kompleks son mos qo`yilgan bo`lsa, u holda E to`plamda funksiya berilgan deyiladi va $w=f(z)$ ko`rinishda yoziladi.

Har qanday kompleks o`zgaruvchi $z=x+iy$ ning funksiyasi $w=f(z)$ ni quyidagi ko`rinishda ifodalash mumkin:

$$w=f(x+iy)=u(x,y)+iv(x,y),$$

bu erda $u(x,y)$ va $v(x,y)$ funksiyalar ikki, x va y o`zgaruvchilarning haqiqiy qiymatli funksiyalari bo`lib, $u=u(x,y)$ funksiya $w=f(z)$ funksiyaning haqiqiy qismi, $v(x,y)$ esa - mavhum qismi deyiladi va mos ravishda $Ref(z)$ va $Imf(z)$ kabi belgilanadi.

№24. a) $w=z^2-3z+i$ funksiyaning haqiqiy va mavhum qismini toping.

Yechish: $u+iv=w=z^2-3z+i=(x+yi)^2-3(x+yi)+i=x^2-y^2-3x+i(2xy-3y+1)$, bundan $u=x^2-y^2-3x$, $v=2xy-3y+1$ hosil bo`ladi.

b) $w=\frac{1}{z}$ funksiyaning haqiqiy va mavhum qismini toping.

Yechish: $w=\frac{1}{z}=\frac{1}{x+yi}=\frac{x-yi}{x^2+y^2}=\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}$, Bundan $u=\frac{x}{x^2+y^2}$ va $v=-\frac{y}{x^2+y^2}$.

Geometrik nuqtai nazardan $w=f(z)$ funksiya z-tekislikdagi E to`plamni $w$-tekislikning biror $E'$ nuqtalar to`plamiga akslantirishni beradi. E' to`plam $E$ to`plamning $w=f(z)$ akslantirishdagi obrazi deyiladi.

№25. $|z|=1$ birlik aylananing $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ akslantirishdagi obrazini toping.

Yechish: $|z|=1$ birlik aylananing tenglamasini parametrik ko`rinishda yozib olamiz: $z=e^{it}$, $0\leq t\leq 2\pi$. U holda uning obrazi tenglamasi quyidagicha bo`ladi:

$$w=\frac{1}{2}\left(e^{it}+\frac{1}{e^{-it}}\right)=\frac{1}{2}\left(e^{it}+e^{it}\right)=e^{it},\ 0\leq t\leq 2\pi.$$

Demak, $|z|=1$ birlik aylananing $w=\frac{1}{2}\left(z+\frac{1}{z}\right)$ akslantirishdagi obrazi $|w|=1$ birlik aylanadan iborat bo`ladi.

Faraz qilaylik, $f(z)$ funksiya E tuplamda berilgan va z_0 nuqta bu to`plamning limit nuqtasi bo`lsin.

Agar $\forall\varepsilon>0$ son uchun shunday $\delta(\varepsilon)>0$ son topilib, $|z-z_0|<\delta$ tengsizlikni qanoatlantiradigan hamma z $(z\neq z_0)$ lar uchun $|f(z)-A|<\varepsilon$ tengsizlik bajarilsa, A kompleks son $f(z)$ funksiyaning $z\to z_0$ dagi limiti deyiladi va quyidagicha yoziladi:

$$\lim_{z\to z_0} f(z)=A.$$

$ReA=V$, $ImA=S$, $Ref(z)=u(x,y)$, $Imf(z)=v(x,y)$ bo`lsin. U holda $\lim_{z\to z_0} f(z)=A$ munosabat quyidagi ikkita

$$\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y)=B, \qquad \lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y)=C$$

munosabatlarga ekvivalent bo`ladi. Demak

$$\lim_{z\to z_0} f(z)=\lim_{\substack{x\to x_0\\ y\to y_0}} (u(x,y)+iv(x,y))=\lim_{\substack{x\to x_0\\ y\to y_0}} u(x,y)+i\lim_{\substack{x\to x_0\\ y\to y_0}} v(x,y) \qquad (8)$$

o`rinli bo`ladi.

№26. $f(z)=az+b$, bu erda a, b-kompleks sonlar, funksiyaning z_0 nuqtadagi limiti $A=az_0+b$ ga teng ekanligini isbotlang.

Yechish: Ixtiyoriy $\varepsilon>0$ son olamiz. $|f(z)-A|=|(az+b)-(az_0+b)|=|az-az_0|=$ $=|a||z-z_0|$ bo`lganligi sababli, $\delta(\varepsilon)>0$ son sifatida $\delta=\varepsilon/|a|$ sonni tanlaymiz. U holda $|z-z_0|<\delta$ shartni qanoatlantiruvchi z larda $|f(z)-A|<\varepsilon$ bo`ladi. Bu degani $A= az_0+b$ son $f(z)=az+b$ funksiyaning z_0 nuqtadagi limiti ekanligini bildiradi.

$w=f(z)$ funksiya E to`plamda berilgan va $z_0$ nuqta $E$ to`plamga tegishli bo`lgan limit nuqtasi bo`lsin.

Agar $\lim_{z\to z_0} f(z)=f(z_0)$ bo`lsa, $f(z)$ funksiya z_0 nuqtada uzluksiz deyiladi, boshqacha aytganda $\forall\varepsilon>0$ son uchun shunday $\delta>0$ son topilib, $|z-z_0|<\delta$ tengsizlikni qanoatlantiradigan barcha $z\in E$ lar uchun $|f(z)-f(z_0)|<\varepsilon$ tengsizlik bajarilsa, u holda $f(z)$ funksiya z_0 nuqtada uzluksiz deyiladi.

№27. $f(z)=az+b$, bu erda a, b-kompleks sonlar, funksiyaning kompleks tekislikning ixtiyoriy z_0 nuqtasida uzluksiz ekanligini isbotlang.

Yechish. z_0 - kompleks tekislikning ixtiyoriy nuqtasi bo`lsin. 26 -misolda $f(z)=az+b$ funksiyaning $z_0$ nuqtadagi limiti $A=az_0+b$ ga teng ekanligini ko`rsatilgan edi. Endi $f(z_0)= az_0+b$ ekanligini e'tiborga olsak, $f(z)=az+b$ funksiyaning z_0 nuqtada uzluksizligi kelib chiqadi.

Teorema. $f(z)=u(x,y)+iv(x,y)$ funksiyaning $z_0=x_0+iy_0$ nuqtada uzluksiz bo`lishi uchun uning haqiqiy $u(x,y)$ va mavhum $v(x,y)$ qismlarinig $(x_0,u_0)$ nuqtada uzluksiz bo`lishi zarur va etarli.

Agar $f(z)$ funksiya E to`plamning har bir nuqtasida uzluksiz bo`lsa, u holda $f(z)$ funksiya E to`plamda uzluksiz deyiladi.

№28. $f(z)=z^2$ funksiyaning kompleks tekislikda uzluksiz ekanligini ko`rsating.

Yechish. Faraz qilaylik $z_0=x_0+iy_0$ - kompleks tekislikning ixtiyoriy nuqtasi bo`lsin. Haqiqiy analizdan ma'lumki $u(x,y)=Rez^2=x^2-y^2$ va $v(x,y)=Imz^2=2xy$ funksiyalar (x_0,u_0) nuqtada uzluksiz, demak yuqoridagi teoremaga asosan $f(z)=z^2$ funksiya $z_0=x_0+iy_0$ nuqtada uzluksiz. Bu nuqta ixtiyoriy tanlanganligi sababli qaralayotgan funksiya kompleks tekislikda uzluksiz bo`ladi.

№29. $f(z)=argz$ funksiyaning $z_0=-1$ nuqtada uzluksiz emasligini ko`rsating.

Yechish. Avval $f(z)=argz$ funksiyaning kompleks tekislikning 0 nuqtasidan boshqa barcha nuqtalarida aniqlangan va uning qiymatlar tuplami $(-\pi,\pi]$ oraliqdan iborat. Funksiyaning –1 nuqtadagi qiymati $f(-1)=arg(-1)=\pi$ ga teng. $f(z)=argz$ funksiyaning $z_0=-1$ nuqtada uzluksiz emasligini ko`rsatish uchun shunday $\varepsilon>0$ son mavjud bo`lib, $z_0=-1$ nuqtaning ixtiyoriy atrofida $|f(z_1)-f(z_0)|\geq\varepsilon$ tengsizlikni qanoatlantiruvchi $z1$ nuqtaning borligini ko`rsatish etarli. $\varepsilon=\pi$ deb olamiz. Ixtiyoriy $\delta>0$ son uchun $z_1=-1-0,5\delta i$ nuqtani qaraymiz.

Ravshanki, $|z_1-z_0|=|-1-0,5\delta i-(-1)|=|0,5\delta i|=0,5\delta<\delta$, $f(z_1)$ uchun quyidagi qo`sh tengsizlik o`rinli: $-\pi<arg(-1-0,5\delta i)<-\pi/2$. Bundan $|f(z_1)-f(z_0)|>3\pi/2>\varepsilon$ ekanligi kelib chiqadi. Demak $argz$ funksiya –1 nuqtada uzilishga ega.

Izoh. $f(z)=argz$ funksiyaning boshqa manfiy qiymatlarida ham uzluksiz emasligini yuqoridagi kabi ko`rsatish mumkin. Uning 0 nuqtada uzluksiz emasligi ta’rifdan kelib chiqadi. Bu funksiya qolgan barcha nuqtalarda uzluksiz bo`ladi. Buni mustaqil isbotlashni tavsiya qilamiz.

№30. Quyida berilgan funksiyalarning haqiqiy va mavhum qismlarini ajrating:

a) $w=z^2+2i$; b) $w=\dfrac{i}{z+2}$; c) $w=z^2-\dfrac{1}{z}$;

d) $w=iz+|z|$; e) $w=e^{i\alpha}z+1$; $(\alpha\in R)$; e) $w=\dfrac{\bar{z}}{z}$.

№31. Berilgan nuqtaning ko`rsatilgan akslantirishdagi obrazini toping:

a) $z_0=1+i$, $w=z^2$; b) $z_0=2-i$, $w=(z+i)^2$;

c) $z_0=1$, $w=(z-i)-1$; d) $z_0=3+4i$, $w=z^2+7$.

№32. $|z|=1$ birlik aylananing birinchi chorakdagi qismining a) $w=z^2$; b) $w=z^3$; c) $w=z^4$ akslantirishdagi obrazini toping.

№33. Limitning ta’rifidan foydalanib, quyidagi tengliklarni isbotlang:

a) $\lim\limits_{z\to1+i}(2z-i)=2+i$; b) $\lim\limits_{z\to3-4i}|z|=5$.

№34. (8) dan foydalanib, agar $\lim\limits_{z\to z_0}f_1(z)=A_1$ va $\lim\limits_{z\to z_0}f_2(z)=A_2$ bo`lsa, u holda

a) $\lim\limits_{z\to z_0}(f_1(z)+f_2(z))=A_1+A_2$; b) $\lim\limits_{z\to z_0}(f_1(z)-f_2(z))=A_1-A_2$;

c) $\lim\limits_{z\to z_0}(f_1(z)\cdot f_2(z))=A_1\cdot A_2$; d) $\lim\limits_{z\to z_0}(f_1(z)/f_2(z))=A_1/A_2$, $A_2\neq0$

ekanligini isbotlang.

№35. Quyidagi funksiyalarning kompleks tekislikda uzluksiz ekanligini isbotlang:

a) $f(z)=Rez$; b) $f(z)=Imz$; c) $f(z)=|z|$; d) $f(z)=(1+3i)z+5$

4-§. Kompleks o`zgaruvchili funksiyaning hosilasi. Analitik funksiya tushunchasi

Kompleks tekislikdagi biror E sohada aniqlangan bir qiymatli $w=f(z)$ funksiya berilgan bo`lcin. $\forall z_0 \in E$ nuqta olib, unga shunday $\Delta z$ orttirma beraylikki, natijada $z= z_0+\Delta z$ ham $E$ to`plamga tegishli bo`lsin. U holda $w=f(z)$ funksiyaning orttirmasi $\Delta w=f(z_0+\Delta z)-f(z_0)$ bo`ladi.

Agar Δz ni har qanday yo`l (qonun) bilan nolga yaqinlashtirilganda ham $\frac{\Delta w}{\Delta z}$ nisbat faqat birgina aniq limitga intilsa, $w=f(z)$ funksiya z_0 nuqtada differensiallanuvchi, limit esa $f(z)$ funksiyaning z_0 nuqtadagi hosilasi deyiladi va $w'(z_0)$, $f'(z_0)$, $\frac{df(z_0)}{dz}$, $\frac{dw(z_0)}{dz}$ ko`rinishda belgilanadi. Demak, ta'rif bo`yicha

$$f'(z_0)=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}.$$

Teorema. Biror E sohada aniqlangan $f(z)=u(x,y)+iv(x,y)$ funksiyaning, shu sohaga tegishli $z_0=x_0+iy_0$ nuqtada differensiallanuvchi bo`lishi uchun $u(x,y)$ va $v(x,y)$ funksiyalarning $(x_0,y_0)$ nuqtada differensiallanuvchi bo`lishlari, shuningdek

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \qquad \text{(S-R)}$$

shartlarning bajarilishi zarur va etarlidir.

(C-R) shart Koshi-Riman shartlari deyiladi.

Agar $f(z)$ funksiya $z\in E$ nuqtada va uning biror atrofida differensiallanuvchi bo`lsa, u holda bu funksiya $z$ nuqtada analitik funksiya deyiladi. Agar $f(z)$ funksiya $E$ sohaning har bir nuqtasida differensiallanuvchi bo`lsa, u holda $f(z)$ funksiya E sohada analitik deyiladi. Har qanday analitik funksiya uchun quyidagi formula o`rinli:

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}. \qquad (9)$$

№36. $f(z)=z^2+1$ funksiyaning kompleks tekislikda analitik ekanligini ko`rsating.

Yechish. $z^2+1=(x+iy)^2+1=x^2-y^2+1+i2xy$ bo`lganligidan, $u(x,y)=x^2-y^2+1$, $v(x,y)=2xy$. Bu funksiyalar $x$ va $y$ haqiqiy o`zgaruvchining funksiyalari sifatida xOy tekislikning ixtiyoriy (x,y) nuqtasida differensiallanuvchi:

$$\frac{\partial u}{\partial x}=2x,\quad \frac{\partial u}{\partial y}=-2y,\quad \frac{\partial v}{\partial x}=2y,\quad \frac{\partial v}{\partial y}=2x.$$

va (C-R) shartlarini qanoatlantiradi. Demak, $f(z)=z^2+1$ funksiya butun kompleks tekislikda analitik bo`ladi va (9) formulaga asosan

$$f'(z)=(z^2+1)'=(x^2-y^2+1)x'+i(2xy)x'=2x+i2y=2(x+iy)=2z$$

bo`ladi.

№37. $w=z\bar{z}$ funksiya biror nuqtada analitik bo`ladimi?

Yechish: $z\bar{z}=x^2+y^2$ bo`lganligi sababli $u(x,y)=x^2+y^2$ va $v(x,y)\equiv 0$ o`rinli. Bu holda Koshi-Riman shartlari quyidagicha bo`ladi:

$$\begin{cases} 2x=0, \\ 2y=0 \end{cases}$$

va faqat (0,0) nuqtadagina bajariladi. Demak, $w=z\bar{z}$ funksiya faqat $z=0$ nuqtadagina differensiallanuvchi bo`lib, hech erda analitik emas. Funksiyaning $z=0$ nuqtadagi hosilasi 0 ga teng.

№38. $w=x^2+y^2+ixy^2$ ning analitik yoki analitik emasligi tekshirilsin.

Yechish: w funksiyaning haqiqiy va mavhum qismlari mos ravishda $u(x,y)=x^2+y^2$ va $v(x,y)=xy^2$ ga teng va

$$\frac{\partial u}{\partial x}=2x,\ \frac{\partial u}{\partial y}=2y,\ \frac{\partial v}{\partial x}=y^2,\ \frac{\partial v}{\partial y}=3xy^2$$

o`rinli. Bulardan ko`rinadiki, (S-R) shartlarining bajarilishi uchun $x=0$, $u=0$ bo`lishi kerak. Demak, $w$ funksiya (0,0) nuqtadagina differensiallanuvchi, boshqa nuqtalarda hosilasi yo`q, ya'ni berilgan funksiya analitik emas.

Agar E sohaning har bir nuqtasida

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}=0 \qquad (10)$$

tenglik bajarilsa, $F=F(x,y)$ funksiya E sohada garmonik funksiya deyiladi.

(10) tenglama Laplas tenglamasi deyiladi.

№39. $u(x,y)=x^2-y^2$ garmonik funksiya bo`ladimi?

Yechish: Berilgan funksiyaning Laplas tenglamasini qanoatlantirishini tekshiramiz. $\frac{\partial u}{\partial x}=2x,\ \frac{\partial^2 u}{\partial x^2}=2,\ \frac{\partial u}{\partial y}=-2y,\ \frac{\partial^2 u}{\partial y^2}=-2$ va $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=2+(-2)=0$.

Demak, $u(x,y)=x^2-y^2$ funksiya garmonik funksiya ekan.

Analitik funksiyaning haqiqiy va mavhum qismlari har doim garmonik funksiya bo`ladi.

Biror E sohada garmonik bo`lgan ikkita $u(x,y)$ va $v(x,y)$ funksiyalar yordamida tuzilgan $f(z)=u(x,y)+iv(x,y)$ funksiya har doim ham analitik funksiya bo`lavermaydi. Bu funksiya faqat $u(x,y)$ hamda $v(x,y)$ funksiyalar Koshi-Riman shartlarini qanoatlantirgandagina analitik funksiya bo`ladi. Bu holda $u(x,y)$ va $v(x,y)$ funksiyalar qo`shma garmonik funksiyalar deyiladi. E sohada garmonik bo`lgan har qanday funksiyaning qo`shma garmonik funksiyasi mavjud bo`ladi, boshqacha aytganda biror sohada garmonik bo`lgan ixtiyoriy $u(x,y)$ funksiya uchun haqiqiy (mavhum) qismi $u(x,y)$ ga teng bo`lgan analitik funksiyani tiklash mumkin.

Berilgan z_0 nuqtaning biror atrofida analitik bo`lgan $f(z)$ funksiyani uning haqiqiy qismi $u(x,y)$ ga ko`ra

$$f(z)=2u\left(\frac{z+\overline{z_0}}{2},\frac{z-\overline{z_0}}{2i}\right)-\overline{C_0}, \qquad (11)$$

mavhum qismi $v(x,y)$ ga ko`ra

$$f(z)=2iv\left(\frac{z+\overline{z_0}}{2},\frac{z-\overline{z_0}}{2i}\right)+\overline{C_0}, \qquad (12)$$

formulalar orqali tiklash mumkin, bu erda $\overline{C_0}$ son $S_0=f(z_0)$ ga qo`shma kompleks son.

№40. Haqiqiy qismi $Ref(z)=x^2+y^2$ bo`lgan analitik funksiya mavjudmi?

Yechish: Haqiqiy qismi $Ref(z)=x^2+y^2$ bo`lgan analitik funksiya mavjud bo`lishi uchun $Ref(z)=x^2+y^2$ funksiya garmonik funksiya bo`lishi shart. Lekin

x^2+y^2 funksiya garmonik funksiya emas. Demak, haqiqiy qismi $Ref(z)=x^2+y^2$ bo`lgan analitik funksiya mavjud emas.

№41. Mavhum qismi $v(x,y)=4x+2xy$ va $f(-i)=2$ bo`lgan $f(z)$ analitik funksiyani tiklang.

Yechish: 1-usul. (12) formuladan foydalanamiz. Bizning misolda $v(x,y)=4x+2xy$, $z_0=-i$, $C=2$, demak

$$f(z)=2i\left(4\frac{z+i}{2}+2\frac{z+i}{2}\cdot\frac{z-i}{2i}\right)+2=4iz-4+z^2+1+2=4iz+z^2-1.$$

2-usul. $v(x,y)$ va $u(x,y)$ funksiyalarning qo`shma garmonik funksiyalar ekanligidan foydalanib, izlanayotgan funksiyaning haqiqiy qismi $u(x,y)$ funksiyani topamiz. Koshi-Rimanning birinchi shartidan $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=2x$, bundan $u(x,y)=\int 2x\,dx+\varphi(y)=x^2+\varphi(y)$ ga ega bo`lamiz. Endi noma'lum $\varphi(y)$ funksiyani topish uchun KoshiRimanning ikkinchi $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ shartidan foydalanamiz.

$$\frac{\partial v}{\partial x}=(4x+2xu)x'=4+2y,\ \frac{\partial u}{\partial y}=(x2+\varphi(y))y'=\varphi'(y).$$

Yuqoridagi shartdan $\varphi'(y)=-4-2y$, bundan $\varphi(y)=\int(-4-2y)dy+S=-4y-y^2+C.$

Demak, $u(x,y)=x^2+\varphi(y)=x^2-4y-y^2+C$ va $f(z)=u(x,y)+iv(x,y)=x^2-4y-y^2+C+i(4x+2xy)=(x+iy)^2+i4(x+iy)+C=z^2+4iz+C.$

S o`zgarmasni $f(-i)=2$ shartdan foydalanib topamiz.
$2=(-i)^2+4i(-i)+C$, $C=2-3=-1$. Natijada, $f(z)=z^2+4iz-1$ funksiyaga ega bo`lamiz.

№42. Koshi-Riman shartlaridan foydalanib, quyidagi funksiyalardan qaysilari kamida bitta nuqtada analitik ekanligini aniqlang:

a) $w=z^2\bar{z}$; b) $w=\frac{1}{z}$; c) $w=\bar{z}$; d) $w=zRez$;

e) $w=cosx+isiny$; f) $w=x^2+iy^2$; g) $w=yx+i(x^2-y^2)$

№43. Quyidagi funksiyalar garmonik funksiya bo`ladimi?

a) $3xy+1$; b) x^2+y^3; c) $\frac{y}{x}$; d) $xy(x-y)$.

№44. a) $Imf(z)=x^2-2y^2$; b) $Ref(z)=excosy$; c) $Imf(z)=xy^2$ bo`lgan $f(z)$ analitik funksiya mavjudmi?

№45. Berilgan haqiqiy yoki mavhum qismiga ko`ra $f(z)=u(x,y)+iv(x,y)$ analitik funksiyani tiklang:

a) $v(x,y)=x^3-3xy^2$; b) $v(x,y)=3x^2y-y^3$, $f(0)=0$;

c) $u(x,y)=x^2-y^2+2x$, $f(i)=2i-1$; d) $u(x,y)=\frac{x^2-x+y^2}{x^2+y^2}$, $f(1)=0$.

4-§. Hosila moduli va argumentining geometrik ma'nosi. Konform akslantirish haqida tushuncha

Faraz qilaylik, $w=f(z)$ funksiya z_0 nuqtada analitik va $f'(z_0)\neq 0$ bo`lsin. U holda $|f'(z_0)|$ hosila moduli $w=f(z)$ akslantirishning $z_0$ nuqtadagi chiziqli cho`zilish koeffitsienti k ga, hosila argumenti $\arg f'(z_0)$ shu akslantirishda z_0 nuqtadan o`tuvchi silliq chiziqning bo`rilish burchagi φ ga teng bo`ladi.

№46. $w=z^2$ akslantirishning $z_0=1+i$ nuqtadagi cho`zilish koeffitsienti va burilish burchagini toping.

Yechish: $w=z^2$ funksiyaning $z_0=1+i$ nuqtadagi hosilasini topamiz: $w'(z)=2z$, $w'(1+i)=2+2i$. Bundan cho`zilish koeffitsientini topamiz $k=|w'(1+i)|=|2+2i|=2\sqrt{2}$. Burilish burchagi $\varphi=\arg w'(z_0)=\operatorname{arctg}1=\pi/4$ ga teng bo`ladi.

Agar $w=f(z)$ akslantirish

1) markazi z_0 nuqtada bo`lgan kichik aylanani aylanaga o`tkazish;

2) z_0 nuqtadan o`tuvchi har qanday ikkita chiziq orasidagi burchak miqdorini ham, yo`nalishini ham saqlash xossalariga ega bo`lsa, $w=f(z)$ akslantirish z_0 nuqtada konform akslantirish deyiladi.

Teorema. Agar $w=f(z)$ funksiya sohada analitik va undagi z_0 nuqtada $f'(z_0)\neq 0$ bo`lsa, u holda $w=f(z)$ funksiya yordamida bajariladigan akslantirish $z_0$ nuqtada konform bo`lib, $\arg f'(z_0)$ burilish burchagini, $|f'(z_0)|$ esa z_0 nuqtadagi chiziqli cho`zilish koeffitsientini bildiradi.

№47. $w=z^2+z$ akslantirishda z tekislikning qaysi qismi kengayib, qaysi qismi torayib akslanadi?

Yechish: $w=z^2+z$ funksiya butun kompleks tekislikda analitik funksiya ekanligini tekshirib ko`rish qiyin emas. $z$ tekislikning $|w'|>1$ ($|w'|<1$)tengsizlikni qanoatlantiruvchi nuqtalar to`plami akslantirish natijasida kengayib (torayib) akslanadi. $w'=2z+2$ va $k=|2z+2|$. Agar $|z+1|>0,5$ bo`lsa kengayib, $|z+1|<0,5$ bo`lsa torayib akslanadi. Demak, $|z+1|<0,5$ doiraning ichki qismi torayib, tashqi qismi kengayib akslanadi.

№48. $w=z^3$ akslantirishda quyidagi nuqtalarda cho`zilish koeffitsienti k va burilish burchagi φ ni toping:

a) $z=-1$; b) $z=1+i$; c) $z=4+3i$; d) $z=\sqrt{3}+i$.

№39. Quyidagi akslantirishlar natijasida tekislikning qaysi qismi kengayib, qaysi qismi torayib akslanadi?

a) $w=z^2-4z$; b) $w=\frac{1}{z}$; c) $w=iz^2$; d) $w=\frac{1+i}{z}$.

6-§. Asosiy elementar funksiyalar

6.1. Chiziqli va kasr chiziqli funksiyalar

Ushbu

$$w=az+b \qquad (13)$$

(bu erda a va b o`zgarmas kompleks sonlar, $a\neq 0$) ko`rinishdagi funksiya chiziqli funksiya (akslantirish) deyiladi.

$a=me^{i\alpha}$ deb ifodalasak (bu erda $m=|a|$, $\alpha=\arg a$), $w=me^{i\alpha}z+b$ bo`ladi va bu akslantirishni $w=t_2+b$, $t_2=mt_1$, $t_1=e^{i\alpha}z$ ko`rinishdagi uchta sodda akslantirishni ketma-ket bajarishga keltirish mumkin: bunda t_1 - koordinatalar boshi atrofida α burchakka burish; t_2 – markazi koordinatalar boshida va koeffitsienti m ga teng bo`lgan gomotetiya; $w=t_3$ - $\vec{b}$ vektor qadar parallel ko`chirish.

(13) chiziqli funksiyani ($a\neq 1$ bo`lganda) quyidagi ko`rinishda ham yozish mumkin:

$$w-c=a(z-c), \qquad (14)$$

shu sababli chiziqli funksiya vositasida bajariladigan akslantirishni $a=1$ bo`lganda faqat parallel ko`chirishga, $a\neq 1$ bo`lganda $s$ nuqta atrofida burish, so`ngra markazi s nuqtada bo`lgan gomotetiyaga keltirish mumkin.

Izoh. (14) tenglikdagi c songa mos keluvchi nuqta w akslantirishning qo`zg`almas nuqtasi bo`ladi, ya'ni $f(z)=z$ tenglamaning yechimi bo`ladi.

Chiziqli funksiya butun kompleks tekislikda analitik bo`ladi va $a\neq 0$ bo`lganligi sababli, uning vositasida bajariladigan akslantirish barcha nuqtalarda konform bo`ladi.

№50. Uchlari $A=2+3i$, $B=5-i$, $C=6+2i$ nuqtalarda bo`lgan ABC uchburchakning $w=(1+i)z-2$ akslantirishdagi aksini toping.

Yechish: Berilgan $w=(1+i)z-2$ chiziqli funksiya ABC uchburchakni $A_1B_1C_1$ akslantiradi, bunda $A_1=w(A)$, $B_1=w(B)$, $C_1=w(C)$. Bu nuqtalarni topamiz:

$A_1=w(A)=w(2+3i)=(1+i)(2+3i)-2=-3+5i;$

$B_1=w(B)=w(5-i)=(1+i)(5-i)-2=4+4i;$

$C_1=w(C)=w(6+2i)=(1+i)(6+2i)-2=2+8i.$

Demak, ABC uchburchak w akslantirish natijasida uchlari $A_1=-3+5i$, $B_1=4+4i$ va $C_1=2+8i$ nuqtalarda bo`lgan $A_1B_1C_1$ uchburchakga akslanadi.

№51. Berilgan $|z-1+i|<2$ doiraning $w=(4-3i)z+2+3i$ akslantirishdagi aksini toping.

Yechish: Avval $w=(4-3i)z+2+3i$ tenglikdan z ni topamiz: $z=\frac{w-2-3i}{4-3i}$. z ning ifodasini doirani aniqlovchi tengsizlikka ko`yamiz:

$\left|\frac{w-2-3i}{4-3i}-1+i\right|<2 \iff |w-2-3i+(-1+i)(4-3i)|<2\cdot|4-3i| \iff |w-3+4i|<10$, demak berilgan doiraning $w=(4-3i)z+2+3i$ akslantirishdagi aksi markazi $3-4i$ nuqtada radiusi 10 ga teng doiradan iborat ekan.

№52. $w=(1+i)z-2+3i$ akslantirishni sodda akslantirishlarning superpozitsiyasi ko`rinishda tavsiflang. Uning qo`zg`almas nuqtasini toping.

Yechish: $a=1+i$ ni ko`rsatkichli ko`rinishda yozib olamiz: $1+i=|1+i|\cdot e^{arg(1+i)}=\sqrt{2}\, e^{\frac{\pi}{4}}$. Demak berilgan akslantirishni koordinatalar boshi atrofida $\pi/4$ burchakka burish, markazi koordnatalar boshida koeffitsienti $\sqrt{2}$ ga teng gomotetiya va $\vec{b}=-2+3i$ vektorga parallel ko`chirishni ketma-ket qo`llash orqali bajarish mumkin.

Akslantirishning qo`zg`almas nuqtasini topish uchun $z=(1+i)z-2+3i$ tenglamani echamiz. Bundan $z=-3-2i$. Berilgan akslantirishni (14) ko`rinishda ifodalaymiz:

$$w+2+3i=(1+i)(z+3+2i) \text{ yoki } w+2+3i=\sqrt{2}\, e^{\frac{\pi}{4}}(z+3+2i).$$

Bu tenglikdan esa w akslantirishni $-3-2i$ nuqta atrofida $\pi/4$ burchakka burish va markazi $-3-2i$ nuqtada koeffitsienti $\sqrt{2}$ ga teng bo`lgan gomotetiyalar superpozitsiyasi kabi ifodalash mumkinligi kelib chiqadi.

№53. Ushbu $z_0=2-3i$ nuqtani qo`zg`almas qoldirib, $z_1=1$ nuqtani $w_1=i$ nuqtaga akslantiradigan chiziqli funksiyani toping.

Yechish: Chiziqli akslantirishni $w=az+b$ ko`rinishda izlaymiz. Berilgan shartlardan quyidagi tenglamalar sistemasiga ega bo`lamiz:

$$\begin{cases} a(2-3i)+b=2-3i, \\ a\cdot 1+b=i \end{cases}$$

Bu sistemani yechib, quyidagilarga ega bo`lamiz:
$a=1,4+0,2i;$ $b=-1,4+0,8i.$ Demak izlanayotgan akslantirish (chiziqli funksiya) $w=(1,4+0,2i)z+(-1,4+0,8i)$ dan iborat.

№54. Uchlari $A=2i,$ $B=2+3i,$ $C=6$ va $D=1$ nuqtalarda bo`lgan $ABCD$ to`rtburchakning $w=iz+3-2i$ akslantirishdagi aksini toping.

№55. Berilgan $|z+2-i|<1$ doiraning $w=3iz+1-3i$ akslantirishdagi aksini toping.

№56. Berilgan $\{Rez<2\}$ sohani $w=2iz+1$ akslantirishdagi aksini toping.

№57. $w=3+4i-iz$ akslantirishni sodda akslantirishlarning superpozitsiyasi ko`rinishda tavsiflang. Uning qo`zg`almas nuqtasini toping.

6.2. Kasr chiziqli funksiya

$w=\dfrac{az+b}{cz+d}$ (bu erda $a,$ $b,$ c va d o`zgarmas kompleks sonlar, $bc-ad\neq 0$) ko`rinishdagi funksiya kasr-chiziqli funksiya deyiladi. $w=\dfrac{az+b}{cz+d}=\dfrac{a}{c}+\dfrac{bc-ad}{c}\cdot\dfrac{1}{cz+d}$ bo`lganligi sababli, kasr chiziqli funksiya vositasida bajariladigan akslantirishni unga ekvivalent bo`lgan $w=cz+d,$ $\eta=\dfrac{1}{w},$ $w=a_1\eta+b_1$ funksiyalar vositasida bajariladigan akslantirishlar superpozitsiyasiga keltirish mumkin, bunda: $a_1=\dfrac{bc-ad}{c}$, $b_1=\dfrac{a}{c}$.

Kasr chiziqli $w=\dfrac{az+b}{cz+d}$, $ad-bc\neq 0$ funksiya quyidagi xossalarga ega:

1. Kengaytirilgan (z) tekislikni kengaytirilgan (w) tekislikga o`zaro bir qiymatli va konform akslantiradi ($z=-\dfrac{d}{c}$ bo`lganda $w=\infty$, va $w=\dfrac{a}{c}$ bo`lganda $z=\infty$ bo`ladi).

2. Unga teskari bo`lgan $z=\dfrac{dw-b}{-cw+a}$ funksiya kasr chiziqli funksiya bo`ladi.

3. z tekislikdagi har xil uchta $z_1,$ $z_2,$ z_3 nuqtalarni w tekislikdagi berilgan har xil uchta $w_1,$ $w_2,$ w_3 nuqtalarga akslantiruvchi kasr-chiziqli funksiya quyidagi formuladan topiladi:

$$\frac{w-w_1}{w-w_2}\cdot\frac{w_3-w_2}{w_3-w_1}=\frac{z-z_1}{z-z_2}\cdot\frac{z_3-z_2}{z_3-z_1} \qquad (15)$$

Bu formulada $z_k,$ w_k ($k=1,2,3$) sonlardan biri ∞ bo`lgan holda ∞ qatnashgan ayirmani 1 bilan almashtirish kerak.

4. Kasr-chiziqli akslantirish aylana yoki to`g`ri chiziqni aylana yoki to`g`ri chiziqqa akslantiradi.

5. Kasr-chiziqli akslantirish z tekislikdagi γ aylanaga yoki to`g`ri chiziqqa nisbatan simmetrik bo`lgan $z_1$ va $z_2$ nuqtalar juftini $w$ tekislikda $w(\gamma)$ aylanaga yoki to`g`ri chiziqqa nisbatan simmetrik bo`lgan $w(z_1)$ va $w(z_2)$ nuqtalarga o`tkazadi.

6. Agar kasr-chiziqli akslantirish natijasida γ aylana γ' aylanaga akslansa, u holda γ aylana bilan chegaralangan D soha γ' aylana bilan chegaralangan ikki sohaning biriga o`tadi. Bunda biror yo`nalishda γ chiziq bo`ylab harakatlanganda $D$ soha chapda (o`ngda) joylashgan bo`lsa, bu harakat yo`nalishiga mos γ' bo`ylab harakatlanganda $D$ ga mos $D'$ soha chapda (o`ngda) joylashgan bo`lishi kerak.

№58. z tekislikdagi 0, 1, ∞ nuqtalarni mos ravishda w tekislikdagi –1, 0, 1 nuqtalarga o`tkazuvchi kasr-chiziqli funksiyani toping.

Yechish: $z_1=0,\ z_2=1,\ z_3=\infty$ va $w_1=-1,\ w_2=0,\ w_3=1$ belgilashlar kiritib, (15) formuladan foydalanamiz.

$$\frac{w-(-1)}{w-0}\cdot\frac{1-0}{1-(-1)}=\frac{z-0}{z-1}\cdot\frac{1}{1} \Leftrightarrow \frac{w+1}{w}=\frac{2z}{z-1} \Leftrightarrow 1+\frac{1}{w}=\frac{2z}{z-1} \Leftrightarrow \frac{1}{w}=\frac{z+1}{z-1} \Leftrightarrow$$

$$\Leftrightarrow w=\frac{z-1}{z+1}.$$

№59. $|z-(1-i)|\le\sqrt{2}$ doirani quyi yarim tekislikka akslantiruvchi biror kasr-chiziqli funksiyani toping.

Yechish: Bu doira chegarasida uchta nuqta tanlab olamiz, masalan $z_1=0$, $z_2=2,\ z_3=2i.$ Bu nuqtalar ketma-ketligi doira chegarasi aylana bo`ylab harakatlanganda doira o`ng tomonda qoladigan (joylashadigan) harakat yo`nalishini aniqlaydi.

w tekislikda quyi yarim tekislikning chegarasi bo`lgan haqiqiy o`qda $w_1,\ w_2,\ w_3$ nuqtalarni shunday tanlaymizki, mos harakat yo`nalishida quyi yarim tekislik o`ngda joylashsin. Bunday nuqtalar sifatida $w_1=0$, $w_2=1$, $w_3=\infty$ nuqtalarni olish mumkin. Hosil qilingan uchta nuqtalar jufti orqali (15) formuladan foydalanib kasr-chiziqli funksiyani tuzamiz:

$$\frac{z-0}{z-2}\cdot\frac{-2i-2}{-2i-0}=\frac{w-0}{w-1}\cdot\frac{1}{1} \Leftrightarrow \text{w}=\frac{(1+i)z}{z+2i}.$$

Izoh. $z_1,\ z_2,\ z_3$ va $w_1,\ w_2,\ w_3$ nuqtalar ixtiyoriy tanlanganligi sababli, masala yechimi yagona emas, ya'ni bunday akslantirishni bajaradigan kasr-chiziqli funksiyalar cheksiz ko`p.

№60. z tekislikdagi 1, i, ∞ nuqtalarni mos ravishda w tekislikdagi –1, 0, 1 nuqtalarga o`tkazuvchi kasr-chiziqli funksiyani toping.

№61. z tekislikdagi 1, i, -1 nuqtalarni mos ravishda w tekislikdagi i, -1, - i nuqtalarga o`tkazuvchi kasr-chiziqli funksiyani toping.

№62. z tekislikdagi birlik doiraning $w=\frac{2z-1}{z-2}$ akslantirishdagi aksini toping. Bu doiraning markazi qaysi nuqtaga akslanadi?

№63. $Rez<1$ sohaning $w=\frac{z}{z-2}$ akslantirishdagi aksini toping.

№64. $1<|z|<2$ sohaning $w=\frac{2}{z-1}$ akslantirishdagi aksini toping.

6.3. Ko`rsatkichli funksiya

Ixtiyoriy $z=x+iy$ kompleks son uchun e^z ko`rsatkichli funksiya ushbu

$$w=e^z=e^{x+iy}=e^x(\cos y+i\sin y)$$

munosabat bilan aniqlanadi.

$w=e^z$ funksiya quyidagi xossalarga ega:

1) z haqiqiy bo`lganda e^z funksiya e^x ning barcha xossalariga ega;
2) bu funksiya butun tekislikda analitik va $(e^z)'=e^z$ tenglik o`rinli;
3) Ixtiyoriy z uchun $(e^z)'\neq 0$ bo`lib, $w=e^z$ funksiya yordamida bajariladigan akslantirish $z$ tekislikning har bir nuqtasida konform bo`ladi.
4) Ko`rsatkichli funksiyaning asosiy xossasi- qo`shish teoremasi saqlanadi:

$$e^{z_1}\cdot e^{z_2}=e^{z_1+z_2} \qquad (16)$$

5) Funksiya davriy bo`lib, uning asosiy davri sof mavhum son $2\pi i$ ga teng:

$e^{z+2\pi i}=e^z$

№65. $w=e^z$ funksiyaning $z=1+\pi i$ nuqtadagi qiymatini hisoblang.

Yechish: (16) formuladan foydalanamiz. $e^{1+\pi i}=e^1(\cos\pi+i\sin\pi)=-e$.

№66. $w=e^z$ funksiyaning a) $z=1-\pi i$; b) $z=\pi i$; c) $z=-\pi i/2$ nuqtalardagi qiymatlarini hisoblang.

№67. $w=e^z$ funksiya hosilasining ko`rsatilgan nuqtalardagi moduli va argumentini toping: a) $z=2+i$; b) $z=3-4i$; c) $z=3-2i$; d) $z=4+3i$.

6.4. Trigonometrik funksiyalar

Ixtiyoriy kompleks son z uchun trigonometrik funksiyalar quyidagi

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i}, \quad \cos z=\frac{e^{iz}+e^{-iz}}{2}$$

tengliklar bilan aniqlanadi. tgz va $ctgz$ funksiyalar ushbu

$$tgz=\frac{\sin z}{\cos z}=\frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}, \quad ctgz=\frac{\cos z}{\sin z}=\frac{i(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}}$$

formulalar bilan aniqlanadi.

$\sin z$ va $\cos z$ funksiyalar kompleks tekislikda analitik funksiyadir. tgz va $ctgz$ funksiyalar mos ravishda $\cos z$ va $\sin z$ nolga aylanmaydigan barcha nuqtalarda analitik funksiya bo`ladi.

Haqiqiy o`zgaruvchili trigonometrik funksiyalar orasidagi munosabatlarni ifodalovchi barcha formulalar kompleks sohada ham o`rinli bo`ladi.

Kompleks tekislikda giperbolik funksiyalar quyidagicha aniqlanadi:

$$chz=\frac{1}{2}(e^z+e^{-z}); \quad shz=\frac{1}{2}(e^z-e^{-z}); \quad thz=\frac{shz}{chz}$$

Trigonometrik va giperbolik funksiyalar orasida quyidagi munosabatlar o`rinli: $\cos iz=chz$; $\sin iz=ishz$.

№68. Quyidagi ayniyatni isbotlang:

$$\sin z_1\cos z_2+\cos z_1\sin z_2=\sin(z_1+z_2) \qquad (17)$$

Yechish: $\sin z_1 \cos z_2 + \cos z_1 \sin z_2 = \frac{e^{iz_1} - e^{-iz_1}}{2i} \cdot \frac{e^{iz_2} + e^{-iz_2}}{2} + \frac{e^{iz_1} + e^{-iz_1}}{2} \cdot \frac{e^{iz_2} - e^{-iz_2}}{2i} = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} = \sin(z_1+z_2)$.

№69. *sinz* funksiyaning haqiqiy va mavhum qismlarini toping.

Yechish: $z=x+iy=z_1+z_2$, bu erda $z_1=x$, $z_2=iy$ deb, (17) formula va giperbolik funksiyalar bilan trigonometrik funksiyalar orasidagi munosabatdan foydalanamiz. U holda $\sin(x+iy)=\sin x\cos iy+\cos x\sin iy= =\sin x\, ch y - i\cos y\, sh x$. Bundan $Re\sin z=\sin x\, ch y$; $Im\sin z=-sh y\cos x$.

№70. *sinz*=0 tenglamani yeching.

Yechish: *sinz* funksiyaning ta'rifidan $e^{iz}-e^{-iz}=0$ yoki $e^{2iz}=1$. $z=x+iy$ desak, $e^{2i(x+iy)}=1$ yoki $e^{-2y}\, e^{2ix}=1$. Bundan

$$e^{-2y}=1,\ 2x=2k\pi,\ k\in Z$$

tengliklarni hosil qilamiz. Demak, $y=0$, $x=k\pi$. Shunday qilib, *sinz* funksiya $z=k\pi$, $k\in Z$ nuqtalarda 0 ga aylanadi.

Kompleks sohada har qanday *z* uchun |*sinz*| va |*cosz*| birdan katta bo`lmaydi deb tasdiqlash mumkin emas.

Haqiqatan ham, $z=i$ bo`lganda $\sin i=\frac{e^{-1}-e^{1}}{2i}\approx 1{,}17i$, $\cos i=\frac{e+e^{-1}}{2}\approx 1{,}54$, ya'ni: $|\sin i|>1$ va $|\cos i|>1$.

№71. *cosz* funksiyaning haqiqiy va mavhum qismlarini toping.

№72. Quyidagi sonlarning haqiqiy va mavhum qismlarini toping:

a) $\cos\frac{i\pi}{2}$; b) $tg(1-i)$; c) $\sin(2-3i)$; d) $ch(1+i)$. Bu sonlarning moduli va argumentini toping.

№73. Quyidagi ayniyatlarni isbotlang:

a) $\cos z_1\cos z_2 - \sin z_1\sin z_2=\cos(z_1+z_2)$; b) $\sin(z+\frac{\pi}{2})=\cos z$;

c) $\sin(z+\pi)=-\sin z$; d) $\cos(z+\frac{\pi}{2})=-\sin z$; e) $\sin 2z+\cos 2z=1$

№74. $\cos z=0$ tenglamani yeching.

№75. $w=e^{z}$ funksiya kompleks tekislikning qaysi nuqtalarida haqiqiy (mavhum) qiymatlar qabul qiladi?

№76. $w=\cos z$ funksiya kompleks tekislikning qaysi nuqtalarida haqiqiy (mavhum) qiymatlar qabul qiladi?

№77. $w=\sin z$ funksiya kompleks tekislikning qaysi nuqtalarida haqiqiy (mavhum) qiymatlar qabul qiladi?

6.5. Logarifmik funksiya

Berilgan $e^{w}=z$ tenglamani qanoatlantiruvchi har qanday *w* son *z* sonning *e* asosli (natural) logarifmi deyilib, quyidagicha yoziladi:

$$w=Lnz \qquad (18)$$

Ma'lumki, $e^w \neq 0$. Shu sababli (18) dagi $z=x+iy\neq 0$, ya'ni nolning logarifmi mavjud emas. $z\neq 0$ sonning logarifmi quyidagi formula bilan topiladi:

$$Lnz=lnr+i(\varphi+2k\pi) \qquad (19)$$

Bu erda $r=|z|$, $\varphi=argz$. Shuning uchun ba'zan (19) tenglik

$$Lnz=ln|z|+iArgz$$

ko`rinishda ham yoziladi: $k=0$ bo`lgan holni, ya'ni $lnr+i\varphi$ ni logarifning bosh qiymati deyiladi va lnz orqali belgilanadi:

$$lnz=ln|z|+i\varphi$$

Buni e'tiborga olib, (19) formulani quyidagicha ham yozish mumkin:

$$Lnz=lnz+2k\pi i.$$

Bundan logarifmik funksiyaning ko`p qiymati funksiya ekanligi ko`rinib turibdi.

Elementar matematikada isbot qilingan logarifmlarga tegishli teoremalar shaklan bu joyda ham o`z kuchida qoladi. Ular quyidagilardan iborat:

1. $Ln(z_1z_2)=Lnz_1+Lnz_2$

2. $Ln\frac{z_1}{z_2}=Lnz_1-Lnz_2$

№78. Lni ni algebraik ko`rinishda yozing.

Ravshanki, $i=cos\frac{\pi}{2}+isin\frac{\pi}{2}$, ya'ni $r=|i|=1$, $\varphi=argi=\frac{\pi}{2}$.

$Lni=ln1+i(\frac{\pi}{2}+2k\pi)=(2k+\frac{1}{2})\pi i,\ k\in Z.$

Umumiy darajali funksiya z^a, bu erda $a=\alpha+i\beta$ ixtiyoriy kompleks son, quyidagicha aniqlanadi:

$$z^a=e^{aLnz} \qquad (20)$$

Bu ko`p qiymatli funksiyaning bosh qismi $z^a=e^{alnz}$ dan iborat.

№79. $(1+i)^{-i}$ ni hisoblang.

Yechish: (20) ga asosan $(1+i)^{-i}=e^{-iLn(1+i)}=e^{-i(ln|1+i|+iArg(1+i))}=$

$=e^{-i(\ln\sqrt{2}+i(\frac{\pi}{4}+2k\pi))}=e^{\frac{\pi}{4}+2k\pi}\cdot e^{-i\ln\sqrt{2}}$. Xususan, $k=0$ bo`lganda

$(1+i)\text{-}i=e^{\frac{\pi}{4}}(\cos\ln\sqrt{2}+i\sin\ln\sqrt{2})$ bo`ladi.

Umumiy ko`rsatkichli funksiya $a^z$, bu erda $a\neq 0$ bo`lgan ixtiyoriy kompleks son, quyidagicha aniqlanadi:

$$a^z=e^{zLna}.$$

Bu ko`p qiymatli funksiyaning bosh qismi $a^z=e^{zlna}$ dan iborat.

№80. Quyidagi sonlarning logarifmlarini hisoblang:

a) -5; b) $-1+i\sqrt{3}$; c) $1-i\sqrt{3}$; d) $-3-i\sqrt{3}$; e) $-i$; f) $-1-i\sqrt{3}$;

№81. Ixtiyoriy musbat haqiqiy x son uchun Lnx ning faqat birgina qiymati haqiqiy, ixtiyoriy manfiy yoki mavhum z son logarifmi mavhum ekanligini isbotlang.

№82. Quyidagi darajalarni hisoblang:

a) i^i; b) $(-1)^i$; c) $(3-4i)^{-i}$; d) $(1+i)^{1-i}$.

7-§. Integral va uning xossalari. Koshi teoremasi. Koshining integral formulasi

Kompleks (z) tekislikdagi biror E sohada uzluksiz va bir qiymatli

$$w=f(z)=u(x,y)+iv(x,y)$$

funksiyaning E sohada yotgan ixtiyoriy G silliq yoki bo`lakli silliq chiziq bo`yicha integrali quyidagi formula bilan hisoblanadi:

$$\int_{\Gamma} f(z)dz = \int_{\Gamma} u(x,y)dx - v(x,y)dy + i\int_{\Gamma} v(x,y)dx + u(x,y)dy \qquad (21)$$

Bu formulani qisqaroq qilib,

$$\int_{\Gamma} f(z)dz = \int_{\Gamma} (u+iv)(dx+idy)$$

ko`rinishda yozish mumkin. (21) ning o`ng tomoni haqiqiy argumentli funksiyalardan olingan egri chiziqli integrallardan iborat.

Agar G chiziq $z=z(t)$ ($\alpha \le t \le \beta$) tenglama bilan berilsa, u holda $f(z)$ funksiyaning G chiziq bo`yicha (parametrning o`sish yo`nalishida) olingan integrali quyidagi formula yordamida hisoblanadi:

$$\int_{\Gamma} f(z)dz = \int_{\alpha}^{\beta} f(z(t))z'(t)dt \qquad (22)$$

Kompleks o`zgaruvchining integrali quyidagi xossalarga ega:

1. O`zgarmas ko`paytiruvchini integral belgisi tashqarisiga chiqarish mumkin:

$$\int_{\Gamma} af(z)dz = a\int_{\Gamma} f(z)dz .$$

2. Integrallash chizig`ining yo`nalishi qarama-qarshisiga o`zgartirilsa integral belgisi oldidagi ishora ham o`zgaradi:

$$\int_{\Gamma^-} f(z)dz = -\int_{\Gamma} f(z)dz.$$

3. Chekli sondagi funksiyalar yig`indisidan olingan integral uning har bir hadidan olingan integrallar yig`indisiga teng:

$$\int_{\Gamma} (f_1(z) + f_2(z)+\ldots+f_m(z))dz = \int_{\Gamma} f_1(z)dz + \int f_2(z)dz+\ldots+\int_{\Gamma} f_m(z)dz.$$

4. Agar uzunligi L bo`lgan $G$ chiziqning hamma nuqtalarida $M$>0 son uchun $|f(z)|\le M$ bo`lsa, u holda

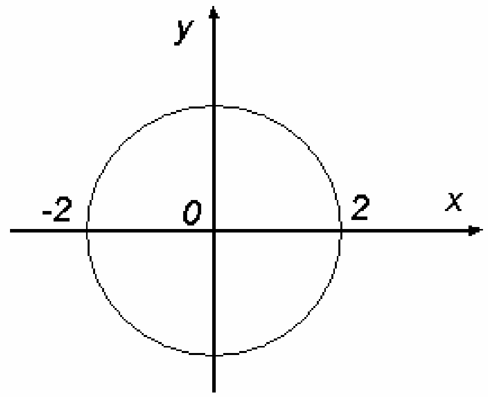
$$\left|\int_{\Gamma} f(z)dz\right| < LM$$

bo`ladi.

5. $$\int_{\Gamma_1+\Gamma_2+\ldots+\Gamma_m} f(z)dz = \int_{\Gamma_1} f(z)dz+\ldots+\int_{\Gamma_m} f(z)dz$$

bunda $\Gamma_1+\Gamma_2+\ldots+\Gamma_m$ egri chiziq Γ_k yoylardan tuzilgan bo`lib, $\Gamma_k$ ning oxirgi nuqtasi $\Gamma_{k+1}$ ning boshlang`ich nuqtasi bilan ustma-ust tushgan, ya'ni ikkalasi bir nuqta.

№83. $\int\limits_{\Gamma}|z|dz$ integralni hisoblang, bunda *G* chiziq
a) –2 va 2 nuqtalarni tutushtiruvchi kesma;
b) $|z|=2$ aylananing yuqori qismi, bu erda integrallash yo`lining boshlang`ich nuqtasi $z=2$ nuqtadan iborat.



Yechish: a) –2 va 2 nuqtalarni tutushtiruvchi kesma tenglamasini parametrik ko`rinishda quyidagicha ifodalash mumkin:

$z=t,\ -2\le t\le 2$. U holda (22) ga asosan

$$\int\limits_{\Gamma}|z|\,dz=\int\limits_{-2}^{2}|t|\,dt=\int\limits_{-2}^{0}(-t)dt+\int\limits_{0}^{2}tdt=-\frac{t^2}{2}\Big|_{-2}^{0}+\frac{t^2}{2}\Big|_{0}^{2}=-2+2=0.$$

b) $|z|=2$ aylananing yuqori qismining parametrik tenglamasini $z=2e^{it},\ 0\le t\le\pi$ ko`rinishda yozib olamiz. U holda $dz=2ie^{it}dt$ va

$$\int\limits_{\Gamma}|z|\,dz=2\int\limits_{0}^{\pi}|2e^{it}|\,ie^{it}dt=4\int\limits_{0}^{\pi}|e^{it}|\,e^{it}d(it)=4\,e^{it}\Big|_{0}^{\pi}=4(e^{i\pi}-e^{0})=4(-1-1)=-8.$$

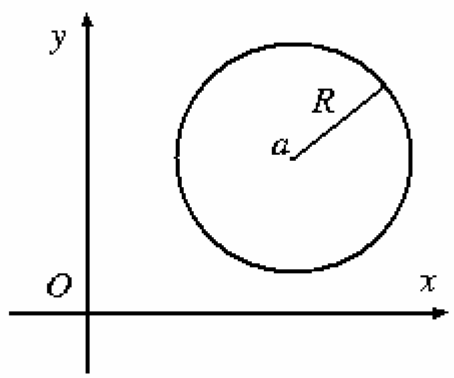
Bu misoldan ko`rinadiki, integral integrallash yo`liga bog`liq ekan.

№84. $\int\limits_{\Gamma}\frac{dz}{z}$ integralni hisoblang, bu erda Γ- $|z|=R$ aylanadan iborat. (soat mili yo`nalishiga qarama-qarshi yo`nalishda).

Yechish: $|z|=R$ aylananing parametrik tenglamasini quyidagicha $z=Reit$, $0\le t\le 2\pi$ yozib olamiz. U holda $dz=Rie^{it}dt$ va (22) formulaga asosan

$$\int\limits_{\Gamma}\frac{dz}{z}=\int\limits_{0}^{2\pi}\frac{i\,\mathrm{Re}^{it}}{\mathrm{Re}^{it}}dt=i\int\limits_{0}^{2\pi}dt=2\pi i.$$

№85. $\int\limits_{\Gamma}\frac{dz}{(z-a)^m}$, *m*-butun son, integralni hisoblang, bu erda Γ- $|z-a|=R$ aylanadan iborat. (soat mili yo`nalishiga qarama-qarshi yo`nalishda).



Yechish: $|z-a|=R$ aylananing parametrik tenglamasi $z-a=Re^{it}$, $0\le t\le 2\pi$, va $dz=Rie^{it}dt$ va (22) formulaga asosan

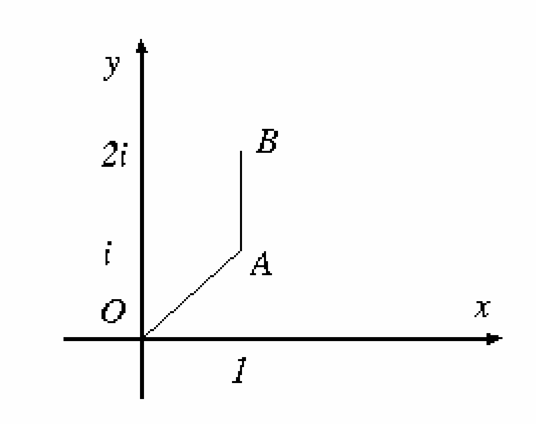
$$\int\limits_{\Gamma}\frac{dz}{(z-a)^m}=\int\limits_{0}^{2\pi}\frac{Rie^{it}}{R^m e^{imt}}dt=\frac{i}{R^{m-1}}\int\limits_{0}^{2\pi}e^{i(1-m)t}dt.$$

Endi a) $m\ne1$ va b) $m=1$ hollarni alohida qaraymiz. $m\ne1$ bo`lgan holda

$\int_0^{2\pi} e^{i(1-m)t}dt = \left.\frac{e^{i(1-m)t}}{i(1-m)}\right|_0^{2\pi} = \frac{e^{i(1-m)2\pi}-1}{i(1-m)}$=0, ikkinchi holda ($m$=1) $\int_\Gamma \frac{dz}{z-a} = i\int_0^{2\pi} dt$=2πi.
Demak,

$$\int_\Gamma \frac{dz}{(z-a)^m} = \begin{cases} 0, & \text{agar } m \neq 1, \\ 2\pi i, & \text{agar } m = 1 \end{cases}$$

№86. $\int_\Gamma \bar{z}^2 dz$ integralni hisoblang, bu erda Γ uchlari $O(0,0)$, $A(1,1)$ va $B(2,1)$ nuqtalarda bo`lgan siniq chiziqdan iborat.



Yechish: OA kesmaning tenglamasi $y=x$ ($0\le x\le 1$) yoki parametrik ko`rinishda $x=t$, $y=t$, $0\le t\le 1$, bundan $z=x+iy=(1+i)t$ va $dz=(1+i)dt$

$\int_{OA} \bar{z}^2 dz = \int_0^1 ((1-i)t)^2(1+i)dt = 2(1-i)\int_0^1 t^2 dt =$

$= \frac{2}{3}(1-i)$.

AB kesmaning tenglamasi y=1, yoki parametrik tenglamasi $x=t$, y=1, $1\le t\le 2$. Bu holda $z=x+iy=t+i$ va $dz=dt$

$\int_{AB} \bar{z}^2 dz = \int_1^2 (t-i)^2 dt = \left.\frac{(t-i)^3}{3}\right|_1^2 = \frac{4-9i}{3}$ bo`ladi.

Butun siniq chiziq bo`yicha olingan integral

$\int_\Gamma \bar{z}^2 dz = \int_{OA} \bar{z}^2 dz + \int_{AB} \bar{z}^2 dz = \frac{2}{3}(1-i) + \frac{4-9i}{3} = 2 - \frac{11}{3}i$ ga teng bo`ladi.

№87. $\int_\Gamma \mathrm{Re}\, z dz$ integralni hisoblang, bu erda Γ- uchlari $A(0,1)$, $B(2,1)$ va $C(3,0)$ nuqtalarda bo`lgan siniq chiziqdan iborat.

№88. $\int_\Gamma z dz$ integralni hisoblang, bu erda Γ-uchlari $A(-2,0)$, $B(-1,1)$, $C(1,1)$ va $D(2,0)$ nuqtalarda bo`lgan $ABCD$ to`rtburchak konturidan iborat.

№89. $\int_\Gamma (x-y)dx - iydy$ integralni hisoblang, bu erda Γ- $|z|=1$ birlik aylanadan iborat.

№90. $\int_\Gamma z\,\mathrm{Im}\, z^2 dz$ integralni hisoblang, bu erda Γ - $|z|=1$ $(-\pi\le \arg z\le 0)$.

№91. $\int_\Gamma z\bar{z} dz$ integralni hisoblang, bu erda Γ: $|z|=1$, $(-\pi\le \arg z\le \pi)$

1-teorema. (Koshi) Agar bir bog`lamli $E$ sohada $f(z)$ analitik bo`lsa, u holda E da yotuvchi har qanday Γ yopiq kontur bo`ylab $f(z)$ funksiyadan olingan integral nolga teng bo`ladi:

$$\int_\Gamma f(z)dz = 0$$

Agar $f(z)$ funksiya bir bog`lamli $E$ sohada analitik va $z_0, z_1 \in E$ bo`lsa, u holda Nьyuton-Leybnits formulasi

$$\int_{z_0}^{z_1} f(\xi)d\xi = F(z)\Big|_{z_0}^{z_1} = F(z_1)-F(z_0)$$

o`rinli bo`ladi, $F(z)$ funksiya $f(z)$ funksiyaning biror boshlang`ich funksiyasi, ya'ni $F'(z)=f(z)$.

№92. $\int_{1+i}^{-1-i}(2z+1)dz$ integralni hisoblang.

Yechish: $\int_{1+i}^{-1-i}(2z+1)dz = z^2+z\Big|_{1+i}^{-1-i}$ $=(-1-i)2+(-1-i)-(1+i)2-(1+i)=-2(1+i)$.

№93. $\int_{\Gamma}\cos z dz$ integralni hisoblang, bu erda Γ $z_1=0$ va $z_2=\pi+i$ nuqtalarni tutashtiruvchi kesma.

Yechish. $\cos z$ funksiya butun kompleks tekislikda analitik bo`lganligi sababli, bu funksiyadan olingan integral integrallash yo`liga bog`liq emas. Shu sababli $\int_{\Gamma} \cos z dz = \int_{\frac{\pi}{2}}^{\pi+i}\cos z dz = \sin z\Big|_{\frac{\pi}{2}}^{\pi+i} = \sin(\pi+i)-\sin\frac{\pi}{2} = -\cos i-1 = ch1-1$.

Agar $f(z)$ funksiya chegarasi Γ silliq yoki bo`lakli silliq yopiq chiziq bilan chegaralangan yopiq $\overline{E}$ sohada bir qiymatli va analitik bo`lsa, u holda quyidagi formula o`rinli bo`ladi:

$$f(z_0) = \frac{1}{2\pi i}\oint_{\Gamma}\frac{f(z)dz}{z-z_0} \qquad (23)$$

Bunda integral musbat yo`nalishda, ya'ni $\Gamma$ bo`ylab harakatlanganda E soha har doim chap tomonda qoladi.

Bu formula E sohada analitik funksiyaning shu sohaning ixtiyoriy ichki nuqtasidagi qiymatini hisoblash uchun uning E soha chegarasidagi qiymatlarini bilish etarli ekanligini bildiradi. Analitik funksiyaning bu ajoiyb xossasidan yopiq $\overline{E}$ sohada analitik bo`lgan har qanday $f(z)$ funksiya shu sohaning ixtiyoriy ichki $z_0$ nuqtasida istalgan tartibli hosilalarga ega bo`lib, ular

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_{\Gamma}\frac{f(z)dz}{(z-z_0)^{n+1}} \qquad (24)$$

formula bilan ifodalanishi, ya'ni hosilalarning ham qiymatlarini funksiyaning chegaradagi qiymatlari orqali hisoblash mumkinligi isbotlanadi.

(23) va (24) formulalar Koshining integral formulalari deb yuritiladi.

Koshining integral formulalari ba'zi integrallarni hisoblashda qo`l keladi.

№94. $\int_{|z|=2}\frac{chiz}{z-1}dz$ integralni hisoblang.

Yechish: Berilgan integral tashqi ko`rinishidan Koshi formulasi o`ng tomonidagi integralga o`xshash, bu erda $z_0=1$, $f(z)=chiz$. Bu funksiya $|z|\le 2$ yopiq doirada analitik va $z_0=1$ shu doira ichida yotganligi uchun

$\frac{1}{2\pi i}\int\limits_{|z|=2}\frac{chiz}{z-1}dz$ ifoda *f(z)* funksiyaning z_0=1 nuqtadagi qiymatiga teng:

$f(1)=ch(i\cdot 1)=\cos 1$, bundan $\cos 1=\frac{1}{2\pi i}\int\limits_{|z|=2}\frac{chiz}{z-1}dz \Rightarrow \int\limits_{|z|=2}\frac{chiz}{z-1}dz=2\pi i\cos 1.$

№95. $\int\limits_{\Gamma}\frac{dz}{z^2+4}$ integralni hisoblang, bu erda Γ a) $|z|=1$; b) $|z+2i|=2$;

c) $|z-2i|=2$; d) $|z|=5$ aylanadan iborat.

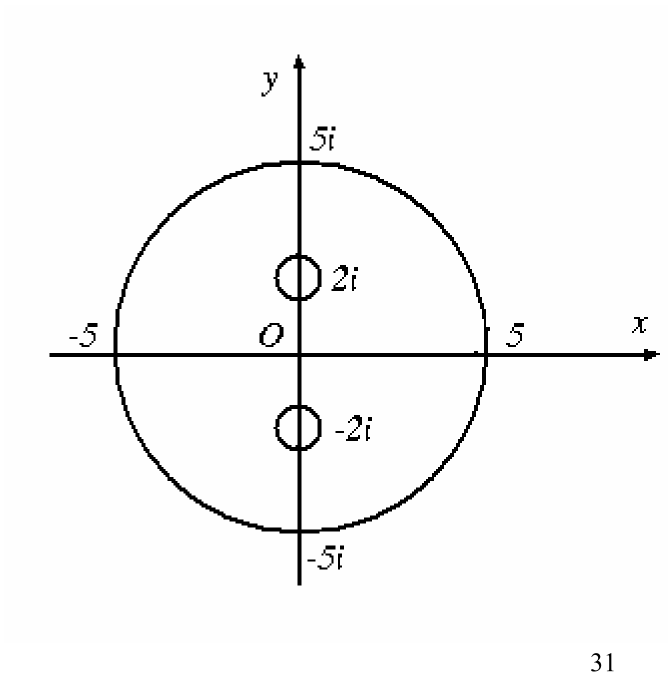
Yechish: a) Integral ostidagi $\frac{1}{z^2+4}$ funksiya $|z|\le 1$ yopiq doirada analitik. Bu holda Koshi teoremasiga ko`ra integral nolga teng bo`ladi.

b) $|z+2i|\le 2$ doirada integral ostidagi funksiya analitik emas, chunki bu funksiya $z=-2i$ nuqtada aniqlanmagan. Integralni quyidagi ko`rinishda yozib olamiz:

$\int\limits_{\Gamma}\frac{dz}{z^2+4}=\int\limits_{\Gamma}\frac{dz}{(z+2i)(z-2i)}$ va Koshining integral formulasi bilan taqqoslaymiz. $\frac{1}{z-2i}$ funksiya qaralayotgan sohada analitik bo`lganligi sababli, $f(z)=\frac{1}{z-2i}$, $z_0=-2i$ deb olish mumkin. Demak, izlanayotgan integral $2\pi i f(z_0)=2\pi i\frac{1}{-2i-2i}=-\frac{\pi}{2}$ teng bo`ladi.

c) Bu holda $f(z)=\frac{1}{z+2i}$, $z_0=2i$ deb olamiz. Natijada integral $2\pi i\frac{1}{2i+2i}=\frac{\pi}{2}$ ga teng bo`ladi.

d) $\frac{1}{z-2i}$ va $\frac{1}{z+2i}$ funksiyalardan hech biri $|z|\le 5$ doirada analitik emas, demak Koshining integral formulasini bevosita tatbiq qilib bo`lmaydi. Integral ostidagi funksiyani sodda kasrlarga ajratamiz:



$$\frac{1}{z^2+4}=\frac{1}{4i}\left(\frac{1}{z-2i}-\frac{1}{z+2i}\right).$$

U holda $\int\limits_{\Gamma}\frac{dz}{z^2+4}=\frac{1}{4i}\int\limits_{\Gamma}\frac{1}{z-2i}dz-\frac{1}{4i}\int\limits_{\Gamma}\frac{1}{z+2i}dz$ bo`lib, o`ng tomondagi integrallarning har biri Koshining integral formulasi yordamida hisoblanadi, bunda $f(z)=1$ va birinchi integralda $z_0=2i$, ikkinchisida esa $z_0=-2i$ ga teng. Shunday qilib,

$\int_{\Gamma}\frac{dz}{z^2+4}=\frac{1}{4i}2\pi i f(2i)-\frac{1}{4i}2\pi i f(-2i)=0$ natijaga ega bo`lamiz.

Bu integralni 2-usulda ham hisoblash mumkin edi. Uning uchun markazlari $z=-2i$ va $z=2i$ nuqtalarda bo`lib, o`zaro kesishmaydigan hamda $|z|\le 5$ doirada yotadigan etarlicha kichik radiusli γ_1 va γ_2 aylanalar chizamiz.

$|z|=5$, γ_1 va γ_2 aylanalar bilan chegaralangan uch bog`lamli sohada integral ostidagi funksiya analitik. Ko`p bog`lamli soha uchun qoshi teoremasiga ko`ra tashqi kontur bo`yicha olingan integral ichki konturlar bo`yicha olingan integrallar yig`indisiga teng:

$$\int_{\Gamma}\frac{dz}{z^2+4}=\int_{\gamma_1}\frac{dz}{z^2+4}+\int_{\gamma_2}\frac{dz}{z^2+4}$$

O`ng tomondagi har bir integralga Koshining integral formulasini qo`llash mumkin. Natijada quyidagiga ega bo`lamiz:

$$\int_{\Gamma}\frac{dz}{z^2+4}=2\pi i\left.\frac{1}{z-2i}\right|_{z=-2i}+2\pi i\left.\frac{1}{z+2i}\right|_{z=2i}=0.$$

№96. $\int\limits_{|z-1|=4}\frac{\cos z}{(z-\frac{\pi}{4})^3}dz$ integralni hisoblang.

Yechish: Bu integralni hisoblash uchun (24) formuladan foydalanamiz. $f(z)=\cos z$ funksiya $|z-1|\le 4$ doirada analitik va $\frac{\pi}{4}$ nuqta shu doira ichida yotganligi sababli $f(z)=\cos z$, $z_0=\frac{\pi}{4}$ va $n=2$ deb olamiz. U holda

$$\int\limits_{|z-1|=4}\frac{\cos z}{(z-\frac{\pi}{4})^3}dz=\frac{2\pi i}{2!}\cos''z\Big|_{z=\frac{\pi}{4}}=\frac{-2\pi i\cos\frac{\pi}{4}}{2}=\frac{\pi i\sqrt{2}}{2}.$$

№97. Faraz qilaylik Γ chiziq $|z-1|=1$ aylanadan iborat bo`lsin.

a) Nima uchun $\int_{\Gamma}(z^2-1)dz$, $\int_{\Gamma}\frac{dz}{z^2+1}$, $\int_{\Gamma}((x-y)+i(x+y))dz$ integrallarga Koshi teoremasini qo`llash va ularni nolga teng deyish mumkinligini asoslang.

b) Nima uchun $\int_{\Gamma}\frac{dz}{z^2-1}$, $\int_{\Gamma}((x+y)+i(x-y))dz$ integrallarga Koshi teoremasini qo`llash mumkin emasligini va ularni nolga teng deya olmasligimizni asoslang.

№98. Nima uchun $\int_{\Gamma}\frac{dz}{z^2+2}$ integralni Γ chiziq $|z|=1$ bo`lganda nolga teng deb tasdiqlash mumkin, ammo $\Gamma$ chiziq $|z|=2$ bo`lganda bunday tasdiq o`rinli emas? Javobingizni asoslang.

№99. Quyidagi integrallarni hisoblang:

a) $\int\limits_{1}^{1+i} z^3 dz$; b) $\int\limits_{1}^{i} (3z^4 - 2z^3 + 1)dz$;

c) $\int\limits_{\Gamma} e^z dz$, bu erda *G*- z_1=0 va $z_2=1+i$ nuqtalarni tutashtiruvchi $y=x^2$ parabola yoyi;

d) $\int\limits_{\Gamma} chzdz$ integralni hisoblang, bu erda Γ chiziq z_1=0 va $z_2=i$ nuqtalarni tutashtiruvchi kesma.

e) $\int\limits_{1}^{i} z\sin z dz$; f) $\int\limits_{1+i}^{2i} (z^3 - z)e^{\frac{z^2}{2}} dz$;

g) $\int\limits_{0}^{i} (z-i)e^{-z} dz$; i) $\int\limits_{1+i}^{0} \sin z \cos z dz$.

№100. Koshining integral formulasi yordamida quyidagi integrallarni hisoblang:

a) $\int\limits_{|z|=1} \frac{e^z}{z^2+2z} dz$; b) $\int\limits_{|z+i|=1} \frac{e^{iz}}{z^2+1} dz$; c) $\int\limits_{|z|=2} \frac{\sin iz}{z^2-4z+3} dz$;

d) $\int\limits_{|z-1|=2} \frac{\cos z}{z^2+2z-3} dz$; e) $\int\limits_{|z|=6} \frac{zdz}{z^2+25}$; f) $\int\limits_{|z|=3} \frac{dz}{(z^2+4)(z+5)}$.

№101. Quyidagi integrallarni hisoblang:

a) $\int\limits_{\Gamma} \frac{z^3+5z-4}{(1-z)^3} dz$, bu erda Γ- $\frac{x^2}{4} + \frac{y^2}{9} = 1$ ellips; b) $\int\limits_{|z|=1} \frac{shz}{z^2} dz$;

c) $\int\limits_{|z-3|=6} \frac{zdz}{(z-2)^3(z+4)}$; d) $\int\limits_{|z|=1/3} \frac{1-\sin z}{z^2} dz$.

8-§. Darajali qator. Analitik funksiyani Teylor qatoriga yoyish

Quyidagi ko`rinishdagi

$$c_0+c_1z+c_2z^2+\ldots+c_nz^n+\ldots=\sum_{n=1}^{\infty} c_n z^n,$$

qator darajali qator deyiladi, bu erda c_0, c_1, ... - o`zgarmas kompleks sonlar (darajali qator koeffitsientlari deyiladi), *z*-kompleks o`zgaruvchi.

Bu qatorning yaqinlashish sohasi markazi koordinatalar boshida bo`lgan doiradan iborat bo`ladi.

Darajali qatorning yaqinlashish radiusi

$$R=\lim_{n\to\infty}\frac{|c_n|}{|c_{n+1}|},\ c_n\neq 0 \qquad (25)$$

yoki

$$R=\lim_{n\to\infty}\frac{1}{\sqrt[n]{|c_n|}} \qquad (26)$$

formulalar yordamida topiladi (ko`rsatilgan limitlar mavjud bo`lgan holda)

№102. $\sum_{n=1}^{\infty} e^{in} z^n$ qatorning yaqinlashish radiusini toping.

Yechish: Bu erda $c_n=e^{in}$ va $|c_n|=|e^{in}|=1$. Yaqinlashish radiusini topish uchun (26) formuladan foydalanamiz. U holda

$R=\lim_{n\to\infty}\frac{1}{\sqrt[n]{|c_n|}}=\lim_{n\to\infty}\frac{1}{\sqrt[n]{1}}=1$, demak berilgan qatorning yaqinlashish radiusi 1 ga teng.

№103. $\sum_{n=1}^{\infty}(n+i)z^n$ qatorning yaqinlashish radiusini toping.

Yechish: Bu erda $c_n=n+i$ va $|cn|=|n+i|=\sqrt{n^2+1}$. Yaqinlashish radiusini topish uchun (25) formuladan foydalanamiz. U holda

$$R=\lim_{n\to\infty}\frac{|c_n|}{|c_{n+1}|}=\lim_{n\to\infty}\frac{|n+i|}{|n+1+i|}=\lim_{n\to\infty}\frac{\sqrt{n^2+1}}{\sqrt{(n+1)^2 1}}=\lim_{n\to\infty}\sqrt{\frac{n^2+1}{n^2+2n+2}}=$$

$=\sqrt{\lim_{n\to\infty}\frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}}}=1$, demak berilgan qatorning yaqinlashish radiusi 1 ga teng.

Berilgan a nuqtada bir qiymatli va analitik bo\`lgan har qanday $f(z)$ funksiyani shu nuqtaning biror atrofida ($z-a$) ning darajalari bo\`yicha Teylor qatoriga yoyish mumkin:

$$f(z)=\sum_{n=1}^{\infty}c_n(z-a)^n$$

Bu qatorning koeffitsientlari

$$c_n=\frac{f^{(n)}(a)}{n!}=\frac{1}{2\pi i}\oint_C\frac{f(z)dz}{(z-a)^{n+1}} \qquad (27)$$

formulalar bilan hisoblanadi, bu erda C- markazi a nuqtada va $f(z)$ funksiya analitik bo\`lgan sohada yotuvchi aylana. Teylor qatorini yaqinlashish doirasining markazi $a$ nuqtada bo\`lib, $f(z)$ funksiyaning a nuqtaga eng yaqin joylashgan ζ maxsus nuqtasidan o\`tadi, ya'ni qatorning yaqinlashish radiusi $a$ nuqtadan funksiya maxsus nuqtalarigacha bo\`lgan masofalarning eng kichigiga teng bo\`ladi.

№104. e^{3z} funksiyani i nuqtaning atrofida Teylor qatoriga yoying.

Yechish: Berilgan funksiyaning n-tartibli hosilasini topamiz: $f(n)(z)=(e^{3z})^{(n)}=3^n e^{3z}$, bundan $f^{(n)}(i)=3^n e^{3i}$ va $s_n=\frac{3^n\cdot e^{3i}}{n!}$, $n=0, 1, 2, \ldots$ Demak,

$$e^{3z}=\sum_{n=0}^{\infty}\frac{3^n\cdot e^{3i}}{n!}(z-i)^n .$$

(27) formula yordamida qator koeffitsientlarini topish orqali funksiyani Teylor qatoriga yoyish hamma vaqt ham qo\`lay bo\`lavermaydi, chunki hosilalarni (integrallarni) hisoblash ancha murakkabliklarga olib kelishi mumkin. Shuning uchun (27) formulani tatbiq etishni talab qilmaydigan qo\`shimcha usullardan foydalaniladi. Ular quyidagilardan iborat:

a) elementar funksiyalarning Teylor qatorlaridan foydalanish:

1) $\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n$, |z|<1; (28)

2) $e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$, $|z|<\infty$; (29)

3) $sinz=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^{2n-1}}{(2n-1)!}$, $|z|<\infty$; (30)

4) $cosz=\sum_{n=0}^{\infty}(-1)^{n}\frac{z^{2n}}{(2n)!}$, $|z|<\infty$; (31)

5) $shz=\sum_{n=1}^{\infty}\frac{z^{2n-1}}{(2n-1)!}$, $|z|<\infty$; (32)

6) $chz=\sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}$, $|z|<\infty$; (33)

7) $(1+z)^{\alpha}=1+\sum_{n=1}^{\infty}\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}z^n$, $|z|<1$; (34)

8) $ln(1+z)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}$, $|z|<1$; (35)

b) qatorlarni hadma-had differensiallash va integrallash usuli;
c) noaniq koeffitsientlar usuli.

№105. $f(z)=\frac{1}{z+5}$ funksiyani z=-2 nuqta atrofida Teylor qatoriga yoying, uning yaqinlashish radiusini toping.

Yechish: Bu funksiyani Teylor qatoriga yoyish uchun (28) formuladan foydalanamiz. Buning uchun uni quyidagicha yozib olamiz: $\frac{1}{z+5}=\frac{1}{3+z+2}=\frac{1}{3(1-(-\frac{z+2}{3}))}$. Natijada

$\frac{1}{z+5}=\frac{1}{3}(1-\frac{z+2}{3}+\frac{(z+2)^2}{3^2}-\cdots+(-1)^n\frac{(z+2)^n}{3^n}+\cdots$, yoki $\frac{1}{z+5}=\sum_{n=0}^{\infty}(-1)^n\frac{(z+2)^n}{3^{n+1}}$

yoyilmaga ega bo`lamiz. $\frac{1}{z+5}$ funksiyaning maxsus nuqtasi z=-5 dan iborat. Bu nuqtadan z=-2 nuqtagacha bo`lgan masofa 3 ga teng. Demak, hosil bo`lgan qatorning yaqinlashish radiusi R=3 ga teng.

№106. $\frac{2}{(1-z)^3}$ funksiyani z ning darajalari bo`yicha qatorga yoying.

Yechish: |z|<1 doirada yaqinlashuvchi (28) qatorni qaraymiz:

$$\frac{1}{1-z}=1+z+z2+z3+...+zn+...$$

Bu qatorni ikki marta hadma-had differensiallaymiz:

$$\left(\frac{1}{1-z}\right)'=\frac{1}{(1-z)^2}=1+2z+3z^2+4z^3+\ldots+(n+1)z^n+\ldots,$$

$$\left(\frac{1}{1-z}\right)''=\frac{2}{(1-z)^3}=2+6z+12z^2+\ldots+(n+2)(n+1)z^n+\ldots$$

Natijada $|z|<1$ doirada yaqinlashuvchi $\sum_{n=0}^{\infty}(n+2)(n+1)z^n$ qatorga ega bo`lamiz.

№107. *tgz* funksiyaning z=0 nuqta atrofidagi Teylor qatorining noldan farqli birinchi uchta hadini va bu qatorning yaqinlashish doirasini toping.

Yechish. Noaniq koeffitsientlar usulidan foydalanamiz. *tgz* funksiya z=0 nuqtada analitik bo`lganligi sababli uni $z$ ning darajalari bo`yicha qatorga yoyish mumkin:

$$tgz=c_0+c_1z+c_2z^2+c_3z^3+c_4z^4+\ldots$$

$tg0=0$ va yuqoridagi tenglikdan $c_0=0$ ekanligi kelib chiqadi. Qatorning boshqa koeffitsientlarini topish uchun $tgz\cdot cosz=sinz$ munosabatdan foydalanamiz. Bizga kosinus va sinuslarning qatorlari ma’lum. Shu sababli

$$z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots=\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\cdots\right)(c_1z+c_2z^2+c_3z^3+c_4z^4+\ldots),$$

$$z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots=c_1z+c_2z^2+\left(c_3-\frac{c_1}{2!}\right)z^3+\left(c_4-\frac{c_2}{2!}\right)z^4+\left(c_5-\frac{c_3}{2!}+\frac{c_1}{4!}\right)z^5+\ldots$$

Funksiyani darajali qatorga yoyishning yagonaligidan quyidagi munosabatlar bajarilishi shart:

$c_1=1,\ \ c_2=0,\ \ \left(c_3-\frac{c_1}{2!}\right)=-\frac{1}{6},\ \ \left(c_4-\frac{c_2}{2!}\right)=0,\ \ \left(c_5-\frac{c_3}{2!}+\frac{c_1}{4!}\right)=\frac{1}{5!}$. Bulardan $c_1=1,\ \ c_2=0,$ $c_3=\frac{1}{3},\ c_4=0,\ c_5=\frac{2}{15}$ ekanligi kelib chiqadi. Demak, $tgz=z+\frac{1}{3}z^3+\frac{2}{15}z^5+\ldots$.

tgz funksiyaning maxsus nuqtalari $\frac{\pi}{2}+\pi n,\ n\in Z$ dan iborat. Ulardan z=0 nuqtaga eng yaqini $\pm\frac{\pi}{2}$ dir. Demak, qator $|z|<\frac{\pi}{2}$ doirada yaqinlashuvchi bo`ladi.

№108. Quyidagi qatorlarning yaqinlashish radiuslarini toping:

a) $\sum_{n=1}^{\infty}\left(\frac{z}{in}\right)^n$; b) $\sum_{n=0}^{\infty}(\cos in)z^n$; c) $\sum_{n=1}^{\infty}(4+3i)^n z^n$;

d) $\sum_{n=1}^{\infty}\frac{z^n}{\sin^n n}$; e) $\sum_{n=1}^{\infty}\left(\frac{z}{\ln in}\right)^n$; f) $\sum_{n=1}^{\infty}ch\frac{i}{n}z^n$.

№109. *cos4z* funksiyani a) z=0; b) z=π nuqta atrofida Teylor qatoriga yoying.

№110. Quyidagi funksiyalarni ko`rsatilgan nuqta atrofida Teylor qatoriga yoying:

a) $cos2z,\ z=0$; b) $sinzcosz,\ z=0$; c) $e^z,\ z=-1$;

d) $\frac{1}{z+i},\ z=1$; e) $\frac{z}{z+2},\ z=-1$; f) $\frac{1}{z^2+9},\ z=0$.

№111. Quyidagi funksiyalarni z ning darajalari bo`yicha qatorga yoying:

a) $(z+1)\ln(z+1)-z$; b) $\frac{1}{(z^2+1)^2}$;

c) $\frac{z^2}{(1-z^3)^2}$; d) $\ln(1+z)$.

№112. Quyidagi funksiyalarning $z=0$ nuqta atrofidagi Teylor qatorining noldan farqli birinchi uchta hadini va bu qatorlarning yaqinlashish doirasini toping.

a) $\frac{z}{\cos z}$; b) $\frac{z}{\ln(1-z)}$.

9-§. Loran qatori. Analitik funksiyani Loran qatoriga yoyish

Musbat hamda manfiy darajali hadlardan tuzilgan quyidagi umumiy qator

$$\ldots + s_{-n}(z-a)^{-n} + \ldots + c_{-2}(z-a)^{-2} + c_{-1}(z-a)^{-1} + s_0 + s_1(z-a) +$$

$$+c_2(z-a)^2+\ldots+c_n(z-a)^n+\ldots=\sum_{n=-\infty}^{\infty} c_n(z-a)^n \qquad (35)$$

Loran qatori deyiladi. Bunda c_n, $n=0, \pm1, \pm2, \ldots$-kompleks sonlar Loran qatorining koeffitsientlari, a-biror kompleks son.

Loran qatori

$$\sum_{n=0}^{\infty} c_n(z-a)^n \qquad (36)$$

va

$$\sum_{n=-1}^{-\infty} c_n(z-a)^n \qquad (37)$$

qatorlar yig`indisidan iborat.

(36) Loran qatorining to`g`ri qismi, (37) bosh qismi deyiladi.

Loran qatorining to`g`ri qismi $|z-a|<R$ doirada yaqinlashadi, bu erda

$R=\lim\limits_{n\to\infty}\frac{|c_n|}{|c_{n+1}|}$, $c_n\neq 0$ yoki $R=\lim\limits_{n\to\infty}\frac{1}{\sqrt[n]{|c_n|}}$ formulalardan topiladi.

Loran qatorining bosh qismi $|z-a|=r$ aylana tashqarisida, ya'ni $|z-a|>r$ sohada yaqinlashadi, bu erda

$r=\lim\limits_{n\to\infty}\frac{|c_{-n-1}|}{|c_{-n}|}$ yoki $r=\lim\limits_{n\to\infty}\sqrt[n]{|c_{-n}|}$ formulalardan topiladi.

Loran qatori $r<R$ bo`lgandagina yaqinlashuvchi bo`ladi va uning yaqinlashish sohasi $r<|z-a|<R$ halqadan iborat bo`ladi. Qator yig`indisi yaqinlashish halqasida analitik funksiya bo`lib, $|z-a|=r$, $|z-a|=R$ aylanalar funksiyaning maxsus nuqtalari orqali o`tadi.

№113. $\sum\limits_{n=-\infty}^{\infty}\frac{z^n}{2^n+1}$ qatorning yaqinlashish sohasini toping.

Yechish: Berilgan qatorning to`g`ri qismi $\sum_{n=0}^{\infty}\frac{z^n}{2^n+1}$ ning yaqinlashish doirasini topamiz. $R=\lim_{n\to\infty}\frac{|c_n|}{|c_{n+1}|}=\lim_{n\to\infty}\frac{2^{n+1}+1}{2^n+1}=\lim_{n\to\infty}\frac{\frac{1}{2^n}+2}{\frac{1}{2^n}+1}$=2 bo`lganligi sababli, to`g`ri qismi $|z|<2$ doirada yaqinlashadi.

Qatorning bosh qismi $\sum_{n=-1}^{-\infty}\frac{z^n}{2^n+1}$ uchun $r=\lim_{n\to\infty}\frac{|c_{-n-1}|}{|c_{-n}|}=\lim_{n\to\infty}\frac{1+2^{-n}}{1+2^{-n-1}}$=1, shu sababli u $|z|>1$ cohada yaqinlashadi.

Demak, berilgan qator $1<|z|<2$ halqada yaqinlashadi.

№114. $\sum_{n=0}^{\infty}\frac{z^n}{2^{n-1}}+\sum_{n=1}^{\infty}\frac{i^{n+1}}{z^n}$ qatorning yaqinlashish sohasini va yig`indisini toping.

Yechish. $\sum_{n=0}^{\infty}\frac{z^n}{2^{n-1}}$ qatorni quyidagicha yozib olamiz: $\sum_{n=0}^{\infty}\frac{z^n}{2^{n-1}}=2\sum_{n=0}^{\infty}\frac{z^n}{2^n}$, u holda $\sum_{n=0}^{\infty}\frac{z^n}{2^n}$ maxraji $\frac{z}{2}$ ga teng bo`lgan geometrik progressiya bo`lib, $|\frac{z}{2}|<1$ da, ya'ni $|z|<2$ doirada yaqinlashadi. Uning yig`indisi $\frac{1}{1-\frac{z}{2}}=\frac{2}{2-z}$ ga teng bo`ladi. Demak $\sum_{n=0}^{\infty}\frac{z^n}{2^{n-1}}$ qatorning yig`indisi $\frac{4}{2-z}$ ga teng.

Qatorning bosh qismi $\sum_{n=1}^{\infty}\frac{i^{n+1}}{z^n}=i\sum_{n=1}^{\infty}\frac{i^n}{z^n}$, $\sum_{n=1}^{\infty}\frac{i^n}{z^n}$ qator esa maxraji $\frac{i}{z}$ ga teng bo`lgan geometrik progressiya bo`lib, $|\frac{i}{z}|<1$ da, ya'ni $|z|>1$ sohada yaqinlashadi. Uning yig`indisi $\frac{\frac{i}{z}}{1-\frac{i}{z}}=\frac{i}{z-i}$ ga teng. Demak $\sum_{n=1}^{\infty}\frac{i^{n+1}}{z^n}$ qatorning yig`indisi $i\cdot\frac{i}{z-i}=-\frac{1}{z-i}$ ga teng.

$\sum_{n=0}^{\infty}\frac{z^n}{2^{n-1}}+\sum_{n=1}^{\infty}\frac{i^{n+1}}{z^n}$ qatorning yaqinlashish cohasi $1<|z|<2$ halqa bo`lib, yig`indisi $\frac{4}{2-z}-\frac{1}{z-i}=-\frac{5z-2-4i}{z^2-(2+i)z+2i}$ funksiyadan iborat.

Teorema. Agar $f(z)$ funksiya K: $r<|z-a|<R$ halqada bir qiymatli va analitik bo`lsa, u funksiyani shu halqada

$$f(z)=\sum_{n=-\infty}^{\infty}c_n(z-a)^n \qquad (38)$$

Loran qatoriga yoyish mumkin bo`lib, sn koeffitsientlar

$$c_n=\frac{1}{2\pi i}\oint_{\gamma}\frac{f(z)dz}{(z-a)^{n+1}}\text{ , n=0, ±1, ±2, ...,} \qquad (39)$$

formulalar orqali topiladi, bundagi γ - $|z-a|=\rho$, $r<\rho<R$ aylana.

Berilgan $f(z)$ funksiyani Loran qatoriga yoyish uchun uning koeffitsientlarini (39) ga asoslanib topish ancha noqo`lay. Amaliyotda, Teylor qatoridagi kabi, sun'iy usullar ko`proq qo`llaniladi. Bunda berilgan funksiyaning berilgan halqada (38) ko`rinishdagi qatori yagonaligidan foydalaniladi. Ko`pincha quyidagicha ish tutiladi: $r<|z-a|<R$ halqada berilgan funksiyani birinchisi $|z-a|<R$ doirada, ikkinchisi $|z-a|>r$ sohada analitik bo`lgan $f_1(z)$ va $f_2(z)$ funksiyalarning yig`indisi ko`rinishda ifodalaniladi. Ularning birinchisini $z-a$ ning musbat darajalari bo`yicha, ikkinchisini $z-a$ ning manfiy darajalari bo`yicha yoyiladi, hosil bo`lgan qatorlar qo`shiladi.

№115-misol. a) $f(z)=\frac{2z-3}{z^2-3z+2}$ funksiyani $2<|z+1|<3$ halqada Loran qatoriga yoying.

b) Bu funksiyani $1<|z|<3$ halqada qatorga yoyish mumkinmi?

Yechish: a) Funksiyani quyidagi ko`rinishda yozib olamiz: $f(z)=\frac{2z-3}{(z-1)(z-2)}$. Bundan funksiyaning $z_1=1$, $z_2=2$ maxsus nuqtalari $2<|z+1|<3$ halqa chegaralarida yotishi ko`rinib turibdi. Demak, funksiya qaralayotgan halqada analitik funksiya bo`lib, uni Loran qatoriga yoyish mumkin (Loran teoremasiga ko`ra). Loran qatoriga yoyish uchun funksiyani $f(z)=\frac{1}{z-1}+\frac{1}{z-2}$ ko`rinishda yozamiz.

$f_1(z)=\frac{1}{z-1}$ funksiyani $z+1$ ning musbat darajalari bo`yicha qatorga yoysak, uning yaqinlashish sohasi $|z+1|<2$ doiradan iborat bo`ladi. Masala shartiga ko`ra biz funksiyani bu doiraning tashqarisida qatorga yoyishimiz kerak. Shu sababli bu funksiyani $z+1$ ning manfiy darajalari bo`yicha yoyishga harakat qilamiz. Buning uchun $f_1(z)$ funksiyani quyidagi ko`rinishda yozib olamiz:

$f_1(z)=\frac{1}{z-1}=\frac{1}{(z+1)-2}=\frac{1}{z+1}\cdot\frac{1}{1-\frac{2}{z+1}}$ va $t=\frac{2}{z+1}$ deb (28) formuladan foydalanamiz.

U holda $\frac{1}{1-\frac{2}{z+1}}=\sum_{n=0}^{\infty}\left(\frac{2}{z+1}\right)^n$ ga bo`lamiz. Bundan $\frac{1}{z-1}=\frac{1}{z+1}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{z+1}\right)^n=$ $=\sum_{n=1}^{\infty}\frac{2^{n-1}}{(z+1)^n}$. Bu qator $|t|<1$ da, ya'ni $|z+1|>2$ sohada yaqinlashadi.

$f_2(z)=\frac{1}{z-2}$ funksiya $|z+1|<3$ doirada analitik bo`lganligi sababli uni $z+1$ ning musbat darajalari bo`yicha Teylor qatoriga yoyish mumkin. Buning uchun $f_2(z)$ funksiyani quyidagicha ifodalab olamiz: $f_2(z)=\frac{1}{z-2}=\frac{1}{(z+1)-3}=-\frac{1}{3}\cdot\frac{1}{1-\frac{z+1}{3}}$.

(24) formulada $t=\frac{z+1}{3}$ deb, quyidagiga ega bo`lamiz: $\frac{1}{z-2}=-\frac{1}{3}\cdot\frac{1}{1-\frac{z+1}{3}}=-$ $\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n=-\sum_{n=0}^{\infty}\frac{(z+1)^n}{3^{n+1}}$. Bu qatorning yaqinlashish sohasi $|z+1|<3$ doiradan iborat.

$f_1(z)$ va $f_2(z)$ funksiyalar yoyilmalaridan foydalanib, $f(z)$ funksiyaning berilgan halqadagi Loran qatoriga ega bo`lamiz.

$$f(z)=\frac{2z-3}{(z-1)(z-2)}=\sum_{n=1}^{\infty}\frac{2^{n-1}}{(z+1)^n}-\sum_{n=0}^{\infty}\frac{(z+1)^n}{3^{n+1}},\ 2<|z+1|<3.$$

b) $1<|z|<3$ halqaga $z=2$ nuqta tegishli. Bu nuqtada berilgan funksiya aniqlanmagan. Demak, berilgan funksiyani shu halqaning barcha nuqtalarida yaqinlashuvchi qator ko`rinishda ifodalab bo`lmaydi.

№116. $f(z)=\frac{2z-3}{z^2-3z+2}$ funksiyani z ning manfiy va musbat darajalari bo`yicha qatorga yoyish mumkinmi? Yoyish mumkin bo`lsa, u holda necha xil usulda yoyish mumkin? Bu yoyilmalarni toping.

Yechish: Markazi $a=0$ nuqtada bo`lgan biror halqada analitik bo`lgan funksiyani $f(z)=\sum_{n=-\infty}^{\infty}c_n z^n$ Loran qatoriga yoyish mumkin.

Berilgan funksiya markazi $a=0$ nuqtada bo`lgan a) $|z|<1$; b) $1<|z|<2$; c) $2<|z|<\infty$ halqalarda analitik va shu halqalarda Loran qatoriga yoyiladi. Shu yoyilmalarni topamiz.

a) $|z|<1$ halqada Loran qatori Teylor qatoridan iborat bo`ladi. Uni quyidagicha hosil qilamiz:

$$f(z)=\frac{1}{z-1}+\frac{1}{z-2}=-\frac{1}{1-z}-\frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}}=-\sum_{n=0}^{\infty}z^n-\frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{z^n}{2^n}=-\sum_{n=0}^{\infty}z^n-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}=$$

$$=\sum_{n=0}^{\infty}\left(1+\frac{1}{2^{n+1}}\right)z^n,\ (|z|<1).$$

b) Bu holda funksiyaning Loran qatori 115-misoldagi kabi hosil qilinadi.

$$\frac{1}{z-2}=-\frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}}=-\frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{z^n}{2^n}=-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}},\ (|z|<2)$$

$$\frac{1}{z-1}=\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}=\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}},\ (|z|>1)$$

Natijada quyidagiga ega bo`lamiz:

$$f(z)=\frac{1}{z-1}+\frac{1}{z-2}=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}},\ (1<|z|<2)$$

c) Bu sohada $\frac{1}{z-1}$ va $\frac{1}{z-2}$ funksiyalarni z ning manfiy darajalari bo`yicha yoyish kerak. $\frac{1}{z-1}$ funksiya uchun $\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$, (|$z$|>1) ekanligini yuqorida ko`rsatdik. $\frac{1}{z-2}$ funksiya uchun yoyilmani topamiz.

$\frac{1}{z-2}=\frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}}=\frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n=\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$, (|$z$|>2). Bu ikki yoyilmadan foydalanib, oxirgi natijaga erishamiz:

$$f(z)=\frac{2z-3}{z^2-3z+2}=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}+\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}=\sum_{n=0}^{\infty}\frac{1+2^n}{z^{n+1}},\ (|z|>2).$$

№117. Quyidagi qatorlarning yaqinlashish sohasini toping:

a) $\sum_{n=-\infty}^{\infty}2^{-|n|}z^n$; b) $\sum_{n=-\infty}^{\infty}3^n z^n$; c) $\sum_{n=-\infty}^{\infty}4^{-n^2}(z+i)^n$; d) $\sum_{n=0}^{\infty}\left(\frac{z^n}{n!}+\frac{n^2}{z^n}\right)$.

№118. Quyidagi qatorlarning yaqinlashish sohasi va yig`indisini toping:

a) $\sum_{n=0}^{\infty}\frac{3^n}{n!z^n}$; b) $\sum_{n=2}^{\infty}\frac{2^{n-1}-i^{n-1}}{z^n}$; c) $\sum_{n=-1}^{\infty}\frac{(z+2i)^n}{3^{n+2}}$; d) $\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{z^{n+1}}+\frac{1}{3}\sum_{n=0}^{\infty}\frac{z^n}{3^n}$.

№119. Berilgan funksiyani ko`rsatilgan halqalarda Loran qatoriga yoying:

a) $f(z)=\frac{z}{(z-i)(z-2)}$, 1<|$z$|<2; b) $f(z)=\frac{1}{z(z-1)}$, 0<|z|<1;

c) $f(z)=\frac{1}{z^2+1}$, 0<|z-i|<1; d) $f(z)=\frac{3}{z^2-1}$, 1<|z+2|<3.

№120. Quyidagi funksiyalarni z-a ning darajalari bo`yicha mumkin bo`lgan barcha yoyilmalarini yozing:

a) $f(z)=\frac{1}{z^2-4z+3}$, a=1; b) $f(z)=\frac{1}{z^2-4z+3}$, a=0;

c) $f(z)=\frac{1}{z^2-z}$, a=0; d) $f(z)=\frac{1}{z^2-z-2}$, a=-1.

10-§. Funksiyaning nollari va maxsus nuqtalari.

Faraz qilaylik, $f(z)$ funksiya z_0 nuqtada analitik bo`lsin. Agar

$$f(z_0)=0,\ f'(z_0)=0,\ \dots,\ f^{(n-1)}(z_0)=0,\ f^{(n)}(z_0)\neq 0$$

bo`lsa, z_0 nuqta $f(z)$ funksiyaning n –tartibli noli deyiladi.

Agar n=1 bo`lsa, z_0 nuqta oddiy nol deyiladi.

z_0 nuqta shu nuqtada analitik bo`lgan $f(z)$ funksiyaning $n$-tartibli noli bo`lishi uchun z_0 nuqtaning biror atrofida

$$f(z)=(z-z_0)^n\varphi(z) \qquad (40)$$

tenglikning o`rinli bo`lishi zarur va etarli, bu erda $\varphi(z)$ funksiya z_0 nuqtada analitik va $\varphi'(z_0)\neq 0$.

№121. $f(z)=z^3(e^z-1)$ funksiya uchun $z_0=0$ nechanchi tartibli nol ekanligini aniqlang.

Yechish. Berilgan funksiyani z ning darajalari bo`yicha Teylor qatoriga yoyamiz, buning uchun e^z funksiyaning (29) yoyilmasidan foydalanamiz.

$$f(z)=z^3(e^z-1)=z^3(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots+\frac{z^n}{n!}+\cdots-1)=z^4(1+\frac{z}{2!}+\frac{z^2}{3!}+\cdots+\frac{z^{n-1}}{n!}+\cdots)=z^4\varphi(z),$$

bu erda $\varphi(0)\neq 0$ va $\varphi(z)$ funksiya $z_0=0$ nuqtada analitik (chunki u darajali qatorning yig`indisi).

Shunday qilib, $f(z)=z^3(e^z-1)$ funksiya uchun (40) ko`rinishdagi formulaga ega bo`ldik, bundan $z_0=0$ to`rtinchi tartibli nol ekan.

Agar $f(z)$ funksiya a nuqtaning biror $0<|z-a|<R$ atrofida analitik bo`lib, $a$ ning o`zida analitik bo`lmasa, u holda a nuqta $f(z)$ funksiyaning ajralgan (yakkalangan) maxsus nuqtasi deyiladi.

№122. $z=2$ nuqta a) $f(z)=cos\frac{1}{z-2}$; b) $f(z)=sec\frac{1}{z-2}$ funksiyaning ajralgan maxsus nuqtasi bo`ladimi?

Yechish. $w_1=\frac{1}{z-2}$ funksiya $z=2$ nuqtadan boshqa barcha nuqtalarda analitik, $f(z)=cosw_1$ funksiya barcha w_1 nuqtalarda analitik bo`lganligi sababli $f(z)=cos\frac{1}{z-2}$ murakkab funksiya $z=2$ nuqtadan boshqa barcha nuqtalarda, demak ixtiyoriy $0<|z-2|<R$ atrofida ham analitik bo`ladi. Bundan $z=2$ nuqta $f(z)=cos\frac{1}{z-2}$ funksiyaning ajralgan maxsus nuqtasi bo`ladi.

b) $f(z)=sec\frac{1}{z-2}=\frac{1}{\cos\frac{1}{z-2}}$ bo`lganligidan, bu funksiya $z=2$ va $cos\frac{1}{z-2}=0$ bo`lgan nuqtalarda, ya'ni $\frac{1}{z-2}=\frac{\pi}{2}+k\pi$, $z=2+\frac{1}{\pi/2+k\pi}$, $k\in Z$ nuqtalarda aniqlanmagan. Bu nuqtalar ketma-ketligi $a=2$ limit nuqtaga ega, demak $a=2$ nuqtaning ixtiyoriy atrofida funksiyaning maxsus nuqtalari mavjud. Shu sababli funksiya analitik bo`ladigan biror $0<|z-2|<R$ atrofni ko`rsatish mumkin emas. Demak, $z=2$ nuqta $f(z)=sec\frac{1}{z-2}$ funksiyaning ajralgan maxsus nuqtasi emas.

$z=a$ nuqtaning $0<|z-a|<R$ atrofini halqaning xususiy holi deb qarash mumkinligidan, $f(z)$ funksiyani $z-a$ ning darajalari bo`yicha Loran qatoriga yoyish mumkin. Bu qatorning ko`rinishiga ko`ra ajralgan maxsus nuqtalar uchta turga bo`linadi.

Agar

1) Loran qatori faqat to`g`ri qismdangina iborat:

$$f(z)=c_0+c_1(z-a)+c_2(z-a)^2+...+c_n(z-a)^n+...$$

bo`lsa, $a$ nuqta $f(z)$ funksiyaning tuzatib bo`ladigan maxsus nuqtasi;

2) Loran qatorining bosh qismi faqat chekli sondagi hadlardan iborat:

$$f(z)=\frac{c_{-m}}{(z-a)^m}+\frac{c_{-m+1}}{(z-a)^{m-1}}+...+\frac{c_{-1}}{z-a}+c_0+c_1(z-a)+...+c_n(z-a)^n+...,$$

bu erda $m\neq 0$, $m\geq 1$ bo`lsa, $a$ nuqta $f(z)$ funksiyaning qutbi ($m$-tartibli qutbi, $m=1$ bo`lganda oddiy qutb);

3) Loran qatorining bosh qismi cheksiz sondagi hadlardan iborat: $f(z)=\sum_{n=-\infty}^{\infty} c_n (z-a)^n$, bu erda cheksiz sondagi manfiy n lar uchun $c_n \neq 0$, bo`lsa a nuqta $f(z)$ funksiyaning muhim maxsus nuqtasi deyiladi.

Amaliyotda maxsus nuqtaning turini aniqlash uchun quyidagi tasdiqdan foydalanish mumkin:

Agar a nuqta analitik $\psi(z)$ funksiyaning k-tartibli noli, $\varphi(z)$ funksiya a nuqtada analitik va $\varphi(z)\neq 0$ bo`lsa, u holda $a$ nuqta $f(z)=\frac{\varphi(z)}{\psi(z)}$ funksiyaning $k$-tartibli qutbi bo`ladi.

№123. Quyidagi funksiyalarning ajralgan maxsus nuqtalarini va ularning turini aniqlang:

a) $f(z)=\frac{\sin z}{z}$; b) $f(z)=\frac{chz}{z-z^3}$; c) $f(z)= e^{\frac{1}{z}}$.

Yechish. a) $f(z)=\frac{\sin z}{z}$ funksiya butun kompleks tekislikda analitik bo`lgan ikkita funksiyaning nisbatidan iborat. Shu sababli bu funksiyaning maxsus nuqtasi maxrajdagi funksiyaning nolidan iborat bo`ladi: a=0. Bu maxsus nuqtaning turini bevosita aniqlab bo`lmaydi, chunki a=0 da kasrning surati ham nolga aylanadi.

Berilgan funksiyani z ning darajalari bo`yicha Loran qatoriga yoyamiz:

$$\frac{\sin z}{z}=\frac{z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots}{z}=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots.$$

Qatorda z ning manfiy darajalari qatnashmayapti, demak a=0 berilgan funksiyaning tuzatib bo`ladigan maxsus nuqtasi ekan.

b) mahrajining nollarini topamiz: $z-z^3=0 \Rightarrow z_1=0,\ z_2=1,\ z_3=-1$. Bu nuqtalarda surat nolga aylanmaydi. Demak bu nuqtalar berilgan funksiyaning oddiy qutblari bo`ladi.

c) $f(z)=e^{\frac{1}{z}}$ funksiya z=0 nuqtada aniqlanmagan, bu nuqtadan boshqa barcha nuqtalarda analitik, demak z=0 funksiyaning ajralgan maxsus nuqtasi bo`ladi. Uning turini aniqlash uchun z=0 atrofida qatorga yoyamiz: $e^{\frac{1}{z}}=1+\frac{1}{z}+\frac{1}{2!z^2}+...+\frac{1}{n!z^n}+....$

Bu qatorda z ning manfiy darajalari cheksiz ko`p. Demak, z=0 nuqta $f(z)=e^{\frac{1}{z}}$ funksiyaning muhim maxsus nuqtasi.

Maxsus nuqtaning turlari funksiyaning shu nuqta atrofida o`zgarishini tavsiflaydi. Maxsus nuqtalar haqida quyidagi tasdiqlar o`rinli:

a nuqta $f(z)$ funksiyaning

a) qutilib bo`ladigan (yoki chetlashtiriladigan) maxsus nuqtasi bo`lishi uchun $\lim_{z\to a} f(z)=A$, bu erda A aniq chekli son;

b) qutbi bo`lishi uchun $\lim_{z\to a} f(z)=\infty$;

c) muhim maxsus nuqtasi bo`lishi uchun $\lim\limits_{z\to a} f(z)$ mavjud bo`lmasligi zarur va etarli.

№124. $f(z)=\dfrac{\sin z}{z-z^2}$ funksiyaning ajralgan maxsus nuqtalari va ularning turini aniqlang.

Yechish. Funksiyaning maxsus nuqtalari $z=0$, $z=1$ dan iborat.

$z\to 0$ da $\dfrac{\sin z}{z-z^2}=\dfrac{\sin z}{z(1-z)}\to 1$ bo`ladi, demak $z$=0 qutilub bo`ladigan maxsus nuqta bo`ladi.

$z\to 1$ da $\dfrac{\sin z}{z-z^2}=\dfrac{\sin z}{z(1-z)}\to\infty$ bo`ladi, demak $z$=1 qutb bo`ladi.

№125. $f(z)=e^{\frac{1}{z}}$ funksiyaning maxsus nuqtalarini toping va turini aniqlang.

Yechish. Funksiyaning maxsus nuqtasi z=0 dan iborat. z=0 nuqtaga haqiqiy o`q bo`ylab yaqinlashamiz. U holda $f(z)=f(x)=e^{\frac{1}{x}}$ bo`lib, $\lim\limits_{x\to 0+0} e^{\frac{1}{x}}=\infty$ va $\lim\limits_{x\to 0-0} e^{\frac{1}{x}}=0$ tengliklar o`rinli bo`ladi. Bundan esa $\lim\limits_{x\to 0} e^{\frac{1}{z}}$ ning mavjud emasligi kelib chiqadi. Demak, z=0 nuqta $f(z)=e^{\frac{1}{z}}$ funksiyaning muhim maxsus nuqtasi ekan.

№126. Quyidagi funksiyalar uchun z_0=0 nechanchi tartibli nol ekanligini aniqlang:

a) $f(z)=4\sin z^3+z^3(z^6-4)$; b) $f(z)=\dfrac{z^8}{z-\sin z}$.

№127. Quyidagi funksiyalarning barcha nollarini va ularning tartibini aniqlang:

a) $f(z)=z(\cos z-1)$; b) $f(z)=(z2+1)\sin z$;

c) $f(z)=\dfrac{(z^2+4)^2}{z^4}$; d) $f(z)=\dfrac{(z^2-\pi^2)\sin z}{z^2}$.

№128. Quyidagi funksiyalarning ajralgan maxsus nuqtalarini toping va ularning turlarini aniqlang:

a) $f(z)=\dfrac{z-1}{z^2}$; b) $f(z)=\dfrac{\cos z-1}{z^2}$; c) $f(z)=\dfrac{z}{1-\cos z}$;

d) $f(z)=\dfrac{z^4-z}{z^3}$; e) $f(z)=z^3\cos\dfrac{\pi}{z}$; f) $f(z)=\dfrac{1-\cos z}{\sin^2 z}$;

№129. Quyidagi funksiyalarning maxsus nuqtalarini toping va turini aniqlang:

a) $f(z)=\dfrac{tgz}{z}$; b) $f(z)=\sin\dfrac{\pi}{2z}$; c) $f(z)=e^{\frac{z-1}{z}}$; d) $f(z)=e-z\cos\dfrac{1}{z}$.

11-§. Chegirma va uni hisoblash. Chegirmalarning ba'zi tatbiqlari

Faraz qilaylik a nuqta $f(z)$ funksiyaning ajralgan maxsus nuqtasi bo`lsin va γ-shunday $|z-a|=r$ aylana bo`lsinki, $|z-a|\leq r$ doirada $f(z)$ funksiyaning a dan farqli boshqa maxsus nuqtalari bo`lmasin. $f(z)$ funksiyaning ajralgan maxsus $a$ nuqtaga nisbatan chegirmasi (qoldig`i) deb, $\frac{1}{2\pi i}\oint_\gamma f(z)dz$ integralning qiymatiga aytiladi va

$$\operatorname*{res}_{z=a} f(z)=\frac{1}{2\pi i}\oint_\gamma f(z)dz \qquad (41)$$

ko`rinishda yoziladi.

Funksiyaning chegirmasini ta'rifdan foydalanib hisoblash ancha murakkabliklarga olib keladi. Amalda quyidagi tasdiqdan foydalaniladi:

$f(z)$ funksiyaning ajralgan maxsus a nuqtaga nisbatan chegirmasi shu funksiyaning a nuqta atrofidagi Loran qatori $(z-a)^{-1}$ hadining c_{-1} koeffitsientiga teng bo`ladi.

Bundan $f(z)$ funksiyaning to`g`ri yoki qutulib bo`ladigan maxsus nuqtasiga nisbatan chegirmasi $\operatorname*{res}_{z=a} f(z)=0$ ekanligi kelib chiqadi.

№130. $f(z)=e^{\frac{1}{z}}$ funksiyaning z=0 nuqtadagi chegirmasini hisoblang.

Yechish. $f(z)=e^{\frac{1}{z}}$ funksiyaning z=0 nuqta atrofidagi Loran qatori $e^{\frac{1}{z}}=1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z}+\cdots$ ekanligini ko`rish qiyin emas. Bundan z=0 nuqta funksiyaning muhim maxsus nuqtasi va chegirmaning 1 ga tengligi kelib chiqadi.

Agar a nuqta $f(z)$ funksiyaning qutbi bo`lsa, u holda chegirmani hisoblash uchun ba'zi hollarda uni Loran qatoriga yoyish shart emas. Chegirmani hisoblashning boshqa sodda usullari mavjud:

Agar a nuqta $f(z)$ funksiyaning oddiy qutbi bo`lsa, u holda chegirmani quyidagi formula yordamida hisoblash mumkin:

$$\operatorname*{res}_{z=a} f(z)=\lim_{z\to a}(z-a)f(z) \qquad (42)$$

Agar $f(z)$ funksiya $f(z)=\frac{\varphi(z)}{\psi(z)}$ kasr shaklida berilgan bo`lib, oddiy a qutbga ega bo`lsa, (bu holda: $\varphi(a)\neq 0,\ \psi(a)=0,\ \psi'(a)\neq 0$ bo`ladi)

$$\operatorname*{res}_{z=a} f(z)=\frac{\varphi(a)}{\psi'(a)} \qquad (43)$$

bo`ladi.

Agar a nuqta $f(z)$ funksiyaning k-tartibli qutbi bo`lsa, u holda chegirmani quyidagi formula yordamida hisoblash mumkin:

$$\operatorname*{res}_{z=a} f(z)=\frac{1}{(k-1)!}\lim_{z\to a}\frac{d^{k-1}}{dz^{k-1}}((z-a)^k f(z)) \qquad (44)$$

№131. Hisoblang: a) $\operatorname*{res}_{z=a}\frac{z+1}{z^2-z^4}$; b) $\operatorname*{res}_{z=a}\frac{z+1}{z^2-z^4}$; c) $\operatorname*{res}_{z=a}\frac{z+1}{z^2-z^4}$.

Yechish. a) Funksiyani $\frac{z+1}{z^2-z^4}=\frac{z+1}{z^2(1-z^2)}$ ko`rinishda yozib olamiz, bundan z=0 ikkinchi tartibli qutb ekanligi kelib chiqadi. Bu nuqtadagi chegirmani (44) formula yordamida hisoblaymiz:

$$\underset{z=a}{res}\frac{z+1}{z^2-z^4}=\frac{1}{1!}\lim_{z\to 0}\frac{d\left(\frac{z^2(z+1)}{z^2(1-z^2)}\right)}{dz}=\lim_{z\to 0}\frac{d\left(\frac{z+1}{1-z^2}\right)}{dz}=\lim_{z\to 0}\frac{1}{(1-z)^2}=1.$$

b) Bu holda funksiyani $\frac{z+1}{z^2-z^4}=\frac{z+1}{z^2(1-z)(1+z)}$ ko`rinishda yozib olamiz, bundan z=1 oddiy qutb ekanligi kelib chiqadi. Bu nuqtadagi chegirmani (43) formula yordamida hisoblaymiz:

$$\underset{z=a}{res}\frac{z+1}{z^2-z^4}=\lim_{z\to 1}((z-1)\cdot\frac{z+1}{z^2(1-z)(1+z)})=-\lim_{z\to 1}\frac{1}{z^2})=-1.$$

c) f(z)= $\frac{z+1}{z^2(1-z)(1+z)}$ dan z=-1 qutilib bo`ladigan maxsus nuqta ekanligi ko`rinib turibdi. Bu nuqtaga nisbatan chegirma nolga teng.

Chegirmalar nazariyasining turli xil tatbiqlari quyidagi teoremaga asoslangan:

Teorema (Chegirmalar haqidagi asosiy teorema). Agar $f(z)$ funksiya Γ chiziq bilan o`ralgan $E$ yopiq sohaning, ajralgan maxsus $a_1, a_2, ..., a_k$ nuqtalaridan boshqa, hamma nuqtalarida analitik bo`lsa, u holda $f(z)$ funksiyadan Γ bo`ylab olingan integralning qiymati $\Gamma$ ichidagi barcha maxsus $a_n$ nuqtalarga nisbatan funksiya chegirmalari yig`indisining $2\pi i$ ga ko`paytirganiga teng:

$$\oint_\Gamma f(z)dz=2\pi i\sum_{n=1}^{k}\underset{z=a_n}{res}f(z) \qquad (45)$$

№132. Tenglamasi $x^2+y^2-2x-2y=0$ dan iborat bo`lgan $\Gamma$ aylana bo`ylab olingan

$$\oint_\Gamma \frac{dz}{(z-1)^2(z^2+1)}$$

integralni hisoblang.

Yechish. Berilgan Γ aylana tenglamasini $(x-1)^2+(y-1)^2=2$ ko`rinishga keltirsak, uning markazi $a=1+i$ nuqtada va radiusi $R=\sqrt{2}$ ga teng ekanini ko`ramiz. Berilgan

$$f(z)=\frac{1}{(z-1)^2(z^2+1)}$$

funksiya uchun z_1=1 -ikkinchi tartibli va $z_2=i$, $z_3=-i$ oddiy qutblar bo`ladi. Bulardan $z_3=-i$ nuqta $C$ dan tashqarida bo`lgani uchun uni e'tiborga olmaymiz.

Dastlab funksiyaning $z_1=1$ ga nisbatan chegirmasini hisoblaymiz:

$$\frac{d}{dz}((z-1)^2f(z))=\frac{d}{dz}(z^2+1)^{-1}=-(z^2+1)^{-2}2z$$

Bundan

$$\underset{z=1}{res} f(z) = -2\lim_{Z\to 1} z(z^2+1)^{-2} = -2\frac{1}{2^2} = -\frac{1}{2}.$$

Endi $z_2=i$ ga nisbatan chegirmani topamiz:

$$(z-i)\cdot f(z) = \frac{1}{(z-1)^2(z+1)}.$$

Bundan,

$$\underset{z=i}{res} f(z) = \lim_{Z\to i}\frac{1}{(z-1)^2(z+i)} = \frac{1}{(i-1)^2 2i} = \frac{1}{4}.$$

Demak,

$$\oint_C \frac{dz}{(z-1)^2(z^2+1)} = 2\pi i(\underset{z=1}{res} f(z) + \underset{z=i}{res} f(z)) = 2\pi i(-\frac{1}{2}+\frac{1}{4}) = -\frac{\pi i}{2}.$$

№133. $\oint_\Gamma \frac{\sin z}{z-i}dz$ integral hisoblansin, bu erda, Γ $|z-i| = \frac{1}{2}$ aylanadan iborat.

Yechish: $f(z)=\frac{\sin z}{z-i}$ funksiya Γ bilan chegaralangan sohada birgina oddiy $a=i$ qutbga ega bo`lgani sababli:

$$\underset{z=a}{res}\frac{\sin z}{z-i} = \frac{\sin i}{1} = \sin i = ish1.$$

Shunga ko`ra:

$$\oint_\Gamma \frac{\sin z}{z-i}dz = 2\pi i\cdot ish1 = -2\pi sh1.$$

Ko`p hollarda yuqoridagi usul yordami bilan haqiqiy argumentli funksiyalardan olingan aniq integrallarni ham hisoblash mumkin. Buning uchun avval bu integrallarni yopiq kontur bo`ylab olingan integrallarga almashtiriladi, so`ngra chegirmalar haqidagi asosiy teorema qo`llaniladi.

Misol sifatida, trigonometrik funksiyali ifodalarni integrallashni, ya'ni:

$$\int_0^{2\pi} R(\sin x, \cos x)dx \qquad (46)$$

ko`rinishdagi integrallarni o`rganamiz, bunda *R(sinx,cosx)* funksiya sinx va sosx ga nisbatan ratsional funksiya.

Buning uchun $z=e^{ix}$ almashtirish bajaramiz. U holda $\sin x = \frac{z^2-1}{2iz}; \cos x = \frac{z^2+1}{2z}$, $dx = \frac{1}{iz}dz$ bo`ladi. Ravshanki, $0\le x\le 2\pi$ da $|z|=1$ bo`ladi. Natijada (42) integral $\oint_C F(z)dz$ ko`rinishga ega bo`ladi.

Bu integral esa *F(z)* funksiyaning birlik aylana ichidagi maxsus nuqtalarga nisbatan chegirmalar yig`indisining $2\pi i$ ko`paytirilganiga teng.

№134. Ushbu $\int_0^{2\pi}\frac{dx}{a+\cos x}$ (*a*-haqiqiy son va *a*>1) integralni hisoblang.

Yechish. Yuqoridagi almashtirishlarga asosan:

$$\oint_C \frac{\frac{1}{iz}dz}{a+\frac{z^2+1}{2z}} = \frac{2}{i}\oint_C \frac{dz}{z^2+2az+1}.$$

Birlik aylana ichidagi $\frac{1}{z^2+2az+1}$ funksiyaning maxsus nuqtalari $z_{1,2}=-a\pm\sqrt{a^2-1}$ nuqtalar oddiy qutblar bo\`ladi. Bu qutblaridan $a>1$ bo\`lgani uchun faqat $z_1=-a+\sqrt{a^2-1}$ C aylana ichida yotadi. Shu nuqtadagi chegirmani hisoblaymiz:

$$\lim_{я\to я_1}(z-z_1)\frac{1}{(z-z_1)(z-z_2)} = \lim_{я\to я_1}\frac{1}{z-z_2} = \frac{1}{z_1-z_2} = \frac{1}{2\sqrt{a^2-1}}.$$

Demak,

$$\int_0^{2\pi}\frac{dx}{a+\cos x} = \frac{2\pi}{\sqrt{a^2-1}}.$$

Chegirmalar yordamida ayrim xosmas integrallarni hisoblash mumkin.

Faraz qilaylik *f(z)* funksiya haqiqiy o\`qdan yuqorida yotuvchi chekli sondagi $a_1, a_2, ..., a_n$ maxsus nuqtalardan tashqari barcha yuqori yarim tekislikda va haqiqiy o\`qda analitik funksiya bo\`lsin.

Shu bilan birga cheksiz uzoqlashgan nuqta *f(z)* funksiya uchun kamida ikkinchi tartibli nol bo\`lsin. Bu shartlar bajarilganda quyidagi formula o\`rinli bo\`ladi:

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i\sum_{k=1}^{n}\underset{z=a_K}{res} f(z) \qquad (47)$$

№135. $\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^3}$ integralni hisoblang.

Yechish. $f(z)=\frac{1}{(z^2+1)^3}$ funksiya uchun $z=\infty$ nuqta oltinchi tartibli noldir; $z=\pm i$ nuqtalar bu funksiya uchun uchinchi tartibli qutb bo\`lib, $z=i$ yuqori yarim tekislikda yotadi. Shu nuqtaga nisbatan chegirma

$\underset{z=i}{res}\frac{1}{(z^2+1)^3} = \frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}\frac{(z-i)^3}{(z^2+1)^3} = \frac{3}{16i}$ ga teng.

Demak, $\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^3} = 2\pi i\,\underset{z=i}{res}\frac{1}{(z^2+1)^3} = 2\pi i\frac{3}{16i} = \frac{3\pi}{8}$

№136. Quyidagi funksiyalarning har bir maxsus nuqtalariga nisbatan chegirmalarini hisoblang.

a) $f(z)=\frac{z^2+1}{z^2(1-z)}$; b) $f(z)=\frac{\sin\pi z}{(z-1)^3}$; c) $f(z)=tgz$;

d) $f(z)=\frac{z}{e^z-1}$; e) $f(z)=\frac{\sin z}{(z-\pi)^3}$; f) $f(z)=z^3\cos\frac{1}{z-3}$.

№137. Quyidagi integrallarni hisoblang.

a) $\oint\limits_{|z|=1} z\,tg\,\pi z\,dz$; b) $\oint\limits_{|z-3|=2} \frac{zdz}{(z-2)^2(z+3)}$; c) $\oint\limits_{|z|=2} \frac{\sin \pi z}{z^2-2}$;

d) $\oint\limits_{|z|=\frac{1}{3}} z^2 \sin\frac{1}{z} dz$; e) $\oint\limits_{|z|=1} (z-1)e^{2/z} dz$; f) $\oint\limits_{|z-1|=\sqrt{2}} \frac{dz}{z^4+1}$.

№138. Quyidagi integrallarni hisoblang.

a) $\int\limits_0^{2\pi} \frac{dx}{6+5\cos x}$; b) $\int\limits_0^{2\pi} \frac{dx}{\sin x+3}$; c) $\int\limits_0^{2\pi} \frac{cos^2 xdx}{1-0{,}5\sin^2 x}$;

d) $\int\limits_0^{2\pi} \frac{\cos xdx}{1-2p\sin x+p^2}$, $(0<p<1)$; e) $\int\limits_0^{2\pi} \frac{\cos 2xdx}{1-2p\cos x+p^2}$, $(p\geq 1)$.

№139. Quyidagi xosmas integrallarni hisoblang.

a) $\int\limits_{-\infty}^{\infty} \frac{dx}{x^4+1}$; b) $\int\limits_{-\infty}^{\infty} \frac{xdx}{(x^2+4x+13)^2}$; c) $\int\limits_0^{\infty} \frac{xdx}{(x^2+2x+2)^2}$; d) $\int\limits_0^{\infty} \frac{x^2+1}{x^4+1} dx$.

Javoblar

№6. a) 11+i. b) -2+8i. c) 5. d) 6-8i. e) 2,5-2,5i. f) 5/3-2i. g) -0,8+3,4i. i). 3-4i.
№7. a) |z|=5, argz=π-arctg0,75. b) |z|=4, argz=π/3. c) |z|=3, argz=0. d) |z|=1, argz=4π/5. №8. a) -3=3(cosπ+isinπ). b) 4i=4(cos(π/2)+isin(π/2)).
c) $-\sqrt{2}+\sqrt{2}\,i$=2(cos(3π/4)+isin(3π/4)). d) $-1-\sqrt{3}\,i$=2(cos(4π/3)+isin(4π/3)).
№9. a) 2985984. b) $13\sqrt{13}$(cos(3arctg1,5)+isin(3arctg1,5)). c) 4096. d) $i/2^{30}$. e) i.
№10. a) $\cos\frac{3\pi+4k\pi}{8}+i\sin\frac{3\pi+4k\pi}{8}, k=0,1,2,3.$ b) 1; $\frac{-1\pm i\sqrt{3}}{2}$.
c) $2(\cos\frac{5\pi/3+2k\pi}{3}+i\sin\frac{5\pi/3+2k\pi}{3}), k=0,1,2.$
d) $\sqrt{2}(\cos\frac{3\pi/4+2k\pi}{2}+i\sin\frac{3\pi/4+2k\pi}{2}), k=0,1.$
e) $2(\cos\frac{\pi+2k\pi}{4}+i\sin\frac{\pi+2k\pi}{4}), k=0,1,2,3.$
№17. (A+iB)$\bar{z}$+(A-iB)z+2C=0.
№18. a) $\bar{z}$-z=0 va $\bar{z}$+z=0; b) 2(z+$\bar{z}$)+i(z-$\bar{z}$)+2b=0.
№19. a) z$\bar{z}$+2(z+$\bar{z}$)+i(z-$\bar{z}$)=0. b) $z^2+\bar{z}^2=2a^2$.c) 2(z-$\bar{z}$)-i$(z+\bar{z})^2$=0.
№20. a) $x^2+y^2-4x=0$- aylana. b) $x^2-y^2=4$-giperbola. c) $x^2-y^2=0,5$- giperbola.
d) $(x+1)^2+(y-0,5)^2=2,25$- aylana. e) $\frac{x^2}{3}+\frac{y^2}{4}=1$-ellips. f) $y^2=2x+1$- parabola.
g) x+y=0-to`g`ri chiziq. №21. a) $x^2+(y-2)^2=1$- aylana. b) y=x/3, x≥0 nur. c) $y=x^2$ - parabola. d) y=1/x – giperbolaning x>0 bolgan qismi.
№22. a) 0<argz<π/2. b) 1≤Imz≤2. c) 5π/4<argz<4π/3. d) |z+2-3i|≤3
№23. a) y=1 to`g`ri chiziqdan pastda joylashgan nuqtalar to`plami.
b) x=1 to`g`ri chiziqdan o`ngda joylashgan nuqtalar to`plami.
c) markazi z_0=i nuqtada radiusi r=1 bo`lgan yopiq doira.
d) markazi z_0=2 nuqtada, radiuslari r=1 va R=2 bo`lgan aylanalar bilan chegaralangan ochiq halqa. e) markazi $z_0$=1-i nuqtada, radiusi R=1 bo`lgan teshik doira. f) y=0 va y=√3x (x>0) nurlar orasidagi nuqtalar to`plami (burchak). g) Rez>0 yarimtekislik
№30. a) Rew=x^2-y^2; Imz=2+2xy. b) Rew=$\frac{y}{(x+2)^2+y^2}$; Imz=$\frac{x+2}{(x+2)^2+y^2}$
c) Rew=$x^2-y^2-\frac{x}{x^2+y^2}$ Imz=2xy+$\frac{y}{x^2+y^2}$. d) Rew=-y+$\sqrt{x^2+y^2}$; Imz=x.
e) Rew=xcosα+ysinα+1; Imz= ycosα+xsinα. f) Rew=$\frac{x^2-y^2}{x^2+y^2}$; Imz=-$\frac{2xy}{x^2+y^2}$.
№31. a) 2i. b) 4. c) –i. d) 24i. №32. a) |w|=1 birlik aylananing birinchi va ikkinchi chorakdagi qismlari; b) |w|=1 birlik aylananing birinchi, ikkinchi va uchinchi chorakdagi qismlari; c) |w|=1 birlik aylana.
№43. a) bo`ladi. b) bo`lmaydi. c) bo`lmaydi. d) bo`lmaydi.

№44. a) mavjud emas. b) mavjud. c) mavjud emas.

№45. a) f(z)=iz^3+C. b) f(z)=z^3. c) f(z)=z(z+2). d) f(z)=$\frac{z-1}{z}$. №48. a) k=3; φ=0. b) k=6; φ=π/2. c) k=75; φ=arctg(24/7). d) k=12; φ=π/3.

№49. a) |z-2|<0,5 doiraning ichi torayib, tashqarisi kengayib akslanadi.
b) |z|<1 doiraning ichi kengayib, tashqarisi torayib akslanadi.
c) |z|<0,5 doiraning ichi torayib, tashqarisi kengayib akslanadi.
d) |z|<$\sqrt[4]{2}$ doiraning ichi kengayib, tashqarisi torayib akslanadi.

№54. Uchlari A_1=1-2i; B_1=0; C_1=3+4i, D_1=3-i nuqtalarda bo\`lgan to\`rtburchak.

№55. |w+2+9i|<3 doira. №56. Imw<4 soha. №57. $w_1=e^{1,5\pi}z$, w=w_1+3+4i. berilgan nuqtani koordinatalar boshiga nisbatan 3π/2 burchakka burish, hosil bo\`lgan nuqtani 3+4i vektor qadar parallel ko\`chirish. z_0=3,5+0,5i -qo\`zg\`almas nuqta. №60. $w=\frac{z-i}{z-2+i}$. №61. w=iz. №62. birlik doiraga; z=0 nuqta w=0,5 nuqtaga akslanadi. №63. |w+1|<1. №64. Rew>-1 va |w-2|>2√2 shartlarni qanoatlantiruvchi nuqtalar to\`plami. №67. a) e^2, 1; b) e^3, 2π-4;
c) e^3, 2π-2; d) e^4, 3; №71. Recosz= cosxchy; Imcosz=-sinxshy.

№72. a) Re cos$\frac{i\pi}{2}$=ch$\frac{\pi}{2}$, Im cos$\frac{i\pi}{2}$=0. *b)* Retg(1-i)=$\frac{tg1}{1+\cos^2 1sh^2 1}$;

Imtg(1-i)=$\frac{sh1ch1}{\cos^2 1+sh^2 1}$. *c)* Resin(2-3i)=sin2ch3; Imsin(2-3i)=-cos2sh3.

№74. z=π/2+kπ, k∈Z. №75. haqiqiy (mavhum) qiymatlar qabul qiladi? z=x+ikπ da haqiqiy; z=x+i(kπ+π/2) da mavhum qiymatlar qabul qiladi, k-butun son. №76. z=kπ+iy da haqiqiy, z=(π/2+kπ)+iy da mavhum qiymatlar qabul qiladi, k-butun son. №77. *w=sinz* funktsiya kompleks tekislikning qaysi nuqtalarida haqiqiy (mavhum) qiymatlar qabul qiladi? z=(π/2+kπ)+iy da haqiqiy, z=kπ+iy da mavhum qiymatlar qabul qiladi, k-butun son.

№80. a) Ln(-5)=ln5 +i(3π/2 +2kπ), k=0, ±1, ±2, …
b) Ln(–1+$i\sqrt{3}$)=ln2+i(2π/3 +2kπ), k=0, ±1, ±2, …
c) Ln(1-$i\sqrt{3}$)=ln2+i(5π/3 +2kπ), k=0, ±1, ±2, …
d) Ln(–3-$i\sqrt{3}$)=0,5ln12+i(7π/6 +2kπ), k=0, ±1, ±2, …
e) Ln(-*i)*=i(3π/2 +2kπ), k=0, ±1, ±2, …
f) Ln(–1-$i\sqrt{3}$)=ln2+i(4π/3 +2kπ), k=0, ±1, ±2, …

№87. 4,5-2,5i. №88. 0. №89. π. №90. -0,5π. №91. 2π. №99. a) -1. b) 1,6(i-1). c) e(cos1+isin1)-1. d) –sin1. e) cos1-sin1-ie^{-1}. f) -7e^{-2}+(3-2i)e^{i}. g) 1-cos1+i(sin1-1). i) 0,25(cos(2+2i)-1). №100. a) πi. b) -πe. c) πsh1. d) 0,5πicos1. e) 2πi. f) -2iπ/29. №101. a) -6πi. b) 2πi. c) -πi/27. d) -2πi. №108. a) R=∞. b) R=1/e. c) R=0,2. d) R=1. e) R=∞. f) R=1.№109. a) cos4z=$\sum_{n=0}^{\infty}(-1)^n\frac{4^{2n}\cdot z^{2n}}{(2n)!}$;

b) cos4z=$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(z-\pi)^{2n}}{(2n)!}$. №110. a) cos2z=$\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}\cdot z^{2n}}{(2n)!}$;

b) sinzcosz= $\frac{1}{2}\cdot\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n-1}\cdot z^{2n-1}}{(2n-1)!}$ .c) e^z=$\sum_{n=0}^{\infty}\frac{e^{-1}(z+1)^n}{n!}$.

d)$\frac{1}{z+i}$= $\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{(1+i)^{n+1}}$. №111. a) $z^2+\sum_{n=2}^{\infty}(-1)^{n+1}\left(\frac{z^{n+1}}{n}+\frac{z^n}{n}\right)$, |z|<1.

b) $\sum_{n=1}^{\infty}(-1)^n nz^{2n-2}$, |z|<1. c) $\sum_{n=1}^{\infty}nz^{3n-1}$, |z|<1. d) $\sum_{n=1}^{\infty}(-1)^n\frac{z^n}{n}$, |z|<1.

№112. a)$\frac{z}{\cos z}$; $z+\frac{1}{2}z^3+\frac{5}{24}z^5+...$; |z|<0,5π.

b) $\frac{z}{\ln(1-z)}$. $-1+\frac{1}{2}z+\frac{1}{12}z^2+...$; |z|<1. №118. a) 0<|z|<+∞, $e^{\frac{3}{z}}$. b) |z|>2, $\frac{2-i}{(z-2)(z-i)}$. c) 0<|z+2i|<3, $\frac{1}{-z^2+(3-4i)z+4+6i}$. d) 2<|z|<3, $\frac{5}{6+z-z^2}$.

№119. a) $-\frac{1}{2-i}\left(\sum_{n=1}^{\infty}\frac{i^n}{z^n}+\sum_{n=0}^{\infty}\frac{z^n}{2^n}\right)$. b) $-\sum_{n=-1}^{\infty}z^n$. c) $-\frac{i}{2(z-i)}+\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\frac{(z-i)^n}{(2i)^n}$.

d) $\frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{(z+2)^n}-\frac{3}{2}\sum_{n=0}^{\infty}\frac{(z+2)^n}{3^{n+1}}$. №120. a) $-\frac{1}{4}\sum_{n=-1}^{\infty}\frac{(z-1)^n}{2^n}, 0<|z-1|<2;$ $\frac{1}{2}\sum_{n=1}^{\infty}\frac{2^n}{(z-1)^{n+1}}, |z-1|>2$.

b)$\frac{1}{2}\sum_{n=0}^{\infty}z^n\left(1-\frac{1}{3^{n+1}}\right),\quad |z|<1;$ $\frac{1}{2}\left(-\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}-\sum_{n=-\infty}^{-1}z^n\right),\quad 1<|z|<3,$

$\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}(3^n-1),\quad |z|>3$. c) $-\sum_{n=-1}^{\infty}z^n$,0<|z|<1; $\sum_{n=-\infty}^{-2}z^n,\ |z|>1$.

№126. a) 3-tartibli. b) 5-tartibli.
№127. a) z=0 – 3-tartibli; z=2πk, k butun son, oddiy nol.
b) z=±i, z=πk, k butun son, oddiy nol. c) z=±2i – 2- tartibli nollar.
d) z=±π– 2- tartibli nollar, z=πk, k=±2, ±3, … oddiy nollar.
№128. a) z=0 2-tartibli qutb. b) z=0 qutilib bo`ladigan maxsus nuqta.
c) z=2πk, k∈Z, oddiy qutb. d) z=0 2-tartibli qutb. e) z=0 oddiy qutb.
f) z=2πk - qutilib bo`ladigan maxsus nuqta, z=π(2k+1)- 2-tartibli qutb,
№136. a) chegf(1)=-2; chegf(0)=1. b) chegf(1)=0. c) chegf(π/2+πk)=-1, k∈Z.
d) chegf(0)=0. e) chegf(π)=0. f) chegf(3)=13$\frac{13}{24}$. №137. a) 0. b) 6πi/25. c) 0. d) -2πi. e) 0. f) -πi√2/2. №138. a) 2π/√11.b) $\frac{\pi\sqrt{2}}{2}$. c) $\pi(2-\sqrt{2})$. d) 0. e) 2π/(p^2(p^2-1)). №139. a) $\frac{\pi\sqrt{2}}{2}$. b)-π/27. c) π/2. d)π/√2.

2-bob. *FURYE QATORI. FURYE INTEGRALI.*

1-§. Furye qatori, Furye koeffitsientlari.

Quyidagi ko`rinishdagi

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n cosnx + b_n \sin nx \qquad (1)$$

funksional qator trigonometrik qator deyiladi. $a_0, a_1, b_1, \ldots, a_n, b_n$ $(n=1, 2, 3, \ldots)$ o`zgarmas sonlar trigonometrik qatorning koeffitsientlari deyiladi.

Agar (1) qator yaqinlashuvchi bo`lsa, uning yig`indisi davri 2π bo`lgan *f(x)* davriy funksiya bo`ladi, chunki *sinnx* va *cosnx* lar davri 2π bo`lgan davriy funksiyalardir: $f(x+2\pi)=f(x)$.

Davri 2π bo`lgan *f(x)* davriy funksiya berilgan bo`lsin. Qanday shartlar bajarilganda *f(x)* funksiya uchun berilgan funksiyaga yaqinlashuvchi trigonometrik qator topish mumkin?- degan savolga javob berish maqsadida koeffitsientlari

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \qquad (2) \qquad n=0, 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \qquad (3) \qquad n=1, 2, 3, \ldots$$

formulalar bilan aniqlangan (1) trigonometrik qatorni qaraymiz, bu qator *f(x)* funksiyaning Furye qatori deyiladi. (2), (3) formulalar bilan aniqlangan a_n, b_n koeffitsientlar esa Furye koeffitsientlari deyiladi.

Agar

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \qquad (4)$$

tenglik o`rinli bo`lsa, funksiya Furye qatoriga yoyilgan deyiladi.

Quyidagi teorema o`rinli:

Teorema. Agar davri 2π ga teng bo`lgan *f(x)* davriy funksiya [-π,π] kesmada bo`lakli monoton va chegaralangan bo`lsa, bu funksiya uchun tuzilgan Furye qatori shu kesmaning barcha nuqtalarida yaqinlashuvchi bo`ladi. Qatorning *S(x)* yig`indisi funksiyaning uzluksizlik nuqtalaridagi qiymatiga teng, funksiyaning uzilish nuqtalarida qatorning yig`indisi funksiyaning o`ng va chap limitlarining o`rta arifmetik qiymatiga teng bo`ladi, ya'ni

$$S(x)=\frac{f(x-0)+f(x+0)}{2}.$$

№1.Quyidagi berilgan funksiyalarni qaralayotgan intervalda Furye qatoriga yoying.

a) $f(x)=x/2$, $(0,2\pi)$

b) $f(x)=|x|$, $(-\pi,\pi)$

c) $f(x)=\begin{cases} 0, & \text{agar } -\pi \le x < 0, \\ 0, & \text{agar } a < x \le \pi \ , \\ 1, & \text{agar } 0 < x < a, \\ 1/2, & \text{agar } x = 0, x = a \end{cases}$

d) $f(x)=\begin{cases} ax, & \text{agar } -\pi < x \le 0, \\ bx, & \text{agar } 0 < x \le \pi \end{cases}$ bu erda a va b lar o\`zgarmas sonlar.

Yechish. №1. a) $f(x)=x/2$ funksiya $0<x<2\pi$ oraliqda yuqoridagi teorema shartlarini qanoatlantiradi, ya'ni berilgan funksiya bo\`lakli monoton va chegaralangan. Shuning uchun bu funksiyani Furye qatoriga yoyish mumkin.

(2), (3) formulalarga asosan Furye koeffitsientlarini topamiz:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi}\frac{x}{2}dx = \frac{1}{2\pi}\cdot \left.\frac{x^2}{2}\right|_0^{2\pi} = \frac{1}{4\pi}(4\pi^2 - 0) = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_0^{2\pi}\frac{x}{2}\cos nxdx = \frac{1}{2\pi}\int_0^{2\pi} x\cos nxdx = \left|\begin{matrix} u = x, du = dx \\ dv = \cos nxdx \\ v = \frac{1}{n}\sin nx \end{matrix}\right| =$$

$$= \frac{1}{2\pi}\left(\left.\frac{x}{n}\sin nx\right|_0^{2\pi} - \frac{1}{n}\int_0^{2\pi}\sin nxdx\right) = \left.\frac{1}{2\pi n^2}\cos nx\right|_0^{2\pi} = \frac{1}{2\pi n^2}(\cos 2n\pi - 1) = 0,$$

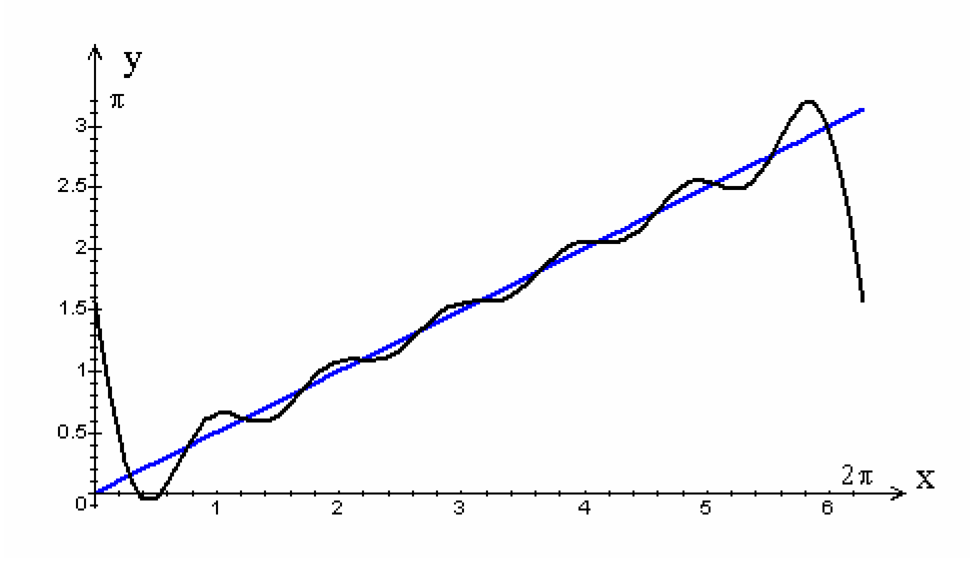
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx = \frac{1}{\pi}\int_0^{2\pi}\frac{x}{2}\sin nxdx = \frac{1}{2\pi}\int_0^{2\pi} x\sin nxdx = \left|\begin{matrix} u = x, du = dx \\ dv = \sin nxdx \\ v = -\frac{1}{n}\cos nx \end{matrix}\right| =$$

$$= \frac{1}{2\pi}\left(\left.-\frac{x}{n}\cos nx\right|_0^{2\pi} + \frac{1}{n}\int_0^{2\pi}\cos nxdx\right) = \frac{1}{2\pi}\left(\left(-\frac{2\pi}{n}\cos 2n\pi - 0\right) + \left.\frac{1}{n^2}\sin nx\right|_0^{2\pi}\right) = -\frac{1}{n}.$$

Topilgan a_0, a_n, b_n koeffitsientlarni (4) ga qo\`ysak quyidagiga ega bo\`lamiz:

$$\frac{x}{2} = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(-\frac{1}{n}\sin nx\right) = \frac{\pi}{2} - \frac{\sin x}{1} - \frac{\sin 2x}{2} - \frac{\sin 3x}{3} - \ldots$$

hosil bo\`lgan Furye qatori $(0,2\pi)$ oraliqda berilgan funksiyaga yaqinlashadi, oraliqning chekka nuqtalarida ya'ni $x=0$, $x=2\pi$ nuqtalarda qator yig\`indisi $\frac{\pi}{2}$ ga teng bo\`ladi. Berilgan funksiya va hosil bo\`lgan qatorning birinchi oltita hadi yig\`indisi grafiklari quyidagicha bo\`ladi:



b) $f(x)=|x|$ funksiya $[-\pi;\pi]$ kesmada yuqoridagi teorema shartlarini qanoatlantiradi. Furye koeffitsientlarini topamiz:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\left(\int_{-\pi}^{0}f(x)dx+\int_{0}^{\pi}f(x)dx\right)=\frac{1}{\pi}\left(\int_{-\pi}^{0}(-x)dx+\int_{0}^{\pi}xdx\right)=$$

$$=\frac{1}{\pi}\left(-\frac{x^2}{2}\Bigg|_{-\pi}^{0}+\frac{x^2}{2}\Bigg|_{0}^{\pi}\right)=\pi,\qquad a_0=\pi\,.$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left(\int_{-\pi}^{0}f(x)\cos nxdx+\int_{0}^{\pi}f(x)\cos nxdx\right)=$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x\cos nxdx)+\frac{1}{\pi}\int_{0}^{\pi}x\cos nxdx=\begin{vmatrix}u=x,du=dx\\ dv=\cos nxdx\\ v=\frac{1}{n}\sin nxdx\end{vmatrix}=$$

$$-\frac{1}{\pi}\left(\frac{1}{n}\sin nx\Bigg|_{-\pi}^{0}-\frac{1}{n}\int_{-\pi}^{0}\sin nxdx\right)+\frac{1}{\pi}\left(\frac{x}{n}\sin nx\Bigg|_{0}^{\pi}-\right.$$

$$\left.-\frac{1}{n}\int_{0}^{\pi}\sin nxdx\right)=-\frac{1}{\pi n^2}\cos nx\Bigg|_{-\pi}^{0}+\frac{1}{\pi n^2}\cos nx\Bigg|_{0}^{\pi}=\frac{1}{\pi n^2}(\cos n\pi-1-1+\cos n\pi)=$$

$$=\frac{2}{\pi n^2}(\cos n\pi-1)=\begin{cases}0, & \text{agar}\quad n=2k,\\ -\dfrac{4}{\pi n^2}, & \text{agar}\quad n=2k+1\end{cases},$$

$$b_n=\frac{1}{\pi}\left(\int_{-\pi}^{0}(-x\sin nxdx)+\int_{0}^{\pi}x\sin nxdx\right)=\left|\begin{array}{l}u=x, du=dx\\ dv=\sin nxdx\\ v=-\frac{1}{n}\cos nx\end{array}\right|=$$

$$=\frac{1}{\pi}\left(\frac{x}{n}\cos nx\Big|_{-\pi}^{0}-\frac{1}{n}\int_{-\pi}^{0}\cos nxdx\right)++\frac{1}{\pi}\left(-\frac{x}{n}\cos nx\Big|_{0}^{\pi}+\right.$$

$$\left.+\frac{1}{n}\int_{0}^{\pi}\cos nxdx\right)=\frac{1}{n}\cos n\pi-\frac{1}{n^2}\sin nx\Big|_{-\pi}^{0}-\frac{1}{n}\cos n\pi+\frac{1}{n^2}\sin nx\Big|_{0}^{\pi}=0$$

Topilgan Furye koeffitsientlarini (2) ga qo`ysak, quyidagiga ega bo`lamiz:

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1}+\frac{\cos 3x}{3^2}+...+\frac{\cos(2n-1)x}{(2n-1)^2}+...\right).$$

Hosil bo`lgan qator qaralayotgan oraliqning barcha nuqtalarida berilgan funksiyaga yaqinlashadi.

c) $f(x)=\begin{cases}0, & \text{agar } -\pi\le x<0,\\ 0, & \text{agar } a<x\le\pi\ ,\\ 1, & \text{agar } 0<x<a,\\ 1/2, & \text{agar } x=0, x=a\end{cases}$

funksiya teoremaning barcha shartlarini qanoatlantiradi.

Furye koeffitsientlarini topamiz:

$$a_0=\frac{1}{\pi}\left(\int_{-\pi}^{0}0dx+\int_{0}^{a}1dx+\int_{a}^{\pi}0dx\right)=\frac{a}{\pi},\quad a_0=\frac{a}{\pi}.$$

$$a_n=\frac{1}{\pi}\int_{0}^{a}\cos nxdx=\frac{1}{n\pi}\sin nx\Big|_{0}^{a}=\frac{\sin na}{n\pi},\quad a_n=\frac{\sin na}{n\pi}.$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\int_{0}^{a}\sin nxdx=-\frac{1}{n\pi}\cos nx\Big|_{0}^{a}=\frac{1-\cos na}{n\pi},$$

$$b_n=\frac{1-\cos na}{n\pi}.$$

Bu holda berilgan funksiya uchun tuzilgan Furye qatori quyidagi ko`rinishda bo`ladi:

$$f(x)=\frac{a}{2\pi}+\sum_{n=1}^{\infty}\left(\frac{\sin na}{n\pi}\cos nx+\frac{1-cosna}{n\pi}\sin nx\right).$$

d) $f(x)=\begin{cases}ax, & \text{agar } -\pi<x\le 0,\\ bx, & \text{agar } 0<x\le\pi\end{cases}$ bu erda *a, b* lar o`zgarmas sonlar bo`lib, $a<0$, $b>0$ bo`lsin.

Furye koeffitsientlarini topamiz:

$$a_0=\frac{1}{\pi}\left(\int_{-\pi}^{0}axdx+\int_{0}^{\pi}bxdx\right)=\frac{1}{\pi}\left(\frac{ax^2}{2}\Big|_{-\pi}^{0}+\frac{bx^2}{2}\Big|_{0}^{\pi}\right)=\frac{1}{\pi}\left(-\frac{a\pi^2}{2}+\frac{b\pi^2}{2}\right)=\frac{(b-a)\pi}{2}$$

$$a_n=\frac{1}{\pi}\left(\int_{-\pi}^{0}ax\cos nxdx+\int_{0}^{\pi}bx\cos nxdx\right)=\begin{vmatrix}u=x, du=dx\\ dv=\cos nxdx\\ v=\frac{1}{n}\sin nx\end{vmatrix}=$$

$$=\frac{a}{\pi}\left(\frac{x}{n}\sin nx\Big|_{-\pi}^{0}-\frac{1}{n}\int_{-\pi}^{0}\sin nxdx\right)+\frac{b}{\pi}\left(\frac{x}{n}\sin nx\Big|_{0}^{\pi}-\frac{1}{n}\int_{0}^{\pi}\sin nxdx\right)=$$

$$=\frac{a}{\pi n^2}\cos nx\Big|_{-\pi}^{0}+\frac{b}{\pi n^2}\cos nx\Big|_{0}^{\pi}=\frac{a}{\pi n^2}(1-\cos n\pi)+\frac{b}{\pi n^2}(\cos n\pi-1)=$$

$$=\frac{a-b}{\pi n^2}(1-\cos n\pi)=\frac{a-b}{\pi n^2}(1-(-1)^2)=\begin{cases}0, & \text{agar } n \text{ guft bo'lsa,}\\ \frac{2(a-b)}{\pi n^2}, & \text{agar } n \text{ toq bo'lsa}\end{cases}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{0}ax\sin nxdx+\frac{1}{\pi}\int_{0}^{\pi}bx\sin nxdx=\begin{vmatrix}u=x, du=dx\\ dv=\sin nxdx\\ v=-\frac{1}{n}\cos nx\end{vmatrix}=\frac{a}{\pi}\left(\frac{x}{n}\cos nx\Big|_{-\pi}^{0}+\right.$$

$$\left.+\frac{1}{n}\int_{-\pi}^{0}\cos nxdx\right)+\frac{b}{\pi}\left(-\frac{x}{n}\cos nx\Big|_{0}^{\pi}+\frac{1}{n}\int_{0}^{\pi}\cos nxdx\right)=-\frac{a}{n}\cos n\pi+\frac{a}{\pi n^2}\sin nx\Big|_{-\pi}^{0}-$$

$$-\frac{b}{n}\cos n\pi+\frac{b}{\pi n^2}\sin nx\Big|_{0}^{\pi}=-\frac{a+b}{n}\cos n\pi=(-1)^{n+1}\frac{a+b}{n}.$$

Bu holda berilgan funksiya uchun tuzilgan Furye qatori quyidagi ko`rinishda yoziladi:

$$f(x)=\frac{b-a}{4}\pi+\frac{2(a-b)}{\pi}\left(\frac{\cos x}{1}+\frac{\cos 3x}{3^2}+...\right)+(a+b)\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-...\right). \quad (5)$$

Hosil kilingan qator (-π,π) oraliqda berilgan funksiyaga yaqinlashadi.

x=-π da $\quad \frac{f(-\pi-0)+f(-\pi+0)}{2}=\frac{(b-a)\pi}{2}$,

x=π da $\quad \frac{f(\pi-0)+f(\pi+0)}{2}=\frac{(b-a)\pi}{2}$ bo`ladi.

Agar a=-1, b=1 deb olsak (5) dan 2 misolning yechimi kelib chiqadi.

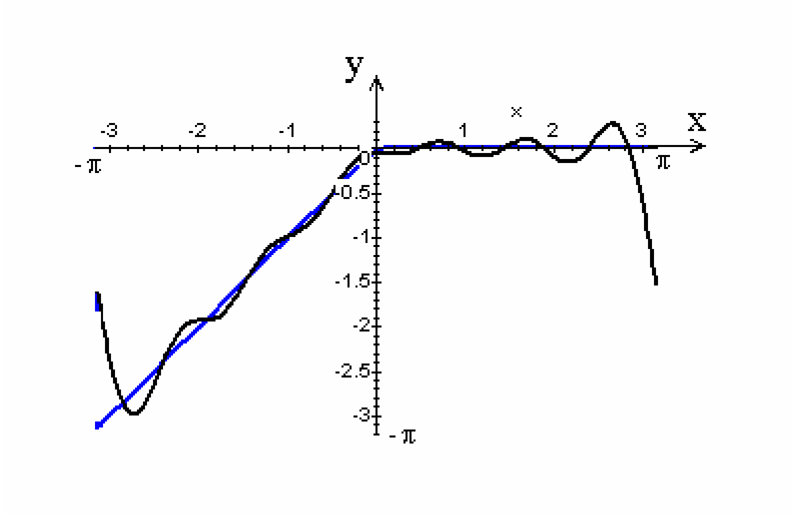
Agar a=0 , b=1 deb olsak funksiya

$$f(x)=\begin{cases}0, & -\pi<x\le 0,\\ x, & 0<x<\pi\end{cases}$$

ko`rinishga keladi, (5) dan bu funksiya uchun tuzilgan Furye qatori osongina kelib chiqadi:

$$f(x)=\frac{\pi}{4}-\frac{2}{\pi}(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+...)+(\sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-...).$$

Agar *a*=1, *b*=0 deb olsak funksiya *f(x)*= $\begin{cases} x, & -\pi < x \le 0, \\ 0, & 0 < x < \pi \end{cases}$ ko`rinishga keladi, bu funksiya va unga mos Furye qatorining birinchi oltita hadi yig`indisining grafiklari quyidagicha bo`ladi:



2-§. Juft va toq funksiyalar uchun Furye qatorlari

Agar *f(x)* funksiya [-*a,a*] kesmada juft funksiya bo`lsa, ya'ni *f(-x)=f(x),* bo`lsa,

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx \qquad (6)$$

bo`ladi.

Agar *f(x)* funksiya [-*a;a*] da toq funksiya bo`lsa, yaьni *f(-x)=-f(x)* bo`lsa,

$$\int_{-a}^{a} f(x)dx = 0 \qquad (7)$$

bo`ladi.

Agar *f(x)* juft funksiya Furye qatoriga yoyilsa, *f(x)sinnx* ko`paytma toq, *f(x)cosnx* juft funksiya bo`ladi. (6) va (7) ga asosan Furye koeffitsientlari quyidagi

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx, \quad a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nxdx, \quad b_n = 0 \qquad (8)$$

ko`rinishga keladi.

Bu holda juft funksiyaning Furye qatori faqat kosinuslarni o`z ichiga oladi.

Agar *f(x)* toq funksiya Furye qatoriga yoyilsa, *f(x)cosnx* ko`paytma toq funksiya, *f(x)sinnx* ko`paytma juft funksiya bo`ladi.

Bu holda (6) va (7) ga asosan:

$$a_0 = 0, a_n = 0, b_n = \frac{2}{\pi}\int_{0}^{a} f(x)\sin nxdx. \qquad (9)$$

Demak, toq funksiya Furye qatoriga yoyilsa, Furye qatori “faqat sinuslarni” o`z ichiga oladi.

№2. Quyidagi funksiyalarni qaralayotgan oraliqda Furye qatoriga yoying.

a) $f(x)=x,\ -\pi<x<\pi;$

b) $f(x)=|x|,\ -\pi<x<\pi;$

c) $f(x)=|\sin x|,\ -\pi\le x\le\pi,$ hosil qilingan yoyilmadan foydalanib, quyidagi

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)(2n+1)}+\ldots$$

qator yig`indisini toping

d) $f(x)=x^2,\ -\pi\le x\le\pi,$ hosil qilingan yoyilmadan foydalanib, quyidagi qatorlarning yig`indilarini toping:

1) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$ 2) $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$

Yechish. a) $f(x)=x,\ (-\pi;\pi)$ oraliqda toq bo`lgani uchun (9) ga asosan Furye koeffitsientlaridan faqat b_n ni topamiz:

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx=\frac{2}{\pi}\int_0^{\pi} x\sin nx dx=\left|\begin{matrix} u=x, du=dx \\ dv=\sin nx dx \\ v=-\frac{1}{n}\cos nx \end{matrix}\right|=$$

$$=\frac{2}{\pi}\left(-\frac{x\cos nx}{n}\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos nx dx\right)=\frac{2}{\pi}\left(-\frac{\pi\cos n\pi}{n}+\frac{1}{n^2}\sin nx\Big|_0^{\pi}\right)=-\frac{2\cos n\pi}{n},$$

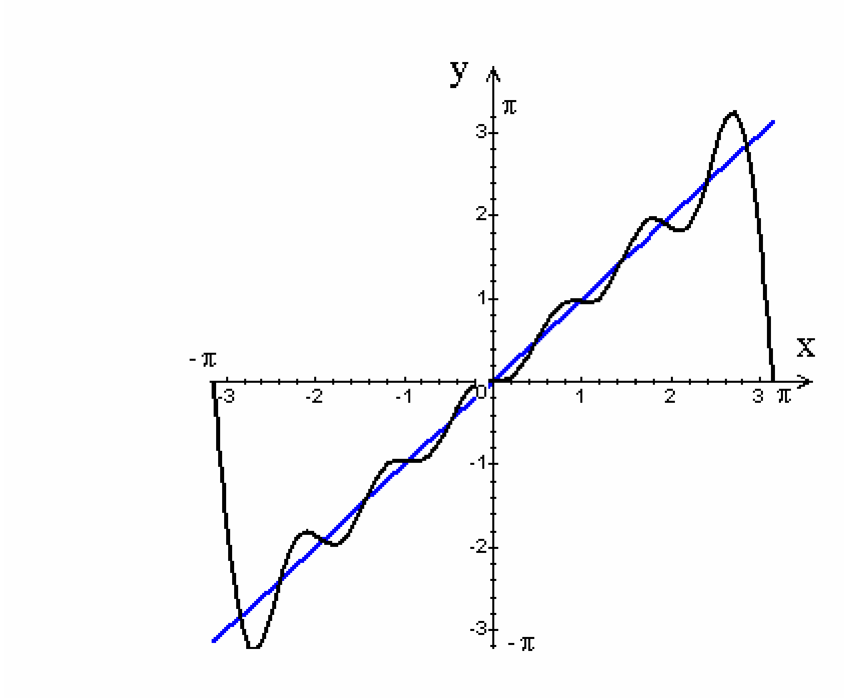
$b_n=-\frac{2\cos n\pi}{n},$ $n=1, 2, 3, \ldots$ da $b_1=2,\ b_2=-2/2,\ b_3=2/3,\ b_4=-2/4,\ldots$

Bu holda berilgan funksiya uchun Furye qatori quyidagi

$$f(x)=2(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\ldots)$$

ko`rinishda bo`ladi. Bu tenglik (-π;π) oraliqda o`rinli $x=-\pi, x=\pi$ da qator yig`indisi 0 ga teng bo`ladi.

Berilgan funksiya va unga mos Furye qatorining dastlabki oltita hadi yig`indisining grafiklari quyida keltirildi:



b) $f(x)=|x|$, funksiya (-π;π) oraliqda juft funksiya bo`lgani uchun (8) ga asosan Furye koeffitsientlaridan fakat a_0, a_n larni topamiz:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \frac{2}{\pi}\cdot\frac{x^2}{2}\Big|_0^{\pi} = \pi, \qquad a_0 = \pi .$$

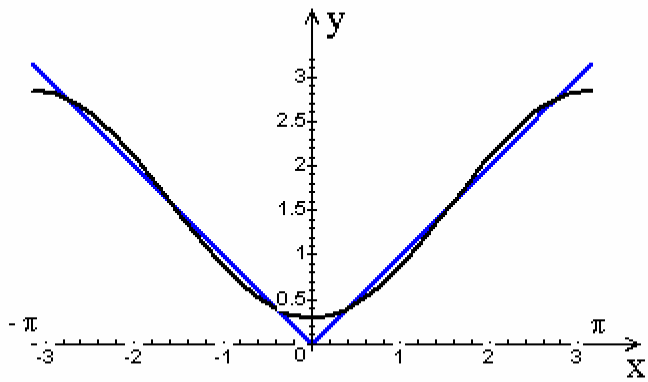
$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx = \left| \begin{matrix} u = x, du = dx \\ dv = \cos nx dx \\ v = \frac{1}{n}\sin nx \end{matrix} \right| = \frac{2}{\pi}\left(\frac{x}{n}\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx dx\right) = \frac{2}{\pi n^2}\cos nx\Big|_0^{\pi} =$$

$$= \frac{2}{\pi n^2}(\cos n\pi - 1) = \begin{cases} 0, \text{agar} \quad n - \text{juft} \quad \text{bo'lsa}, \\ -\frac{4}{\pi n^2}, \text{agar} \quad n - \text{toq} \quad \text{bo'lsa}. \end{cases}$$

Bu holda

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + ...\right)$$

Berilgan funksiya va unga mos Furye qatorining birinchi oltita hadi yig`indisining grafiklari quyidagicha bo`ladi:



x=0 da quyidagi ega bo`lamiz:

$$0=\frac{\pi}{2}-\frac{4}{\pi}(1+\frac{1}{3^2}+\frac{1}{5^2}+...) \qquad \frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+... \qquad .$$

Bu formuladan boshqa formulalar kelib chiqadi, ya'ni

$A_0=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+...,\qquad A_1=1+\frac{1}{3^2}+\frac{1}{5^2}+...,\qquad A_2=\frac{1}{4}(1+\frac{1}{2^2}+\frac{1}{3^2}+...)$ deb olsak,

$A_0=A_1+A_2,\ A_2=\frac{1}{4}A_0=\frac{1}{4}(A_1+A_2),\ 3A_2=A_1=\frac{\pi^2}{8},\ A_2=\frac{\pi^2}{24}.$ Demak, $A_0=\frac{\pi^2}{6}.$

Bu misolni yuqorida ko`rganmiz, ya'ni $y=|x|$ funksiyaning juftligidan foydalanmasdan umumiy holda Furye qatoriga yoyganmiz. Ushbu holda esa berilgan funksiyani juft funksiya ekanligidan foydalanib, Furye qatoriga yoysak maqsadga tezroq olib kelishini ko`rdik.

c) $y=|\sin x|=\begin{cases}\sin x, & 0<x\le\pi,\\ -\sin x, & -\pi\le x\le 0\end{cases}$ funksiya $[-\pi,\pi]$ kesmada juft funksiya. Bu holda Furye koeffitsientlarini (8) formuladan foydalanib hisoblaymiz:

$$a_n=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nx dx=\frac{2}{\pi}\int_0^{\pi}\frac{\sin(1+n)x+\sin(1-n)x}{2}dx=$$

$$\frac{1}{\pi}(-\frac{\cos(1+n)x}{1+n}-\frac{\cos(1-n)x}{1-n})\Big|_0^{\pi}=\frac{1}{\pi}(-\frac{\cos(1+n)\pi}{1+n}+\frac{1}{1+n}-\frac{\cos(1-n)\pi}{1-n}+\frac{1}{1-n})=$$

$$=\frac{1}{\pi}(\frac{1-\cos(1+n)\pi}{1+n}+\frac{1-\cos(1-n)\pi}{1-n})$$

Agar n-juft ($n=2k$) bo`lsa , $\cos(1\pm n)\pi=-1$, bu holda

$$a_n = \frac{1}{\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = -\frac{4}{\pi(n^2-1)} = -\frac{4}{\pi(2k-1)(2k+1)}, \qquad (n = 2k).$$

Agar n-toq ($n \neq 1$) bo`lsa, $a_n=0$ bo`ladi, (n=3,5,7,…). n=1 bo`lganda

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x dx = \frac{2}{\pi}\frac{\sin^2 x}{2}\Big|_0^{\pi} = 0.$$

Furye koeffitsientlarining kiymatlarini qatorga qo`ysak, berilgan funksiya uchun tuzilgan Furye qatori quyidagi

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{1\cdot 3} + \frac{\cos 4x}{3\cdot 5} + ... + \frac{\cos 2kx}{(2k-1)(2k+1)} + ...\right).$$

ko`rinishga keladi. Bu tenglik $[-\pi, \pi]$ kesmaning barcha nuqtalarida o`rinli.

x=0 da , $0 = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + ... + \frac{1}{(2\kappa-1)(2\kappa+1)} + ...\right)$, bundan,

$\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + ... + \frac{1}{(2\kappa-1)(2\kappa+1)} + ...$ hosil bo`ladi.

2. $y=x^2$ funksiya $[-\pi, \pi]$ kesmada juft funksiya bo`lib, yuqoridagi teoremaning barcha shartlarini qanoatlantiradi. Berilgan funksiya juft funksiya bo`lgani uchun $b_n=0$ bo`lib, a_n larni hisoblaymiz.

$$a_0 = \frac{1}{\pi}\int_{-\pi} x^2 dx = \frac{2}{x^2}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\frac{x^3}{3}\Big|_0^{\pi} = \frac{2\pi^2}{3},$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx dx = \begin{vmatrix} u = x^2, du = 2xdx \\ dv = \cos nx dx \\ v = \frac{1}{n}\sin nx \end{vmatrix} = \frac{2}{\pi}\left(\frac{x^2}{n}\sin nx\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x \sin nx dx\right) =$$

$$= \begin{vmatrix} u = x, du = dx \\ dv = \sin nx dx \\ v = -\frac{1}{n}\cos nx \end{vmatrix} = -\frac{4}{\pi n}\left(-\frac{x}{n}\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx dx\right) = -\frac{4}{n\pi}\left(-\frac{\pi}{n}\cos n\pi + \frac{1}{n^2}\sin nx\Big|_0^{\pi}\right) =$$

$$= \frac{(-1)^n \cdot 4}{n^2}.$$

Topilgan Furye koeffitsientlarini qatorga qo`ysak, quyidagiga ega bo`lamiz.

$$f(x) = x^2 = \frac{\pi^2}{3} - \frac{4}{1^2}\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + ... + (-1)^n \frac{4\cos nx}{n^2} + ... \quad ,$$

yoki

$$x^2 = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos nx}{n^2}.$$

Bu yoyilma, berilgan davriy funksiya [-π ,π] kesmada uzluksiz bo`lgani uchun kesmaning barcha nuqtalarida o`rinli, ya'ni tuzilgan Furye qatori [-π;π] kesmaning barcha nuqtalarida berilgan funksiyaga yaqinlashadi. x=0, x=π bo`lganda, yuqoridagi yoyilmadan quyidagilar kelib chiqadi:

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - ... + (-1)^{n-1}\frac{1}{n^2} + ...$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} + ... \quad .$$

Quyidagi funsiyalarni qaralayotgan oraliqda Furye qatoriga yoying.

№3. $f(x) = x, \quad 0 < x < 2\pi.$

№4. $f(x) = \begin{cases} -1, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$

№5. $f(x) = \pi - x, \quad 0 < x < 2\pi$

№6. $f(x) = \cos\frac{x}{2}, \quad 0 < x \le 2\pi.$

№7. $f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ x, & 0 < x < \pi \\ \frac{\pi}{2}, & x = \pi \end{cases}$

№8. $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ 1, & 0 < x < a \\ 0, & a < x \le \pi \\ \frac{1}{2}, & x = 0, x = a \end{cases}$

№9. $f(x) = x^2, \quad 0 < x < 2\pi$

№10. $f(x) = \begin{cases} 1, & -\pi < x < 0 \\ 3, & 0 < x < \pi \end{cases}$

№11. $f(x) = \cos ax, \quad -\pi < x < \pi$ (a-butun son emas)

№12. $f(x) = \sin ax, \quad -\pi < x < \pi$ (a-butun son emas)

№13. $f(x) = \sin\frac{5}{6}x, \quad -\pi < x < \pi$

№14. $f(x) = \begin{cases} c_1, & -\pi < x < 0, \\ c_2, & 0 < x < \pi, \\ \frac{c_1 + c_2}{2}, & x = 0, x = \pm\pi \end{cases}$ ($c_1<0$, $c_2>0$).

№15. $f(x) = \begin{cases} -x, & -\pi < x \le 0 \\ 0, & 0 < x < \pi \end{cases}$

№16. $f(x) = \begin{cases} -\frac{\pi + x}{2}, & -\pi < x < 0 \\ \frac{\pi - x}{2}, & 0 < x < \pi \end{cases}$

№17. $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}, \quad -\pi < x < \pi$

3-§. Davri 2*L* bo`lgan funksiyalar uchun Furye qatori.

Agar *f(x)* funksiyaning davri 2π dan farqli, masalan *2L* ga teng bo`lsa, u holda bu funksiyani Furye qatoriga yoyish uchun quyidagi

$$x = \frac{L}{\pi} t \qquad (10)$$

almashtirishdan foydalanamiz. Ravshanki, $f(\frac{Lt}{\pi})$ funksiya 2π davrli davriy funksiya bo`ladi. Bu funksiyani [-π;π] kesmada Furye qatoriga yoyamiz. $f(\frac{Lt}{\pi})$ funksiya uchun Furye qatori va Furye koeffitsientlari quyidagi ko`rinishda bo`ladi:

$$f(\frac{Lt}{\pi}) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \qquad (11)$$

bunda

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{Lt}{\pi}) dt, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{Lt}{\pi}) \cos ntdt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{Lt}{\pi}\right) \sin ntdt. \qquad (12)$$

(10) dan: $t = \frac{\pi x}{L}$, $dt = \frac{\pi dx}{L}$, bu holda (11) va (12) lar quyidagi ko`rinishga keladi.

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx \qquad (13)$$

Natijada *2L* davrli *f(x)* davriy funksiya uchun Furye qatori quyidagi

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}) \qquad (14)$$

ko`rishga keladi. (14) dagi a_0, a_n, b_n Furye koeffitsientlari (13) formulalar orqali topiladi.

Xususiy holda agar *f(x)* funksiya juft funksiya bo`lsa, Furye qatori quyidagi sodda ko`rinishga keladi.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \qquad (15)$$

Furye koeffitsientlari esa quyidagi formulalar yordamida topiladi:

$$a_0 = \frac{2}{L} \int_{0}^{L} f(x) dx, \quad a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad b_n = 0.$$

Xuddi shuningdek, agar *f(x)* funksiya toq funksiya bo`lsa, Furye qatori

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \quad (a_0 = 0, a_n = 0) \qquad (16)$$

ko`rishga keladi, bundagi b_n Furye koeffitsienti

$b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx$ formula orqali topiladi.

Quyidagi funksiyalar qaralayotgan oraliqda Furye qatoriga yoyilsin.

№18. $f(x)=\begin{cases}0, & -2<x<-1\\ x, & -1\le x<1, \quad -2<x<2.\\ 0, & 1<x\le 2.\end{cases}$

Yechish. $f(x)=\begin{cases}0, & -2<x<-1\\ x, & -1\le x<1\\ 0, & 1<x\le 2.\end{cases}$

funksiyaning davri 4 ga teng bo`lib, toq funksiyadir.

$2L$=4 (L=2) bo`lib, berilgan funksiya toq bo`lgani uchun

$$a_0=0,\quad a_n=0, b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}dx=$$

$$\frac{2}{2}\int_0^2 f(x)\sin\frac{n\pi x}{L}dx=\int_0^1 f(x)\sin\frac{n\pi x}{2}dx+\int_1^2 f(x)\sin\frac{n\pi}{2}xdx=\int_0^1 x\sin\frac{n\pi x}{2}dx=$$

$$\left|\begin{array}{l}u=x, du=dx\\ dv=\sin\frac{n\pi x}{2}dx\\ v=-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\end{array}\right|=-\frac{2x}{n\pi}\cos\frac{n\pi}{2}x\Big|_0^1+\frac{2}{n\pi}\int_0^1\cos\frac{n\pi}{2}xdx=$$

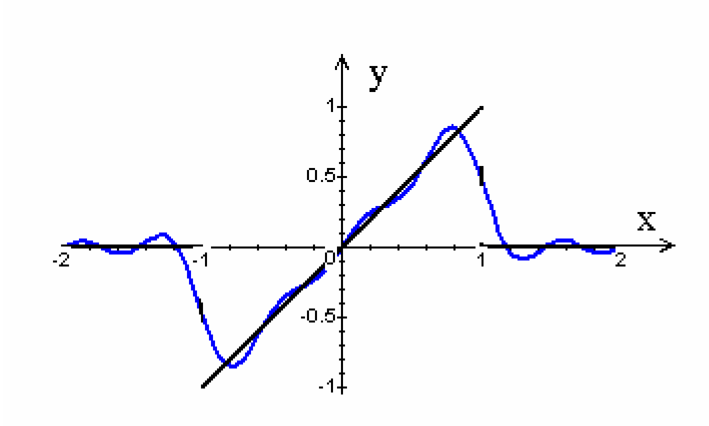
$$=-\frac{2}{n\pi}\cos\frac{n\pi}{2}+\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}x\Big|_0^1=-\frac{2}{n\pi}\cos\frac{n\pi}{2}+\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}=$$

$$=\begin{cases}\dfrac{(-1)^{k+1}}{k\pi}, & \text{agar} \quad n=2k,\\ \dfrac{(-1)^k 4}{(2k+1)^2\pi^2}, & \text{agar} \quad n=2k+1\end{cases}$$

Bu holda berilgan funksiya uchun Furye qatori (16)ga asosan

$$f(x)=\frac{4}{\pi^2}\sin\frac{\pi}{2}x+\frac{1}{\pi}\sin\pi x-\frac{4}{3^2\pi^2}\sin\frac{3\pi}{2}x-\frac{1}{2\pi}\sin 2\pi x+\ldots \quad .$$

Berilgan funksiya va unga mos tuzilgan Furye qatorining birinchi oltita hadi yig`indisining grafiklari quyidagicha bo`ladi:



№19. $f(x)=|x|, \qquad -L\le x\le L.$

Yechish. Berilan *f(x)*=|*x*| funksiya 2*L* davrli davriy funksiya bo\`lib, bu funksiya juft funksiyadir. Bu holda

$$b_n=0, \qquad a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx=\frac{2}{L}\int_0^L f(x)dx=\frac{2}{L}\int_0^L xdx=\frac{2}{L}\cdot\frac{x^2}{2}\Big|_0^L=L, \qquad a_0=L.$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi}{L}xdx=\frac{2}{L}\int_0^L x\cos\frac{n\pi}{L}xdx=\left|\begin{array}{l} u=x, du=dx \\ dv=\cos\frac{n\pi}{L}xdx \\ v=\frac{L}{n\pi}\cos\frac{n\pi}{L}x\end{array}\right|=$$

$$=\frac{2}{L}\Big(\frac{Lx}{n\pi}\sin\frac{n\pi}{L}x\Big|_0^L-\frac{L}{n\pi}\int_0^L\sin\frac{n\pi}{L}xdx\Big)=\frac{2}{n\pi}\cdot\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L=\frac{2L}{n^2\pi^2}(\cos n\pi-1)=$$

$$=\begin{cases}0, & \text{agar} \quad n=2k,\\ -\dfrac{4L}{n^2\pi^2}, & \text{agar} \quad n=2k+1\end{cases}$$

Demak berilgan funksiya uchun Furye qatori quyidagi ko\`rinishda bo\`ladi:

$$f(x)=|x|=\frac{L}{2}-\frac{4L}{\pi^2}\Big(\frac{\cos\frac{\pi x}{L}}{1^2}+\frac{\cos\frac{3\pi x}{L}}{3^2}+...+\frac{\cos\frac{(2k+1)\pi x}{L}}{(2k+1)^2}+...\Big).$$

Xususiy holda *L*=*π* bo\`lgan holda bu misoldan yuqorida yechilgan 2-misol kelib chiqadi.

№20. $f(x)=x-[x], \qquad 0\le x\le 1.$

Yechish.Berilgan *f(x)*=*x*-[*x*] funksiya 2*L*=1 (*L*=1/2) davrli funksiya bo\`lib, [0,1] kesmada *f(x)*=*x* funksiyadan iborat.

Furye koeffitsientlarini hisoblaymiz.

$$a_0=\frac{1}{L}\int_0^{2L} f(x)dx=2\int_0^1 xdx=x^2\Big|_0^1=1,\qquad a_n=\frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}xdx=2\int_0^1 x\cos\frac{n\pi}{\frac{1}{2}}xdx=$$

$$=2\int_0^1 x\cos 2n\pi xdx=\left|\begin{array}{l} u=x, du=dx\\ dv=\cos 2n\pi xdx\\ v=\frac{1}{2n\pi}\sin 2n\pi x\end{array}\right|=2\left(\frac{x}{2n\pi}\sin 2n\pi x\Big|_0^1-\frac{1}{2n\pi}\int_0^1\sin 2n\pi xdx\right)=$$

$$=\frac{1}{2n^2\pi^2}\cos 2n\pi x\Big|_0^1=0.$$

$$b_n=2\int_0^{2L} f(x)\sin\frac{n\pi}{L}xdx=2\int_0^1 x\sin 2\pi xdx=\left|\begin{array}{l} u=x, du=dx\\ v=-\frac{1}{2n\pi}\cos 2n\pi x\end{array}\right|=$$

$$=2\left(-\frac{x}{2n\pi}\cos 2n\pi x\Big|_0^1+\frac{1}{2n\pi}\int_0^1\cos 2n\pi xdx\right)=$$

$$=-\frac{1}{n\pi}+\frac{1}{2n^2\pi^2}\sin 2n\pi x\Big|_0^1=-\frac{1}{n\pi},\qquad b_n=-\frac{1}{n\pi}.$$

Demak , Furye qatori quyidagicha bo`ladi:

$$f(x)=\frac{1}{2}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n};$$

x=0 va x=1 da $f(x)=x-[x]$ funksiya uzilishga ega. Bu nuqtalarda qator yig`indisi 1-§ dagi teoremaga ko`ra $S(x)=\frac{f(x-0)+f(x+0)}{2}$ ga teng, haqiqatan ham $S(1)=S(0)=\frac{1}{2}$.

Quyidagi funksiyalar ko`rsatilgan oraliqda Furye qatoriga yoyilsin.

№21. $f(x)=3-x,\quad -2<x<2\quad (2L=4, L=2).$

№22. $f(x)=\begin{cases}6, & 0<x<2\\ 3x, & 2<x<4\end{cases}$

№23. $f(x)=x^2,\quad -1<x<1.$

4-§. Nodavriy funksiyalarni Furye qatoriga yoyish

Bundan oldingi bandlarda [-π;π], [-L;L], [0;2π], [0;2L] oraliqlarda bo`lakli monoton, va davri 2π, 2L bo`lgan funksiyalar uchun Furye qatorini tuzishni ko`rdik.

Agar *f(x)* funksiya nodavriy bo`lsa, u holda uni Furye qatori yordamida tavsiflash uchun funksiyaning aniqlanish sohasida, masalan *[-L;L]* kesmada, *f(x)* funksiya bilan ustma-ust tushadigan yordamchi davriy *φ(x)* funksiya tuzib olinadi.

Bunday holda *f(x)* funksiya sonlar o`qiga davriy davom ettirilgan deyiladi. Quyidagi hollar bo`lishi mumkin.

1. Agar *f(x)* funksiya *[-L;L]* kesmada berilgan bo`lsa, u holda *[-L;L]* kesmada *f(x)* funksiyaga teng, to`g`ri chiziqning qolgan nuqtalarida uning davriy davomi bo`lgan *t=2L* davrli yordamchi φ(x) funksiya tuziladi.

2. Agar *f(x)* funksiya ixtiyoriy *[a;a+2L]* kesmada berilgan bo`lsa, u holda shu kesmada *f(x)* funksiyaga teng, to`g`ri chiziqning qolgan nuqtalarida uning davriy davomi bo`lgan *t=2L* davrli yordamchi *φ(x)* funksiya tuziladi. Bu holda funksiyaning Furye qatori (14) ko`rinishda bo`ladi, lekin uning koeffitsientlari quyidagi formulalardan topiladi:

$$a_0 = \frac{1}{L}\int_a^{a+2L} f(x)dx, \quad a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\frac{n\pi x}{L}dx$$

3. Agar $[0,L]$ kesmada bo`lakli monoton funksiya berilgan bo`lsa, u holda bu funksiyani yoki faqat sinuslar, yoki faqat kosinuslar, yoki ham sinuslar, ham kosinuslar bo`yicha Furye qatoriga yoyish mumkin.

f(x) funksiyani kosinuslar bo`yicha qatorga yoyish uchun uni $[-L,0]$ kesmaga juft tarzda davom ettiriladi, ya'ni

$$\varphi(x)=\begin{cases} f(-x), & x\in[-L,0], \\ f(x), & x\in[0,L] \end{cases}$$ yordamchi funksiya tuziladi va bu funksiyani sonlar o`qiga davriy davom ettiriladi. Bu holda *f(x)* funksiya uchun Furye qatori quyidagicha bo`ladi:

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}dx, \qquad (17)$$

bu erda

$$a_0 = \frac{2}{L}\int_0^L f(x)dx; \; a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \qquad (18)$$

f(x) funksiyani sinuslar bo`yicha qatorga yoyish uchun uni $[-L,0]$ kesmaga toq tarzda davom ettiriladi, ya'ni

$$\varphi(x)=\begin{cases} -f(-x), & x\in[-L,0], \\ f(x), & x\in[0,L] \end{cases}$$ yordamchi funksiya tuziladi va bu funksiyani sonlar o`qiga davriy davom ettiriladi. Bu holda *f(x)* funksiya uchun Furye qatori quyidagicha bo`ladi:

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}dx, \qquad (19)$$

bu erda

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \qquad (20)$$

№24. Quyidagi funksiyalar Furye qatoriga yoyilsin.

a) $f(x)=\frac{\pi}{4}-\frac{x}{2}$ funksiya $[0;\pi]$ kesmada kosinuslar bo`yicha Furye qatoriga yoyilsin.

b) $f(x)=\begin{cases} 0{,}3, & 0\le x\le 1/2 \\ -0{,}3, & 1/2< x\le 1 \end{cases}$ funksiyani qaralayotgan kesmada faqat kosinuslar bo`yicha va faqat sinuslar bo`yicha Furye qatoriga yoyilsin.

c) *f(x)=xcosx* funksiya [0;π] kesmada faqat kosinuslar bo`yicha Furye qatoriga yoyilsin.

Yechish. a) $f(x)=\dfrac{\pi}{4}-\dfrac{x}{2}$ funksiyani [-π;0] kesmaga juft tarzda davom ettiramiz va natijada [-π;π] kesmada monoton bo`lgan juft funksiyaga ega bo`lamiz. (18) formulalardan foydalanib, Furye koeffitsientlarini hisoblaymiz:

$$a_o=\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{4}-\frac{x}{2}\right)dx=\frac{2}{\pi}\left(\frac{\pi}{4}-\frac{x^2}{4}\right)=\frac{2}{\pi}\left(\frac{\pi^2}{4}-\frac{\pi^2}{4}\right)\Big|_0^{\pi}=0\cdot a_0=0,$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos nxdx=\left|\begin{matrix} u=\dfrac{\pi}{4}-\dfrac{x}{2}, du=-\dfrac{dx}{2} \\ dv=\cos nxdx \\ v=\dfrac{1}{n}\sin x \end{matrix}\right|=$$

$$\frac{2}{n\pi}\left(\frac{\pi}{4}-\frac{x}{2}\right)\sin nx\Big|_0^{\pi}+\frac{2}{n\pi}\frac{1}{2}\int_0^{\pi}\sin nxdx=-\frac{1}{n^2\pi}\cos nx\Big|_0^{\pi}=-\frac{1}{n^2\pi}(\cos n\pi-1)=$$

$$\frac{1}{\pi n^2}(1-cos\, n\pi)=\frac{1}{\pi n^2}\left(1-(-1)^n\right)=\begin{cases} 0, & n=2k, \\ \dfrac{2}{n^2\pi}, & n=2k-1, \end{cases}\quad k\in N$$

Bu holda berilgan funksiya uchun tuzilgan Furye qatori (17) ga asosan quyidagicha bo`ladi:

$$f(x)=\frac{\pi}{4}-\frac{x}{2}=\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}.$$

b) $f(x)=\begin{cases} 0{,}3, & 0\le x\le 1/2 \\ -0{,}3, & 1/2< x\le 1 \end{cases}$ funksiyani [-1,0] kesmaga juft tarzda davom ettiramiz va yuqoridagi kabi Furye koeffitsientlarini hisoblaymiz, bunda *L*=1:

$$a_o=2\int_0^1 f(x)dx=2\int_0^{\frac{1}{2}} f(x)dx+\int_{\frac{1}{2}}^1 f(x)dx=2\int_0^{\frac{1}{2}}0{,}3dx+\int_{\frac{1}{2}}^1(-0.3dx)=2(0{,}3x)\Big|_0^{\frac{1}{2}}-0{,}3x\Big|_{\frac{1}{2}}^1=$$

$$=2\left(\frac{0{,}3}{2}-03+\frac{0{,}3}{2}\right)=0,\quad a_0=0$$

$$a_n = 2\int_0^1 f(x)\cos n\pi x dx = 2\int_0^{\frac{1}{2}} f(x)\cos n\pi x dx + \int_{\frac{1}{2}}^1 f(x)\cos n\pi x dx =$$

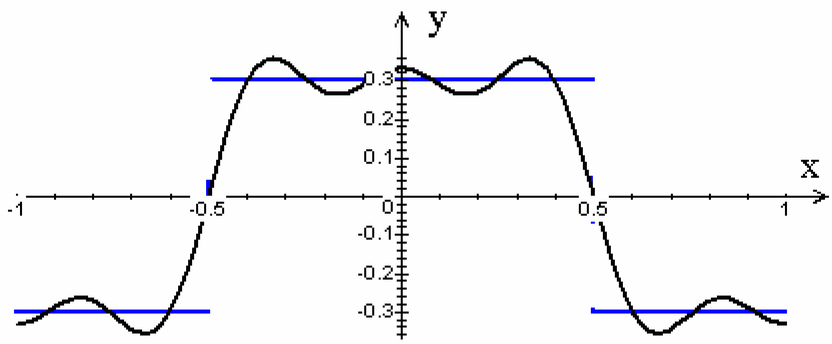
$$= 2\int_0^{\frac{1}{2}} 0{,}3\cos n\pi x dx - 2\int_{\frac{1}{2}}^1 0{,}3\cos n\pi x dx = 0{,}6\frac{1}{n\pi}\sin n\pi x\Big|_0^{\frac{1}{2}} - 0{,}6\frac{1}{n\pi}\sin n\pi x\Big|_{\frac{1}{2}}^1 = \frac{0{,}6}{n\pi}(\sin\frac{n\pi}{2} +$$

$$+\sin\frac{n\pi}{2}) = \frac{1.2}{n\pi}\sin\frac{n\pi}{2} = \begin{cases} 0, & n = 2k, \\ (-1)^{k-1}\dfrac{1{,}2}{(2k-1)\pi}, & n = 2k-1, \end{cases} \quad k \in N$$

Bu holda faqat kosinuslarni o`z ichiga oluvchi Furye qatori quyidagi ko`rinishga keladi:

$$f(x) = \frac{1{,}2}{\pi}\left(\frac{\cos\pi x}{1} - \frac{\cos 3\pi x}{3} + \frac{\cos 5\pi x}{1} - ... + (-1)^{k-1}\frac{\cos(2k-1)\pi x}{2k-1} + ...\right)$$

Berilgan funksiya va unga mos tuzilgan Furye qatorining dastlabki oltita hadi yig`indisi grafiklari quyidagicha bo`ladi:



Berilgan funksiyaning faqat sinuslar qatnashgan Furye qatorini yozish uchun bu funksiyani [-1;0] kesmaga toq tarzda davom ettiramiz va (20) formuladan foydalanamiz:

$$b_n = 2\int_0^1 f(x)\sin n\pi x dx = 2(\int_0^{\frac{1}{2}} 0{,}3 f(x)\sin n\pi x dx - \int_{\frac{1}{2}}^1 0{,}3 f(x)\sin n\pi x dx) =$$

$$= -0{,}6\frac{\cos n\pi x}{n\pi}\Big|_0^{\frac{1}{2}} + 0{,}6\frac{\cos n\pi x}{n\pi}\Big|_{\frac{1}{2}}^1 = \frac{0{,}6}{n\pi}(\cos n\pi - 2\cos\frac{n\pi}{2} + 1),$$

Bundan n-toq bo`lsa, $b_n$=0 va $n$-juft n=2k bo`lsa

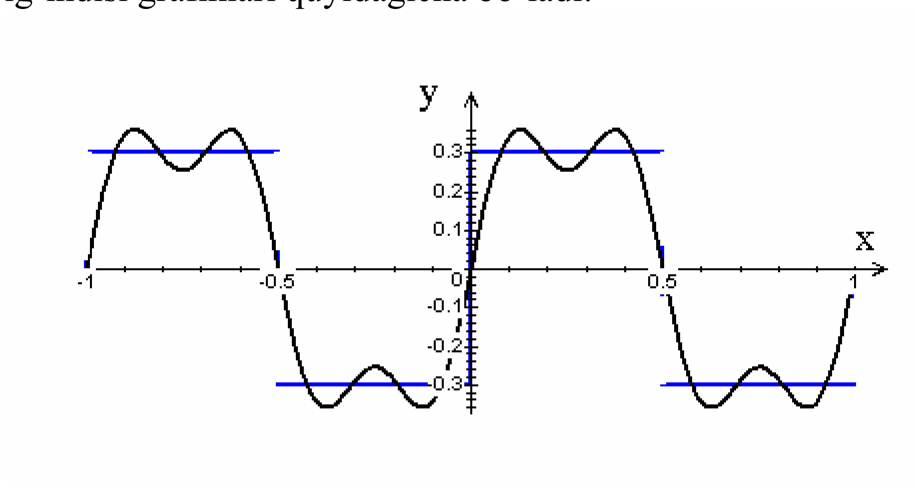
$$b_{2k}=\frac{0,6}{2k\pi}(2-2\cos\frac{2n\pi}{2})=\frac{0,6(1-\cos k\pi)}{k\pi}=\begin{cases}0, & k=2m,\\ \frac{1,2}{\kappa\pi}, & k=2m-1,\end{cases}\qquad m\in N$$

ekanligi kelib chiqadi.

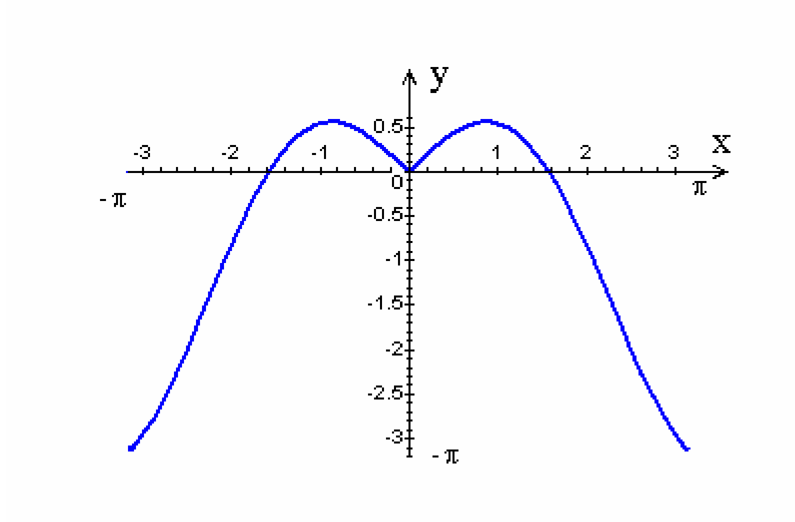
(19) ga asosan funksiyaning Furye qatori quyidagi ko`rinishda bo`ladi:

$$f(x)=\frac{1,2}{\pi}\left(\frac{\sin 2\pi x}{1}+\frac{\sin 6\pi x}{3}+\ldots+\frac{\sin 2(2m-1)\pi x}{2m-1}+\ldots\right)$$

Berilgan funksiya va unga mos tuzilgan Furye qatorining dastlabki oltita hadi yig`indisi grafiklari quyidagicha bo`ladi:



c) *f(x)=xcosx* funksiyani [0;π] kesmada faqat kosinuslar buyicha Furye qatoriga yoyish uchun bu funksiyani [-π;0] kesmaga juft tarzda davom ettiramiz va (18) formulalardan foydalanib, Furye koeffitsientlarini hisoblaymiz:



$$a_n=\frac{2}{\pi}\int_0^{\pi}x\cos x\cos n\pi x dx=\frac{2}{\pi}\int_0^{\pi}x\frac{\cos(n+1)x+\cos(n-1)x}{2}dx=$$

$$=\frac{1}{\pi}\int_0^{\pi}x\cos(n+1)xdx+\frac{1}{\pi}\int_0^{\pi}x\cos(n-1)xdx=$$

$$=\begin{vmatrix} u=x, du=dx \\ v=\frac{1}{n\pm 1}\sin(n\pm 1)x \end{vmatrix}=\frac{1}{\pi}\left(\frac{x}{n+1}\sin(n+1)x\Big|_0^{\pi}-\frac{1}{n+1}\int_0^{\pi}\sin(n+1)xdx\right)+$$

$$+\frac{1}{\pi}\left(\frac{x}{n-1}\sin(n-1)x\Big|_0^{\pi}-\frac{1}{n-1}\int_0^{\pi}\sin(n-1)xdx=\frac{1}{(n+1)^2\pi}\cos(n+1)x\Big|_0^{\pi}+\right.$$

$$+\frac{1}{\pi(n-1)^2}\cos(n-1)x\Big|_0^{\pi}=\frac{1}{\pi}+\left(\frac{\cos(n+1)\pi-1}{(n+1)^2}+\left(\frac{\cos(n-1)\pi-1}{(n-1)^2}\right)\right).$$

Agar $n=2k$, bo`lsa, $\cos(n\pm1)\pi=-1$ bo`lib,

$a_n=a_{2k}=\dfrac{4(4k^2+1)}{\pi(4k^2-1)^2}$ bo`ladi.

Agar $n=2k-1$ bo`lsa, $\cos(n\pm1)\pi=1$ bo`lib, $a_n=a_{2k-1}=0$, $(n\neq1)$, $n=1$ bo`lganda

$$a_1=\frac{2}{\pi}\int_0^{\pi}x\cos^2 xdx=\frac{2}{\pi}\int_0^{\pi}x\frac{1+\cos 2x}{2}dx=\frac{1}{\pi}\left(\int_0^{\pi}xdx+\int_0^{\pi}x\cos 2xdx=\right.$$

$$\begin{vmatrix} u=x, du=dx \\ dv=\cos 2xdx \\ v=\frac{1}{2}\sin 2x \end{vmatrix}=\frac{1}{\pi}\left(\frac{x^2}{2}\Big|_0^{\pi}+\frac{x}{2}\sin 2x\Big|_0^{\pi}-\frac{1}{2}\int_0^{\pi}\sin 2xdx\right)=$$

$=\dfrac{1}{\pi}\left(\dfrac{\pi^2}{2}+\dfrac{1}{4}\cos 2x\Big|_0^{\pi}=\dfrac{\pi}{2}\right.$, demak $a_1=\dfrac{\pi}{2}$ bo`ladi. Natijada

$f(x)=x\cos x=-\dfrac{2}{\pi}+\dfrac{\pi}{2}\cos x-\dfrac{4}{\pi}\sum\limits_{k=0}^{\infty}\dfrac{4k+1}{(4k^2-1)^2}\cos 2kx,\ 0\le x\le\pi$ yoyilmaga ega bo`lamiz.

Yuqoridagi yoyilmada $x=0$ bo`lganda $0=\dfrac{\pi}{2}-\dfrac{4}{\pi}\sum\limits_{k=0}^{\infty}\dfrac{4k^2+1}{(4k^2-1)^2}$, bundan esa

$\sum\limits_{k=1}^{\infty}\dfrac{4k^2+1}{(4k^2-1)^2}=\dfrac{\pi^2}{8}-\dfrac{1}{2}$ hosil bo`ladi.

Berilgan funksiyani [-π;0] kesmaga toq tarzda davom ettirib, funksiyaning faqat sinuslarni o`z ichiga oladigan Furye qatorini hosil qilamiz. Bu masalani talabalarga mustaqil ish sifatida taklif qilamiz.

Quyidagi funksiyalar qaralayotgan kesmada Furye qatoriga yoyilsin.

№25. $f(x)=x^2$ funksiya [0;π] oraliqda sinuslar buyicha qatorga yoyilsin.

№26. $f(x)=x$ funksiya [0,π] kesmada sinuslar bo`yicha qatorga yoyilsin.

№27. *f(x)=x* funksiya [0,π] kemada kosinuslar buyicha qatorga yoyilsin.

№28. *f(x)=cosx* funksiya [0,π/2] kesmada kosinuslar bo`yicha qatorga yoyilsin va

$$-\frac{1}{1\cdot 3}+\frac{1}{1\cdot 5}-\frac{1}{5\cdot 7}+...+(-1)^n\frac{1}{(2n-1)(2n+1)}+...$$ qator yig`indisi topilsin.

№29. $f(x)=\begin{cases}1, & 0\le x\le 1,\\ 0, & 1<x\le\pi\end{cases}$ funksiya Furye qatoriga yoyilsin. Yoyilmadan foydalanib,

$a)\ \sum_{n=1}^{\infty}\frac{\sin n}{n};\quad b)\ \sum_{n=1}^{\infty}(-1)^n\frac{\sin n}{n}$ qatorlarning yig`indilari topilsin.

№30. *f(x)*=1 funksiya [0,*l*] oraliqda sinuslar bo`yicha Furye qatoriga yoyilsin.

№31. $f(x)=\begin{cases}x, & 0\le x<1\\ 2-x, & 1\le x\le 2\end{cases}$ funksiya kosinuslar va sinuslar bo`yicha qatorga yoyilsin.

5-§. Kompleks formadagi Furye qatori.

Yuqoridan ma'lumki 2π davrli *f(x)* davriy funksiya uchun Furye qatori

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx \qquad (21)$$

ko`rinishda bo`ladi.

Eyler formulasiga asosan:

$$\cos nx=\frac{e^{inx}+e^{-inx}}{2},\quad \sin nx=\frac{e^{inx}-e^{-inx}}{2i}=-i\frac{e^{inx}-e^{-inx}}{2}$$

bo`ladi. *cosnx, sinnx* ning qiymatlarini (21) ga qo`ysak

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\frac{a_n-ib_n}{2}e^{inx}+\frac{a_n+ib_n}{2}e^{-inx}\right)$$ yoki

$$f(x)=c_0+\sum_{n=1}^{\infty}(c_ne^{inx}+c_{-n}e^{-inx}) \qquad (22)$$

hosil bo`ladi, bu erda $c_0=\frac{a_0}{2},\quad c_n=\frac{a_n-ib_n}{2},\quad c_{-n}\frac{a_n+ib_n}{2}.$

(22) ni

$$f(x)=\sum_{n=-\infty}^{\infty}c_ne^{inx} \qquad (23)$$

ko`rinishda ham yozish mumkin.

(22) yoki (23) kompleks formadagi Furye qatori deyiladi. Uning koeffitsentlari

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \qquad (24)$$

$$c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \qquad (25)$$

formulalar orqali topiladi. (24), (25) larni quyidagicha ham

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{inx}dx \qquad (26)$$

yozish mumkin, bu erda $n = 0, \quad \pm 1, \quad \pm 2, \ldots$.

Agar *f(x)* funksiya $2L$ davrli davriy funksiya bo`lsa *f(x)* funksiya uchun Furye qatori

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x$$

ko`rinishda bo`ladi.

Bu holda kompleks formadagi Furye qatori

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{L}x} \qquad (27)$$

ko`rinishda bo`lib, c_n koeffitsentlar esa

$$c_n = \frac{1}{2L}\int_{-L}^{Ll} f(x)e^{\frac{-n\pi}{L}x}dx, \quad (n = 0, \pm 1, \pm 2, \ldots)$$

formulalar orqali topiladi.

6-§. Furye integrali.

Faraz qilaylik

1) $(-\infty, \infty)$ oraliqda aniqlangan *f(x)* funksiya shu oraliqda absolyut integrallanuvchi, ya'ni

$$\int_{-\infty}^{\infty} |f(x)|dx = Q \qquad (28)$$

xosmas integral mavjud;

2) *f(x)* funksiya (-*L,L*) oraliqda bo`lakli monoton bo`lsin.

Bu holda *f(x)* funksiya uchun Furye qatori quyidagi

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x \qquad (29)$$

ko`rinishda bo`ladi. Bu erdagi a_n, b_n Furye koeffitsientlari

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}tdt\,, b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi}{L}tdt \qquad (30)$$

ko`rinishda bo`ladi.

Yuqoridagi 1), 2) shartlar bajarilsa, $L \to \infty$ da (29) dan quyidagi

$$f(x)=\frac{1}{\pi}\int_0^\infty\Big(\int_{-\infty}^{\infty}f(t)\cos\alpha(t-x)dt\Big)\,d\alpha \qquad (31)$$

tenglik kelib chiqadi. (31) ning o`ng tomonidagi ifoda *f(x)* funksiya uchun Furye integrali deyiladi. (31) tenglik *f(x)* funksiya uzluksiz bo`ladigan barcha nuqtalar uchun o`rinli bo`lib, uzilish nuqtalarida esa (masalan $x=x_0$ nuqta funksiyaning uzilish nuqtasi bo`lsa) quyidagi

$$\frac{1}{\pi}\int_0^\infty\Big(\int_{-\infty}^{\infty}f(t)\cos\alpha(t-x)dt\Big)d\alpha=\frac{f(x_0+0)+f(x_0-0)}{2} \qquad (32)$$

tenglik o`rinli bo`ladi.

$\cos\alpha(t\text{-}x)=\cos\alpha t\cos\alpha x+\sin\alpha t\sin\alpha x$ ekanligini hisobga olsak, (31) quyidagi ko`rinishga keladi.

$$f(x)=\frac{1}{\pi}\int_0^\infty A(\alpha)\cos\alpha x d\alpha+\frac{1}{\pi}\int_0^\infty B(\alpha)\sin\alpha x d\alpha=\frac{1}{\pi}\int_0^\infty(A(\alpha)\cos\alpha x+B(\alpha)\sin\alpha x)d\alpha \qquad (33)$$

bunda

$$A(\alpha)=\int_{-\infty}^{\infty}f(t)\cos\alpha t dt,\quad B(\alpha)=\int_{-\infty}^{\infty}f(t)\sin\alpha t dt,\quad \text{yoki}$$

$$f(x)=\frac{1}{\pi}\int_0^\infty\Big(\int_{-\infty}^{\infty}f(t)\cos\alpha t dt\Big)\cos\alpha x d\alpha+\frac{1}{\pi}\int_0^\infty\Big(\int_{-\infty}^{\infty}f(t)\sin\alpha t dt\Big)\sin\alpha x d\alpha \qquad (34)$$

(33) yoki (34) ning muxim xususiy hollarini ko`raylik:

1) *f(x)* juft funksiya bo`lsin. Bu holda $f(t)\cos\alpha t$ funksiya juft, $f(t)\sin\alpha t$ -toq funksiya bo`lib, quyidagi

$$A(\alpha)=\int_{-\infty}^{\infty}f(t)\cos\alpha t dt=2\int_0^\infty f(t)\cos\alpha t dt,\qquad B(\alpha)=\int_{-\infty}^{\infty}f(t)\sin\alpha t dt=0$$

munosabatlar o`rinli bo`ladi.

Bu holda (34) quyidagi ko`rinishga keladi.

$$f(x)=\frac{2}{\pi}\int_0^\infty A(\alpha)\cos\alpha x d\alpha \qquad (35)$$

2) xuddi shuningdek *f(x)* toq funksiya bo`lsa, (35) dan

$$f(x)=\frac{2}{\pi}\int_0^\infty\Big(\int_0^\infty f(t)\sin\alpha t dt\Big)\sin\alpha x d\alpha \qquad (36)$$

tenglik kelib chiqadi.

Agar *f(x)* funksiya $(0,\infty)$ oraliqda aniqlangan bo`lsa, *f(x)* funksiyani $(0,\infty)$ oraliqda (35), (36) formulalar bilan tasvirlash mumkin. Birinchi holda *f(x)* funksiyani $(-\infty,0)$ oraliqda juft, ikkinchi holda esa toq funksiya deb qaraymiz. Uzilish nuqtalarida esa $(x=x_0)$ (36), (37) tengliklardagi *f(x)* o`rniga

$\frac{f(x_0+0)+f(x_0-0)}{2}$ ifodani ko`yish kerak bo`ladi.

Agar

$$F(a)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\cos\alpha t dt \qquad (37)$$

deb olsak, (35) formula quyidagi

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} F(\alpha)\cos\alpha x d\alpha \qquad (38)$$

ko`rinishga keladi.

F(α) funksiya uchun Furyening kosinus almashtirishi deyiladi. Agar

$$\Phi(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(t)\sin\alpha t dt \qquad (39)$$

deb olsak, (36) formula quyidagi ko`rinishga keladi:

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} \Phi(\alpha)\sin\alpha x d\alpha \qquad (40)$$

Φ(α) funksiya *f(x)* funksiya uchun Furyening sinus almashtirishi deyiladi.

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)e^{-ia(t-x)}\, dt\right)d\alpha \qquad (41)$$

tenglikning o`ng qismi *f(x)* funksiya uchun kompleks formadagi Furye integrali deyiladi.

(41) formulani qisqacha quyidagi ko`rinishda yozish mumkin.

$$f(x)=\int_{-\infty}^{\infty} C(\alpha)e^{i\alpha x}d\alpha \qquad (42)$$

bunda

$$C(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\alpha t}dt \qquad (43)$$

C(α) funksiya spektral zichlik yoki spektral funksiya deyiladi. $|C(\alpha)|$-esa *f(x)* funksiyaning amplituda spektri deyiladi.

32. Quyidagi funksiyalarning Furye integrali topilsin.

a) $f(x)=\begin{cases}-e^{x}, x<0\\ e^{-x}, x>0\\ 0, x=0\end{cases}$ b) $f(x)=\begin{cases}1, 0<x<1\\ 1/2, x=0, x=1\\ 0, x<0, x>1\end{cases}$

c) $f(x)=\begin{cases} a, -1 \le x < 0 \\ b, 0 \le x < 1 \\ 0, x < -1, x > 1 (|x| > 1) \end{cases}$ d) $f(x)=\begin{cases} 1-\dfrac{x}{2}, 0 \le x \le 2 \\ 0, x > 2 \end{cases}$

a) $f(x)=\begin{cases} -e^{x}, x < 0 \\ 0, x = 0 \\ e^{-x}, x > 0 \end{cases}$

Funksiya bo`lakli monoton va $x$=0 nuqtada funksiya birinchi tur uzilishga ega. Endi *f(x)* funksiyaning butun sonlar o`qida absolyut integrallanuvchi, ya'ni $\int_{-\infty}^{\infty}|f(x)|dx = Q$ ekanligini ko`rsatamiz:

$$\int_{-\infty}^{\infty}|f(x)|dx = \int_{-\infty}^{0}|f(x)|dx + \int_{0}^{\infty}|f(x)|dx = \int_{-\infty}^{0}e^{x}dx = \int_{0}^{\infty}e^{-x}dx = \lim_{a\to-\infty}\int_{a}^{0}e^{x}dx + \lim_{b\to\infty}\int_{0}^{b}e^{-x}dx =$$

$$= \lim_{a\to-\infty}e^{x}\Big|_{a}^{0} + \lim_{b\to\infty}(e^{-x})\Big|_{0}^{b} = \lim_{a\to-\infty}(1-e^{a}) - \lim_{b\to\infty}(e^{-b}-1) = 1+1=2$$

Demak, berilgan funksiya absolyut integrallanuvchi ekan. Bu holda berilgan funksiyani Furye integrali ko`rinishda ifodalash mumkin. Berilgan funksiya toq funksiya bo`lgani uchun (36) formulaga asosan:

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\Big(\int_{0}^{\infty}f(t)\sin\alpha t dt\Big)\sin\alpha x d\alpha = \frac{2}{\pi}\int_{0}^{\infty}\Big(\int_{0}^{\infty}e^{-t}\sin\alpha t dt\Big)\sin\alpha x d\alpha = \frac{2}{\pi}\int_{0}^{\infty}I\sin\alpha x d\alpha,$$

$$I=\int_{0}^{\infty}e^{-t}\sin\alpha t dt = \left|\begin{array}{c} e^{-t}=u, du=-e^{-t}dt \\ dv=\sin\alpha t dt \\ v=-\dfrac{1}{\alpha}\cos\alpha t \end{array}\right| = -\frac{e^{-t}}{\alpha}\cos\alpha t - \frac{1}{\alpha}\int_{0}^{\infty}e^{-t}\cos\alpha t dt =$$

$$=\frac{1}{\alpha}-\frac{1}{\alpha}\int_{0}^{\infty}e^{-at}\cos\alpha t dt = \left|\begin{array}{c} e^{-t}=u, du=-e^{-t}dt \\ dv=\cos\alpha t dt \\ v=\dfrac{1}{\alpha}\sin\alpha t \end{array}\right| = \frac{1}{\alpha}-\frac{1}{\alpha}\Big(\frac{e^{-t}}{\alpha}\sin\alpha t\Big|_{0}^{\infty}+\frac{1}{a}\int_{0}^{\infty}\sin\alpha t dt\Big) =$$

$$=\frac{1}{\alpha}-\frac{1}{\alpha^{2}}\int_{0}^{\infty}e^{-t}\sin\alpha t dt = \frac{1}{\alpha}-\frac{1}{\alpha^{2}}I, \quad I=\frac{1}{\alpha}-\frac{1}{\alpha^{2}}I,$$

$$\Big(1+\frac{1}{\alpha^{2}}\Big)I=\frac{1}{\alpha}, \quad \frac{1+a^{2}}{\alpha^{2}}I=\frac{1}{\alpha}, \quad I=\frac{\alpha}{1+\alpha^{2}}.$$

Bu holda

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\alpha\sin\alpha x}{1+\alpha^2}d\alpha;\quad f(0)=0$$

b) Berilgan

$$f(x)=\begin{cases}1, & 0<x<1\\ \frac{1}{2}, x=0, x=1 & \\ 0, & x<0, x>1\end{cases}$$

funksiya bo`lakli monoton, sonlar o`qida absolyut integrallanuvchi, ya'ni yuqoridagi shartlar bajariladi.

x=0, x=1 nuqtada 1-tur uzilishga ega.

Demak, berilgan funksiyani Furye integrali ko`rinishda ifodalash mumkin. (-∞,0) va (1,∞) oraliqda *f(x)*=0 bo`lgani uchun (31) ga asosan:

$$f(x)=\frac{1}{\pi}\int_0^{\infty}(\int_{-\infty}^{\infty}f(t)\cos\alpha(t-x)dt)d\alpha=\frac{1}{\pi}\int_0^{\infty}(\int_0^{1}1\cdot\cos\alpha(t-x)dt)d\alpha=$$

$$=\frac{1}{\pi}\int_0^{\infty}\frac{1}{\alpha}\sin\alpha(t-x)\Big|_0^1 d\alpha=\frac{1}{\pi}\int_0^{\infty}\frac{\sin\alpha(1-x)+\sin\alpha x}{\alpha}d\alpha=\frac{2}{\pi}\int_0^{\infty}\frac{\sin\frac{\alpha}{2}\cos\frac{\alpha(1-2x)}{2}}{\alpha}d\alpha$$

Demak,

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\sin\frac{\alpha}{2}\cos\frac{\alpha(1-2x)}{2}}{\alpha}d\alpha \qquad (*)$$

Ma'lumki, berilgan funksiya x=0, x=1 nuqtada uzilishga ega. Bu nuqtalarda (*) tenglik o`rinli, ya'ni, $\frac{f(x-0)+f(x+0)}{2}=\frac{1}{2}=f(x).$

Masalan x=0 da

$$f(0)=\frac{1}{\pi}\int_0^{\infty}\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\alpha}d\alpha=\frac{1}{\pi}\int_0^{\infty}\frac{\sin\alpha}{a}d\alpha=\frac{1}{2}.$$ Bu erda $\int_0^{\infty}\frac{\sin\alpha}{\alpha}d\alpha=\frac{\pi}{2}$ tenglikdan foydalandik.

c) $f(x)=\begin{cases}a, -1\le x<0,\\ b,\ 0\le x<1,\\ 0,\quad |x|>1\end{cases}$ funksiya yuqoridagi shartlarni qanoatlantiradi. Demak, bu funksiyani Furye integrali ko`rinishda ifodalash mumkin. Buning uchun

$$f(x)=\frac{1}{\pi}\int_0^\infty A(\alpha)\cos\alpha x d\alpha+\frac{1}{\pi}\int_0^\infty B(\alpha)\sin\alpha x d\alpha \qquad (**)$$

formuladagi *A(α), B(α)* larni hisoblaymiz:

$$A(\alpha)=\int_{-\infty}^{\infty} f(t)\cos\alpha t dt=\int_{-\infty}^{-1} f(t)\cos\alpha t dt+\int_{-1}^{0} f(t)\cos\alpha t dt+\int_{0}^{1} f(t)\cos\alpha t dt+$$

$$\int_1^\infty f(t)\cos\alpha t dt=\int_{-1}^{0} a\cos\alpha t dt+\int_0^1 b\cos\alpha t dt=\frac{a}{\alpha}\sin t\Big|_{-1}^{0}+\frac{b}{\alpha}\sin\alpha t\Big|_0^1=$$

$$\frac{a\sin a}{\alpha}+\frac{b\sin a}{\alpha}=\frac{(a+b)\sin\alpha}{\alpha}.$$

Xuddi shuningdek

$$B(\alpha)=\int_{-1}^{0} a\sin\alpha t dt+\int_0^1 b\sin\alpha t dt=-\frac{a\cos\alpha t}{\alpha}\Big|_{-1}^{0}-b\frac{b\cos\alpha t}{\alpha}\Big|_0^1=$$

$$=-\frac{a}{\alpha}(1-\cos a)-\frac{b}{\alpha}(\cos\alpha-1)=\frac{(b-a)(1-\cos\alpha)}{\alpha}.$$

A(α), B(α) larni qiymatini (**) qo`ysak

$$f(x)=\frac{1}{\pi}\int_0^\infty \frac{(a+b\_\sin\alpha\cos\alpha x+(b-a)(1-\cos\alpha)\sin\alpha x}{\alpha}d\alpha$$

kelib chiqadi.

x=0 bo`lganda

$$\frac{a+b}{\pi}\int_0^\infty\frac{\sin\alpha}{\alpha}d\alpha=\frac{a+b}{\pi}\cdot\frac{\pi}{2}=\frac{a+b}{2} \qquad (\int_0^\infty\frac{\sin\alpha}{\alpha}d\alpha=\frac{\pi}{2}) \text{ bo`ladi.}$$

d) $f(x)=\begin{cases}1-\frac{x}{2},\ 0\le x<2\\ 0,\ x>2\end{cases}$ funksiya yuqoridagi shartlarni qanoatlantiradi.

Bu funksiyani juft tarzda davom etdiramiz:

(35) formulaga asosan:

$$f(x)=\frac{2}{\pi}\int_0^\infty\left(\int_0^\infty f(t)\cos\alpha t dt\right)\cos\alpha x d\alpha=\frac{2}{\pi}\int_0^\infty A(\alpha)\cos\alpha x d\alpha;$$

$$A(\alpha)=\int_0^\infty f(t)\cos\alpha t dt=\int_0^2(1-\frac{t}{2})\cos\alpha t dt=\left|\begin{matrix}1-\frac{t}{2}=u, du=-\frac{dt}{2}\\ v=\frac{1}{\alpha}\sin\alpha t\end{matrix}\right|=\frac{1}{\alpha}(1-\frac{t}{2})\sin\alpha t\Big|_0^2+$$

$$+\frac{1}{2\alpha}\int_0^2\sin\alpha t dt=-\frac{1}{2\alpha^2}\cos\alpha t\Big|_0^2=-\frac{1}{2\alpha^2}(\cos2\alpha-1)=\frac{1}{2\alpha^2}(1-\cos2\alpha)=\frac{\sin^2\alpha}{\alpha^2};$$

U holda

$$f(x)=\frac{2}{\pi}\int_0^\infty \frac{\sin^2\alpha}{\alpha^2}\cos\alpha x d\alpha$$

Xususiy holda x=0

$$f(0)=1=\frac{2}{\pi}\int_0^\infty \frac{\sin^2\alpha}{\alpha^2}d\alpha, \quad \int_0^\infty \frac{\sin^2\alpha}{\alpha^2}d\alpha=\frac{2}{\pi}$$

Quyidagi funksiyalarning Furye integrali topilsin.

№33. $f(x)=\begin{cases}1, |x|<a,\\ 0, |x|\ge a\end{cases}$

№34 $f(x)=\begin{cases}1, a<x<b,\\ 0, x\le a, x\ge b\end{cases}$

№35 $f(x)=\begin{cases}0, x\le 0,\\ \pi x, 0<x<1,\\ 0, x\ge 1\end{cases}$

№36 $f(x)=\begin{cases}\sin x, |x|<\pi,\\ 0, |x|\ge \pi\end{cases}$

№37 $f(x)=\begin{cases}1, |x|<1,\\ 0, |x|>1,\\ \frac{1}{2}, x=\pm 1\end{cases}$

№38 $f(x)=\begin{cases}\sin x, 0\le x\le \pi,\\ 0, x<0, x>\pi\end{cases}$

№39 $f(x)=\begin{cases}\cos x, 0<x<\pi,\\ 0, x>\pi\end{cases}$

№40 $f(x)=\begin{cases}2, & 0<x<3,\\ 1, & x=3,\\ 0, & x>3\end{cases}$

№41 $a)$ $f(x)=\begin{cases}1, & 0\le x\le 1,\\ 0, & x>1\end{cases}$

$b)$ $f(x)=e^{-ax}$ $(a>0, x\ge 0)$ funksiyani juft tarzda davom etdirib, Furye integrali ko`rinishida ifodalang:

№43. $f(x)=\begin{cases} x-1, & 0\le x\le 1, \\ 0, & x>1 \end{cases}$

funksiyani toq tarzda davom ettirib, Furye integrali orqali ifodalang.

№43. Quyidagi funksiyalarning kompleks o`zgaruvchi bo`yicha Furye integrali yozilsin.

a) $f(x)=\begin{cases} c, & |x|\le\pi, \\ 0, & |x|>\pi \end{cases}$

b) $f(x)=\begin{cases} e^{-ax}, & x>0, \\ 0, & x<0 \end{cases}$ $(a>0)$

Yechish. *a*) funksiya (-∞;+∞) oraliqda aniqlangan. (42), (43) formulalarga asosan: $f(x)=\int\limits_{-\infty}^{\infty}C(\alpha)e^{i\alpha x}d\alpha,\ \ C(\alpha)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}f(t)e^{-i\alpha x}\alpha t$

$(f(x)=0,(-\infty,-\pi),(\pi,\infty)).$

$$C(\alpha)=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}ce^{-i\alpha t}dt=-\frac{c}{2\pi i\alpha}e^{-i\alpha t}\Big|_{-\pi}^{\pi}=-\frac{c}{2\pi\alpha i}(e^{-i\alpha\pi}-e^{i\alpha\pi})=$$

$$=\frac{c}{\pi\alpha}\left(\frac{e^{i\alpha\pi}-e^{-i\alpha\pi}}{2i}\right)=\frac{c}{\pi}\frac{\sin\pi\alpha}{\alpha}$$

Bu holda, $f(x)=\frac{c}{\pi}\int\limits_{-\infty}^{\infty}\frac{\sin\pi\alpha}{\alpha}e^{ix\alpha}da.$

b) $f(x)=\begin{cases} e^{-ax}, x>0, \\ 0, x<0 \end{cases}$ $(a>0)$ funksiya (42), (43) formulalarga asosan kompleks o`zgaruvchi bo`yicha Furye integrali ko`rinishida ifodalaymiz:

$((-\infty,0)\{wf(x)=0)$

$$C(\alpha)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}f(x)e^{-it\alpha}dt=\frac{1}{2\pi}\int\limits_{0}^{\infty}e^{-at}e^{-it\alpha}dt=\frac{1}{2\pi}\lim_{b\to\infty}\int\limits_{0}^{b}e^{-(a+i\alpha)t}dt=$$

$$\frac{1}{2\pi}\lim_{b\to\infty}\left(-\frac{1}{a+i\alpha}e^{(a+i\alpha)t}\Big|_{0}^{b}=-\frac{1}{2\pi(a+i\alpha)}\lim_{b\to\infty}(e^{-(a+i\alpha)b}-)=\frac{1}{2\pi(a+i\alpha)}\right.$$

chunki $\lim\limits_{b\to\infty}e^{-ab}e^{-ib\alpha}=\lim\limits_{b\to\infty}e^{-ab}(\cos\alpha b-i\sin\alpha b)=0$.

Demak, $\lim\limits_{b\to\infty}e^{-(a+i\alpha)b}=0$, $\cos\alpha b$, $\sin\alpha b$ lar chegaralangan.

Amplituda spektri $|c(\alpha)|=\left|\frac{1}{2\pi(a+i\alpha)}\right|=\frac{1}{2\pi}\frac{1}{\sqrt{a^2+\alpha^2}}$

Bu holda berilgan funksiyaning kompleks o`zgaruvchi Furye integrali

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{a + i\alpha} e^{-ix\alpha} d\alpha$$ ko`rinishida bo`ladi .

Quyidagi funksiyalarni kompleks o`zgaruvchi bo`yicha Furye integrali ko`rinishida ifodalang:

№44. $f(x) = \begin{cases} 1, & 0 < x < a, \\ 0, & -\infty < x < 0, x > a \end{cases}$ №45 $f(x) = e^{-\frac{x^2}{2}}$

Foydalanilgan adabiyotlar

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